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# A SHARP MIXED TYPE INTEGRAL INEQUALITY

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ABSTRACT. We give a sharp mixed type integral inequality which may provide simplification and improvement of several recent results. Some special cases are discussed with applications in numerical integration and special means.

## 1. Introduction

In [1], Alomari and Dragomir proved the following mixed type inequalities for mappings of bounded variation, Lipschitzian, and absolutely continuous mappings whose first derivatives belong to $L_p[a,b] (1 \leq p \leq \infty)$ which have provided unified treatments of error estimates for various kinds of well-known quadrature rules.

**Theorem 1.1.** *Let $f : [a,b] \to \mathbf{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequality*

$$\begin{aligned}
&|\tfrac{b-a}{2\delta}[\alpha f(a) + \beta f(x) + 2(\gamma-\delta) f(\tfrac{a+b}{2}) + \beta f(a+b-x) + \alpha f(b)] - \textstyle\int_a^b f(t)\,dt| \\
&\leq \tfrac{1}{2\delta} \max\{2\delta(x-a) - \alpha(b-a), \alpha(b-a), (\alpha+\beta)(b-a) - 2\delta(x-a), \\
&(\delta-\alpha-\beta)(b-a)\} \textstyle\bigvee_a^b(f)
\end{aligned}$$

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*for all* $\frac{(2\delta-\alpha)a+\alpha b}{2\delta} \leq x \leq \frac{(2\delta-\alpha-\beta)a+(\alpha+\beta)b}{2\delta}$, *where,* $\bigvee_a^b(f)$ *denotes the total variation of* $f$ *over* $[a,b]$, *and* $\alpha,\beta,\gamma,\delta$ *are positive constants such that* $\alpha+\beta+\gamma=2\delta$ *with* $\gamma\geq\delta>0$.

**Theorem 1.2.** *Let* $f:[a,b]\to\mathbf{R}$ *be an L-Lipschitzian mapping on* $[a,b]$. *Then we have the inequality*

$$
\begin{aligned}
&|\tfrac{b-a}{2\delta}[\alpha f(a)+\beta f(x)+2(\gamma-\delta)f(\tfrac{a+b}{2})+\beta f(a+b-x)+\alpha f(b)]-\int_a^b f(t)\,dt|\\
&\leq L[\tfrac{(b-a)^2}{16}+(x-\tfrac{3a+b}{4})^2+2(\tfrac{\alpha(b-a)-\delta(x-a)}{2\delta})^2\\
&+(\tfrac{(2\delta-\alpha-\beta)a+(\alpha+\beta)b}{2\delta}-\tfrac{1}{2}(\tfrac{a+b}{2}+x))^2\\
&+(\tfrac{(\alpha+\beta)a+(2\delta-\alpha-\beta)b}{2\delta}-\tfrac{a+b}{2}-\tfrac{1}{2}(\tfrac{a+b}{2}-x))^2],
\end{aligned}
$$

*for all* $\frac{(2\delta-\alpha)a+\alpha b}{2\delta} \leq x \leq \frac{(2\delta-\alpha-\beta)a+(\alpha+\beta)b}{2\delta}$, *where,* $\alpha,\beta,\gamma,\delta$ *are positive constants such that* $\alpha+\beta+\gamma=2\delta$ *with* $\gamma\geq\delta>0$.

**Theorem 1.3.** *Let* $f:I\subset\mathbf{R}\to\mathbf{R}$ *be an absolutely continuous mapping on* $I^\circ$, *the interior of the interval* $I$, *where* $a,b\in I$ *with* $a<b$. *If* $f'\in L_p[a,b], p>1$. *Then we have the inequality*

$$
\begin{aligned}
&|\tfrac{b-a}{2\delta}[\alpha f(a)+\beta f(x)+2(\gamma-\delta)f(\tfrac{a+b}{2})+\beta f(a+b-x)+\alpha f(b)]-\int_a^b f(t)\,dt|\\
&\leq(\tfrac{2}{q+1})^{1/q}[(x-\tfrac{(2\delta-\alpha)a+\alpha b}{2\delta})^{q+1}+(\tfrac{\alpha(b-a)}{2\delta})^{q+1}+(\tfrac{(\delta-\alpha-\beta)(b-a)}{2\delta})^{q+1}\\
&(\tfrac{(2\delta-\alpha-\beta)a+(\alpha+\beta)b}{2\delta}-x)^{q+1}]^{1/q}\|f'\|_p,
\end{aligned}
$$

*for all* $\frac{(2\delta-\alpha)a+\alpha b}{2\delta} \leq x \leq \frac{(2\delta-\alpha-\beta)a+(\alpha+\beta)b}{2\delta}$, *where,* $\alpha,\beta,\gamma,\delta$ *are positive constants such that* $\alpha+\beta+\gamma=2\delta$ *with* $\gamma\geq\delta>0$ *and* $\frac{1}{p}+\frac{1}{q}=1, p>1$.

In this work, we will give a new mixed type integral inequality which provide unified treatment on various error estimations of the following quadrature rules:

(i) Midpoint rule

$$\int_a^b f(t)\,dt \approx (b-a)f(\frac{a+b}{2}).$$

(ii) Trapezoid rule

$$\int_a^b f(t)\,dt \approx (b-a)\frac{f(a)+f(b)}{2}.$$

(iii) Simpson's rule

$$\int_a^b f(t)\,dt \approx \frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)].$$

(iv) Corrected Simpson like rule

$$\int_a^b f(t)\,dt \approx \frac{b-a}{30}[7f(a)+16f(\frac{a+b}{2})+7f(b)].$$

(v) Averaged midpoint-trapezoid rule

$$\int_a^b f(t)\,dt \approx \frac{b-a}{4}[f(a)+2f(\frac{a+b}{2})+f(b)].$$

(vi) Maclaurin's rule

$$\int_a^b f(t)\,dt \approx \frac{b-a}{8}[3f(\frac{5a+b}{6})+2f(\frac{a+b}{2})+3f(\frac{a+5b}{6})].$$

(vii) Newton's 3/8-rule

$$\int_a^b f(t)\,dt \approx \frac{b-a}{8}[f(a)+3f(\frac{2a+b}{3})+3f(\frac{a+2b}{3})+f(b)].$$

(viii) Milne's rule (see e.g. [9], page 127)

$$\int_a^b f(t)\,dt \approx \frac{b-a}{90}[7f(a)+32f(\frac{3a+b}{4})+12f(\frac{a+b}{2})+32f(\frac{a+3b}{4})+7f(b)].$$

(ix) Companion of Ostrowski type quadrature rule (see e.g. [2] or [4])

$$\int_a^b f(t)\,dt \approx (b-a)\frac{f(x)+f(a+b-x)}{2}.$$

Our results simplify and improve Theorem 1.2. Some applications in numerical integration and to special means are also given.

## 2. MAIN RESULT

We first derive a mixed type inequality for functions of Lipschitzian type. Recall that a function $f : [a,b] \to \mathbf{R}$ is said to be $L$-Lipschitzian on $[a,b]$ if

$$|f(x) - f(y)| \leq L|x-y|$$

for all $x, y \in [a,b]$, where $L > 0$ is given, and, it is said to be $(l, L)$-Lipschitzian on $[a,b]$ (see e.g. [6]), if

$$l(x_2 - x_1) \leq f(x_2) - f(x_1) \leq L(x_2 - x_1)$$

for all $a \leq x_1 \leq x_2 \leq b$, where $l, L \in \mathbf{R}$ with $l < L$ (the condition has also been considered in [4] and [5] independently).

Clearly, an $L$-Lipschitzian function is a $(-L, L)$-Lipschitzian function.

It is well known (see e.g. [3]) that if $h, g : [a,b] \to \mathbf{R}$ are such that $h$ is Riemann integrable on $[a,b]$ and $g$ is $L$-Lipschitzian on $[a,b]$, then $\int_a^b h(t)\,dg(t)$ exists and

$$|\int_a^b h(x)\,dg(x)| \leq L\int_a^b |h(x)|\,dx. \tag{2.1}$$

**Theorem 2.1.** *Let $f : [a,b] \to \mathbf{R}$ be $(l, L)$-Lipschitzian on $[a,b]$. Then for any $\alpha \in [0,1]$ and $\beta \in [0,1]$ with $\alpha + \beta \leq 1$ we have the inequality*

$$\begin{aligned} &|\textstyle\int_a^b f(t)\,dt - (b-a)[\alpha\frac{f(a)+f(b)}{2} + \beta\frac{f(x)+f(a+b-x)}{2} + (1-\alpha-\beta)f(\frac{a+b}{2})]| \\ &\leq \tfrac{L-l}{2}\{\tfrac{2\alpha^2+\beta^2+2(\alpha+\beta-1)^2}{8}(b-a)^2 + 2[x - \tfrac{(4-2\alpha-\beta)a+(2\alpha+\beta)b}{4}]^2\}, \end{aligned} \tag{2.2}$$

*where* $0 \le \alpha + \beta \le 1$ *and* $x \in [a + \frac{\alpha}{2}(b-a), a + \frac{\alpha+\beta}{2}(b-a)]$.

*Proof.* For brevity, we first set $a = 0$, $b = 1$, and then we need to prove

$$\begin{aligned} &|\int_0^1 f(t)\,dt - [\alpha\tfrac{f(0)+f(1)}{2} + \beta\tfrac{f(x)+f(1-x)}{2} + (1-\alpha-\beta)f(\tfrac{1}{2})]| \\ &\le \tfrac{L-l}{2}[\tfrac{2\alpha^2+\beta^2+2(\alpha+\beta-1)^2}{8} + 2(x - \tfrac{2\alpha+\beta}{4})^2], \end{aligned} \tag{2.3}$$

where $0 \le \alpha + \beta \le 1$ and $x \in [\frac{\alpha}{2}, \frac{\alpha+\beta}{2}]$.

Integration by parts for Riemann-Stieltjes integral, we get

$$\begin{aligned} &\int_0^1 K(\alpha,\beta,x;t)\,df(t) \\ =\ &\alpha\tfrac{f(0)+f(1)}{2} + \beta\tfrac{f(x)+f(1-x)}{2} + (1-\alpha-\beta)f(\tfrac{1}{2}) - \int_a^b f(t)\,dt \end{aligned} \tag{2.4}$$

where

$$K(\alpha,\beta,x;t) = \begin{cases} t - \frac{\alpha}{2}, & 0 \le t < x, \\ t - \frac{\alpha+\beta}{2}, & x \le t < \frac{1}{2}, \\ t - (1 - \frac{\alpha+\beta}{2}), & \frac{1}{2} \le t < 1 - x, \\ t - (1 - \frac{\alpha}{2}), & 1 - x \le t \le b. \end{cases} \tag{2.5}$$

Observe that

$$\int_0^1 K(\alpha,\beta,x;t)\,dt = 0,$$

and so by (2.4) we have

$$\begin{aligned} &\int_0^1 K(\alpha,\beta,x;t)\,d[f(t) - \tfrac{l+L}{2}t] \\ =\ &\alpha\tfrac{f(0)+f(1)}{2} + \beta\tfrac{f(x)+f(1-x)}{2} + (1-\alpha-\beta)f(\tfrac{1}{2}) - \int_a^b f(t)\,dt. \end{aligned} \tag{2.6}$$

Moreover, for any $x \in [\frac{\alpha}{2}, \frac{\alpha+\beta}{2}]$, we get

$$
\begin{aligned}
&\int_0^1 |K(\alpha,\beta,x;t)|\,dt \\
=\;& \int_0^{\frac{\alpha}{2}} (\tfrac{\alpha}{2}-t)\,dt + \int_{\frac{\alpha}{2}}^{x} (t-\tfrac{\alpha}{2})\,dt \\
+\;& \int_x^{\frac{\alpha+\beta}{2}} (\tfrac{\alpha+\beta}{2}-t)\,dt + \int_{\frac{\alpha+\beta}{2}}^{\frac{1}{2}} (t-\tfrac{\alpha+\beta}{2})\,dt \\
+\;& \int_{\frac{1}{2}}^{1-\frac{\alpha+\beta}{2}} (1-\tfrac{\alpha+\beta}{2}-t)\,dt + \int_{1-\frac{\alpha+\beta}{2}}^{1-x} [t-(1-\tfrac{\alpha+\beta}{2})]\,dt \\
+\;& \int_{1-x}^{1-\frac{\alpha}{2}} (1-\tfrac{\alpha}{2}-t)\,dt + \int_{1-\frac{\alpha}{2}}^{1} [t-(1-\tfrac{\alpha}{2})]\,dt \\
=\;& \tfrac{2\alpha^2+(\alpha+\beta)^2+(\alpha+\beta-1)^2}{4} - (2\alpha+\beta)x + 2x^2 \\
=\;& \tfrac{2\alpha^2+\beta^2+2(\alpha+\beta-1)^2}{8} + 2(x-\tfrac{2\alpha+\beta}{4})^2.
\end{aligned}
\tag{2.7}
$$

Then notice that $f(x)-\frac{l+L}{2}x$ is $\frac{L-l}{2}$-Lipschitzian on $[a,b]$ and by using (2.1), the inequality (2.3) follows from (2.6) and (2.7).

The statement for $[a,b]$ is then derived in a routine way by applying to $F(t)=f[a+(b-a)t]$.

**Remark 1.** It is not difficult to find that the inequality (2.3) is sharp in the sense that we can choose $f$ to attain the equality in (2.3). In fact, we may construct the function $f(t)=\int_a^t j(s)\,ds$ with

$$
j(s)=\begin{cases}
l, & s\in(0,\frac{\alpha}{2}) \\
L, & s\in(\frac{\alpha}{2},x) \\
l, & s\in(x,\frac{\alpha+\beta}{2}) \\
L, & s\in(\frac{\alpha+\beta}{2},\frac{1}{2}) \\
l, & s\in(\frac{1}{2},1-\frac{\alpha+\beta}{2}) \\
L, & s\in(1-\frac{\alpha+\beta}{2},1-x) \\
l, & s\in(1-x,1-\frac{\alpha}{2}) \\
L, & s\in(1-\frac{\alpha}{2},1)
\end{cases}
$$

which satisfies the condition of Theorem 2.1.

**Remark 2.** It is clear that the best inequality in (2.2) is obtained for $x=\frac{(4-2\alpha-\beta)a+(2\alpha+\beta)b}{4}\in[a+\frac{\alpha}{2}(b-a),a+\frac{\alpha+\beta}{2}(b-a)]$ and we get a sharp inequality

$$|\int_a^b f(t)\,dt-(b-a)[\alpha\frac{f(a)+f(b)}{2}+\beta\frac{f(\frac{(4-2\alpha-\beta)a+(2\alpha+\beta)b}{4})+f(\frac{(2\alpha+\beta)a+(4-2\alpha-\beta)b}{4})}{2}$$
$$+(1-\alpha-\beta)f(\tfrac{a+b}{2})]|\leq\frac{L-l}{2}[\frac{2\alpha^2+\beta^2+2(\alpha+\beta-1)^2}{8}(b-a)^2]$$

where $0\leq\alpha+\beta\leq 1$

**Remark 3.** Some special cases of the inequality (2.2) in Theorem 2.1 will be listed and discussed as follows:

(i) If we take $\alpha=1$ and $\beta=0$, then $x=\frac{a+b}{2}$ and we get a sharp trapezoid type inequality

$$|\int_a^b f(t)\,dt-(b-a)\frac{f(a)+f(b)}{2}|\leq\frac{L-l}{8}(b-a)^2. \tag{2.8}$$

(ii) If we take $\alpha=0$ and $\beta=0$, then $x=a$ and we get a sharp midpoint type inequality

$$|\int_a^b f(t)\,dt-(b-a)f(\frac{a+b}{2})|\leq\frac{L-l}{8}(b-a)^2. \tag{2.9}$$

(iii) If we take $\alpha=\frac{1}{3}$ and $\beta=0$, then $x=\frac{5a+b}{6}$ and we get a sharp Simpson type inequality

$$|\int_a^b f(t)\,dt-\frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)]|\leq\frac{5(L-l)}{72}(b-a)^2. \tag{2.10}$$

(iv) If we take $\alpha=\frac{7}{15}$ and $\beta=0$, then $x=\frac{23a+7b}{30}$ and we get a sharp corrected Simpson like type inequality

$$|\int_a^b f(t)\,dt-\frac{b-a}{30}[7f(a)+16f(\frac{a+b}{2})+7f(b)]|\leq\frac{113(L-l)}{1800}(b-a)^2. \tag{2.11}$$

(v) If we take $\alpha=\frac{1}{2}$ and $\beta=0$, then $x=\frac{3a+b}{4}$ and we get a sharp averaged midpoint-trapezoid type inequality

$$(2.12)\qquad \left|\int_a^b f(t)\,dt - \frac{b-a}{4}[f(a)+2f(\frac{a+b}{2})+f(b)]\right| \leq \frac{L-l}{16}(b-a)^2.$$

(vi) If we take $\alpha = 0$ and $\beta = 1$, then $x \in [a, \frac{a+b}{2}]$ and we get a sharp company of Ostrowski type inequality

$$(2.13)\quad \left|\int_a^b f(t)\,dt - (b-a)\frac{f(x)+f(a+b-x)}{2}\right| \leq \frac{L-l}{2}[\frac{(b-a)^2}{8} + 2(x - \frac{3a+b}{4})^2].$$

It is clear that the best inequality in (2.13) is obtained for $x = \frac{3a+b}{4}$ and giving a sharp trapezoid type inequality

$$(2.14)\qquad \left|\int_a^b f(t)\,dt - (b-a)\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}\right| \leq \frac{L-l}{16}(b-a)^2.$$

(vii) If we take $\alpha = 0$ and $\beta = \frac{3}{4}$, then $x \in [a, \frac{5a+3b}{8}]$ and we get a sharp inequality

(2.15)
$$\left|\int_a^b f(t)\,dt - \frac{b-a}{8}[3f(x)+2f(\frac{a+b}{2})+3f(a+b-x)]\right| \leq \frac{L-l}{2}[\frac{11(b-a)^2}{128} + 2(x - \frac{13a+3b}{16})^2].$$

If we further take $x = \frac{5a+b}{6}$, then we get a sharp Maclaurin's type inequality

$$(2.16)\quad \left|\int_a^b f(t)\,dt - \frac{b-a}{8}[3f(\frac{5a+b}{6})+2f(\frac{a+b}{2})+3f(\frac{a+5b}{6})]\right| \leq \frac{25(L-l)}{576}(b-a)^2.$$

However, the best inequality in (2.15) is obtained for $x = \frac{13a+3b}{16}$ and we get a sharp inequality

(2.17)
$$\left|\int_a^b f(t)\,dt - \frac{b-a}{8}[3f(\frac{13a+3b}{16})+2f(\frac{a+b}{2})+3f(\frac{3a+13b}{16})]\right| \leq \frac{11(L-l)}{256}(b-a)^2.$$

(viii) If we take $\alpha = \frac{1}{4}$ and $\beta = \frac{3}{4}$, then $x \in [\frac{7a+b}{8}, \frac{a+b}{2}]$ and we get a sharp inequality

$$|\int_a^b f(t)\,dt - \frac{b-a}{8}[f(a)+3f(x)+3f(a+b-x)+f(b)]| \le \frac{L-l}{2}[\frac{11(b-a)^2}{128}+2(x-\frac{11a+5b}{16})^2]. \tag{2.18}$$

If we further take $x = \frac{2a+b}{3}$, then we get a sharp Newton's 3/8 type inequality

$$|\int_a^b f(t)\,dt - \frac{b-a}{8}[f(a)+3f(\frac{2a+b}{3})+3f(\frac{a+2b}{3})+f(b)]| \le \frac{25(L-l)}{576}(b-a)^2. \tag{2.19}$$

However, the best inequality in (2.18) is obtained for $x = \frac{11a+5b}{16}$ and we get a sharp inequality

$$|\int_a^b f(t)\,dt - \frac{b-a}{8}[f(a)+3f(\frac{11a+5b}{16})+3f(\frac{5a+11b}{16})+f(b)]| \le \frac{11(L-l)}{256}(b-a)^2. \tag{2.20}$$

(ix) If we take $\alpha = \frac{7}{45}$ and $\beta = \frac{32}{45}$, then $x \in [\frac{83a+7b}{90}, \frac{51a+39b}{90}]$ and we get a sharp inequality

$$\begin{aligned} &|\textstyle\int_a^b f(t)\,dt - \frac{b-a}{90}[7f(a)+32f(x)+12f(\frac{a+b}{2})+32f(a+b-x)+7f(b)]| \\ &\le \tfrac{L-l}{2}[\tfrac{199(b-a)^2}{2700}+2(x-\tfrac{67a+23b}{90})^2]. \end{aligned} \tag{2.21}$$

If we further take $x = \frac{3a+b}{4}$, then we get a sharp Milne's type inequality

$$|\int_a^b f(t)\,dt - \frac{b-a}{90}[7f(a)+32f(\frac{3a+b}{4})+12f(\frac{a+b}{2})+32f(\frac{a+3b}{4})+7f(b)]| \le \frac{239(L-l)}{6480}(b-a)^2. \tag{2.22}$$

However, the best inequality in (2.21) is obtained for $x = \frac{67a+23b}{90}$ and we get a sharp inequality

$$\begin{aligned}(2.23)\quad &\left|\int_a^b f(t)\,dt - \tfrac{b-a}{90}[7f(a) + 32f(\tfrac{67a+23b}{90}) + 12f(\tfrac{a+b}{2}) + 32f(\tfrac{23a+67b}{90}) + 7f(b)]\right| \\ &\le \tfrac{199(L-l)}{5400}(b-a)^2.\end{aligned}$$

(x) If we take $\alpha + \beta = 1$, then we get a sharp integral inequality

$$\begin{aligned}(2.24)\quad &\left|\int_a^b f(t)\,dt - (b-a)[\alpha\tfrac{f(a)+f(b)}{2} + (1-\alpha)\tfrac{f(x)+f(a+b-x)}{2}]\right| \\ &\le \tfrac{L-l}{2}\{\tfrac{2\alpha^2+(1-\alpha)^2}{8}(b-a)^2 + 2[x - \tfrac{(3-\alpha)a+(1+\alpha)b}{4}]^2\}\end{aligned}$$

where $0 \le \alpha \le 1$ and $x \in [a + \frac{\alpha}{2}(b-a), \frac{a+b}{2}]$. However, the best inequality in (2.24) is obtained for $x = \frac{(3-\alpha)a+(1+\alpha)b}{4}$ and we get a sharp inequality

$$\begin{aligned}(2.25)\quad &\left|\int_a^b f(t)\,dt - (b-a)[\alpha\tfrac{f(a)+f(b)}{2} + (1-\alpha)\tfrac{f(\frac{(3-\alpha)a+(1+\alpha)b}{4})+f(\frac{(1+\alpha)a+(3-\alpha)b}{4})}{2}]\right| \\ &\le \tfrac{L-l}{16}[2\alpha^2 + (1-\alpha)^2](b-a)^2,\end{aligned}$$

where $0 \le \alpha \le 1$. Clearly, the inequality (2.25) recaptures the inequality (2.14) for $\alpha = 0$.

## 3. APPLICATIONS

We first consider the application in numerical integration and restrict it to the Milne's quadrature rule.

**Theorem 3.1.** *Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a, b]$ such that $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ and let the assumptions of Theorem 2.1 hold. Then we have*

(3.1)

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - \tfrac{h}{90}\sum_{1=0}^{n-1}[7f(x_i) + 32f(\tfrac{3x_i+x_{i+1}}{4}) + 12f(\tfrac{x_i+x_{i+1}}{2}) + 32f(\tfrac{x_i+3x_{i+1}}{4}) + 7f(x_{i+1})]\right| \\ &\le \tfrac{239(L-l)(b-a)^2}{6480n}.\end{aligned}$$

*Proof.* From the inequality (2.22) in Remark 3 we obtain

$$(3.2)\quad \begin{aligned} &|\int_{x_i}^{x_{i+1}} f(t)\,dt - \tfrac{h}{90}[7f(x_i) + 32f(\tfrac{3x_i+x_{i+1}}{4}) + 12f(\tfrac{x_i+x_{i+1}}{2}) + 32f(\tfrac{x_i+3x_{i+1}}{4}) + 7f(x_{i+1})]| \\ &\leq \tfrac{239(L-l)}{6480}(x_{i+1}-x_i)^2 \\ &\leq \tfrac{239(L-l)(b-a)^2}{6480n^2} \end{aligned}$$

By summing (3.2) over $i$ from 0 to $n-1$, we get

$$(3.3)\quad \begin{aligned} &\sum_{i=0}^{n-1} |\int_{x_i}^{x_{i+1}} f(t)\,dt - \tfrac{h}{90}[7f(x_i) + 32f(\tfrac{3x_i+x_{i+1}}{4}) + 12f(\tfrac{x_i+x_{i+1}}{2}) + 32f(\tfrac{x_i+3x_{i+1}}{4}) + 7f(x_{i+1})]| \\ &\leq \tfrac{239(L-l)(b-a)^2}{6480n}. \end{aligned}$$

Consequently, the inequality (3.1) follows from (3.3). Thus the theorem is proved.

Now we turn to applications of the Milne's type inequality (2.22) to the following special means:

(1) The arithmetic mean:

$$A(a,b) := \frac{a+b}{2}, a,b \geq 0.$$

(2) The Geometric mean:

$$G(a,b) := \sqrt{ab}, a,b \geq 0.$$

(3) The harmonic mean:

$$H(a,b) := \frac{2ab}{a+b}, a,b > 0.$$

(4) The logarithmic mean:

$$L(a,b) := \frac{b-a}{lnb - lna}, a \neq b, a,b > 0.$$

(5) The identric mean:

$$I(a,b) := \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}, a \neq b, a, b > 0.$$

(6) The p-logarithmic mean:

$$L_p(a,b) = [\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}]^{\frac{1}{p}}, a \neq b, a, b > 0, p \neq -1, 0.$$

**Proposition 1.** Let $a, b \in \mathbf{R}$, $0 < a < b$ and $n \in \mathbf{N}$, $n \geq 3$. Then we have

$$|\frac{1}{45}[7A(a^n, b^n) + 16A^n(\frac{3a}{2}, \frac{b}{2}) + 16A^n(\frac{a}{2}, \frac{3b}{2}) + 6A^n(a,b)] - L_n^n(a,b)| \leq \frac{239n(b-a)b^{n-1}}{3240},$$

*Proof.* The assertion follows from applied the inequality (2.22) to the mapping $f(x) = x^n, x \in [a,b]$ and $n \in \mathbf{N}$ which implies that $|f'(x)| = nx^{n-1} \leq nb^{n-1}$ on $[a,b]$ and so $f$ is $nb^{n-1}$-Lipschitzian on $[a,b]$.

**Proposition 2.** Let $a, b \in \mathbf{R}$, $0 < a < b$. Then we have

$$|\frac{1}{45}[7H^{-1}(a,b) + 16A^{-1}(\frac{3a}{2}, \frac{b}{2}) + 16A^{-1}(\frac{a}{2}, \frac{3b}{2}) + 6A^{-1}(a,b)] - L^{-1}(a,b)| \leq \frac{239(b-a)}{3240a^2},$$

*Proof.* The assertion follows from applied the inequality (2.22) to the mapping $f(x) = \frac{1}{x}, x \in [a,b]$ which implies that $|f'(x)| = \frac{1}{x^2} \leq \frac{1}{a^2}$ on $[a,b]$ and so $f$ is $\frac{1}{a^2}$-Lipschitzian on $[a,b]$.

**Proposition 3.** Let $a, b \in \mathbf{R}$, $0 < a < b$. Then we have

$$|\frac{1}{45}[7lnG(a,b) + 6lnA(a,b) + 16lnA(\frac{3a}{2}, \frac{b}{2}) + 16lnA(\frac{a}{2}, \frac{3b}{2})] - lnI(a,b)| \leq \frac{239(b-a)}{3240a},$$

*Proof.* The assertion follows from applied the inequality (2.22) to the mapping $f(x) = lnx, x \in [a,b]$ which implies that $|f'(x)| = \frac{1}{x} \leq \frac{1}{a}$, on $[a,b]$ and so $f$ is $\frac{1}{a}$-Lipschitzian on $[a,b]$.

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