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A HÖRMANDER TYPE MULTIPLIER THEOREM ON FRACTIONAL FOURIER MULTIPLIER OPERATORS

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Abstract. In this article, we consider the bilinear and biparameter fractional Fourier multiplier operators T_m, whose symbol m satisfies a certain Hörmander type multiplier condition. We prove the boundedness of T_m on Lebesgue spaces. More precisely, we show that T_m is bounded from $L^{p_1}(\mathbb{R}^{2n}) \times L^{p_2}(\mathbb{R}^{2n})$ to $L^q(\mathbb{R}^{2n})$ for appropriate indices p_1, p_2 and q. This is a bilinear-biparameter extension of the classical Hörmander type multiplier theorem.

1. Introduction

For Schwartz function f on $\mathbb{R}^d$, we define the Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx.$$

and the inverse Fourier transform by

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)d\xi.$$

Let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\begin{aligned} &\operatorname{supp}\Psi \subset \left\{\xi\in\mathbb{R}^d : 1/2 \le |\xi| \le 2\right\}, \\ &\sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1 \text{ for } \ \xi \in \mathbb{R}^d\backslash\{0\}. \end{aligned} \tag{1.1}$$

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For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^s} = \|(I-\Delta)^{s/2} f\|_{L^2} < \infty,$$

where $(I-\Delta)^{s/2} f = \mathcal{F}^{-1}\left[(1+|\xi|^2)^{s/2}\widehat{f}(\xi)\right]$.

The multilinear Calderón-Zygmund theory is originated in several works by Coifman and Meyer[16,17] and [18]. After Lacey and Thiele's work [13,14] on the bilinear Hilbert transform, multilinear operators in harmonic analysis have been well studied by many authors. In this paper, we consider the boundedness of bilinear and biparameter fractional Fourier multiplier.

We first recall the linear case. The Mihlin multiplier theorem says that if $m \in C^{[n/2]+1}(\mathbb{R}^n \backslash \{0\})$ satisfies

$$|\partial^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \tag{1.2}$$

for all $|\alpha| \le [n/2]+1$, then $m(D)$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ (see [5, Corollary 8.11]), where $[n/2]$ is the integer part of $n/2$ and $m(D)f = \mathcal{F}^{-1}[m\widehat{f}]$.

The Hörmander multiplier theorem [11] states that if $m \in L^\infty(\mathbb{R}^n)$ satisfies

$$\sup_{j\in\mathbb{Z}} \|m(2^j \cdot)\Psi\|_{H^s(\mathbb{R}^n)} < \infty \tag{1.3}$$

with $s > n/2$, then $m(D)$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ (see also [5, Theorem 8.10]), where $H^s(\mathbb{R}^n)$ is the Sobolev space and Ψ is as in (1.1) with $d = n$. We note that (1.3) is weaker than (1.2).

We next consider the bilinear case. For $m \in L^\infty(\mathbb{R}^{2n})$, the bilinear fractional Fourier multiplier operator T_m is defined by

$$T_m(f,g)(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi+\eta)} m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)\,d\xi d\eta \tag{1.4}$$

for $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Coifman and Meyer [16,19] and [20] first proved that if $m \in C^L(\mathbb{R}^{2n}\backslash\{0\})$ satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \le C_{\alpha,\beta}(|\xi|+|\eta|)^{-(|\alpha|+|\beta|)} \tag{1.5}$$

for all $|\alpha|+|\beta| \le L$, where L is a sufficiently large natural number, then T_m is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ for all $1 < p, q, r < \infty$ with $1/p + 1/q = 1/r$. Results in [16,19] and [20] have been extended to multilinear Calderón-Zygmund operators by Kenig and Stein[1], Grafakos and Kalton [8] , Grafakos and Torres [9,10] to include $0 < r \le 1$.

Tomita [15] proved that if $m \in L^\infty(\mathbb{R}^{2n})$ satisfies

$$\sup_{k\in\mathbb{Z}} \|m_k\|_{H^s(\mathbb{R}^{2n})} < \infty, \tag{1.6}$$

where $m_k(\xi,\eta) = m(2^k\xi, 2^k\eta)\Psi(\xi,\eta)$, $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, $k \in \mathbb{Z}$, Ψ is as in (1.1) with $d = 2n$, then T_m is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ for all $1 < p, q, r < \infty$ with $1/p + 1/q = 1/r$.

Next, we discuss the L^r estimates for multilinear and multiparameter Fourier multiplier operators. In the bilinear and biparameter case, Muscalu, Pipher, Tao and Thiele [2] proved if $m \in L^\infty(\mathbb{R}^{4n})$ satisfies

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2} m(\xi_1,\xi_2,\eta_1,\eta_2)| \le C_{\alpha_1\alpha_2\beta_1\beta_2}(|\xi_1|+|\eta_1|)^{-(|\alpha_1|+|\beta_1|)}(|\xi_2|+|\eta_2|)^{-(|\alpha_2|+|\beta_2|)},$$

then T_m is bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$, where T_m is defined by

$$\begin{aligned} T_m(f,g)(x_1,x_2) = \frac{1}{(2\pi)^{4n}} \int_{\mathbb{R}^{4n}} & e^{ix_1\cdot(\xi_1+\eta_1)+ix_2\cdot(\xi_2+\eta_2)} m(\xi_1,\xi_2,\eta_1,\eta_2) \\ & \times \widehat{f}(\xi_1,\xi_2)\widehat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2. \end{aligned} \tag{1.7}$$

This theorem was extended to the case of multilinear and multiparameter setting in [3].

Next we introduce the biparameter Sobolev spaces. For $s_1, s_2 \in \mathbb{R}$, the biparameter Sobolev space $H^{s_1,s_2}(\mathbb{R}^{4n})$ consists of all $f \in \mathcal{S}'(\mathbb{R}^{4n})$ such that

$$\|f\|_{H^{s_1,s_2}} = \|(I-\Delta)^{s_1/2,s_2/2} f\|_{L^2} < \infty,$$

where

$$(I-\Delta)^{s_1/2,s_2/2}f = \mathcal{F}^{-1}\left[\left(1+|\xi_1|^2+|\eta_1|^2\right)^{s_1/2}\left(1+|\xi_2|^2+|\eta_2|^2\right)^{s_2/2}\widehat{f}(\xi_1,\xi_2,\eta_1,\eta_2)\right]$$

with $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}^n$.

Recently, Chen and Lu [4] proved if $m \in L^\infty(\mathbb{R}^{4n})$ satisfies

$$\sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})} < \infty,$$

where

$$m_{j,k}(\xi_1,\xi_2,\eta_1,\eta_2) = m(2^j\xi_1, 2^k\xi_2, 2^j\eta_1, 2^k\eta_2)\Psi_1(\xi_1,\eta_1)\Psi_2(\xi_2,\eta_2), \tag{1.8}$$

and Ψ_1, Ψ_2 are the same as (1.1) with $d = 2n$, then T_m is bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$ for all $1 < p, q < \infty$, $0 < r < \infty$ with $1/p + 1/q = 1/r$.

The purpose of this paper is to proved a Hörmander type multiplier theorem for bilinear and biparameter fractional Fourier multiplier operators. The main result of this paper is as follows.

Theorem 1.1. *Let* $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$,

$$\operatorname{supp}\Psi \subset \left\{\xi \in \mathbb{R}^{2n} : 1/2 \le |\xi| \le 2\right\},$$

$$\sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1, \xi \in \mathbb{R}^{2n}\backslash\{0\},$$

$$m_{j,k,\alpha}(\xi_1,\xi_2,\eta_1,\eta_2) = 2^{j\alpha}2^{k\alpha}m(2^j\xi_1, 2^k\xi_2, 2^j\eta_1, 2^k\eta_2)\Psi_1(\xi_1,\eta_1)\Psi_2(\xi_2,\eta_2), \tag{1.9}$$

$$\begin{aligned} T_m(f,g)(x_1,x_2) = \frac{1}{(2\pi)^{4n}} \int_{\mathbb{R}^{4n}} & e^{ix_1\cdot(\xi_1+\eta_1)+ix_2\cdot(\xi_2+\eta_2)} m(\xi_1,\xi_2,\eta_1,\eta_2) \\ & \times \widehat{f}(\xi_1,\xi_2)\widehat{g}(\eta_1,\eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2, \end{aligned}$$

$m \in L^\infty(\mathbb{R}^{4n})$*,* $s_1, s_2 > n$*,* $s = \min\{s_1, s_2\}$*,* $1/q = 1/q_1 + 1/q_2$*,* $1/q_1 = 1/p_1 - \alpha_1/n$*,* $1/q_2 = 1/p_2 - \alpha_2/n$*,* $\alpha = \alpha_1 + \alpha_2$*,* $0 < q < \infty$*,* $1 < p_1, p_2 < \infty$*,* $q_1 > 2n/(s+\alpha)$*,* $q_2 > 2n/(s+\alpha)$*,* *If*

$$\sup_{j,k\in\mathbb{Z}} \|m_{j,k,\alpha}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})} < \infty,$$

then T_m *is bounded from* $L^{p_1}(\mathbb{R}^{2n}) \times L^{p_2}(\mathbb{R}^{2n})$ *to* $L^q(\mathbb{R}^{2n})$.

2. M T

The biparameter fractional maximal operator M_α for a function f on $\mathbb{R}^{2n}$ is defined by

$$M_\alpha f(x,y) = \sup_{r_1,r_2>0} \frac{1}{{r_1}^{n-\alpha}} \frac{1}{{r_2}^{n-\alpha}} \int_R |f(u,v)|dudv,$$

where $R = \{(u,v) \in \mathbb{R}^{2n} : |u-x| < r_1, |v-y| < r_2\}$ and f is a locally integral function on $\mathbb{R}^{2n}$.

Lemma 2.1. *Let* $\epsilon_1, \epsilon_2 > 0$, $0 < \alpha < n$. *Then there a constant* $C > 0$ *such that*

$$\sup_{r_1,r_2>0} \left(\int_{\mathbb{R}^{2n}} \frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}} dudv \right) \le CM_\alpha f(x,y).$$

Proof. Since

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}} dudv \\
&= \int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<1/r_1,|v-y|<1/r_2\}} \frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}} dudv \\
&\quad + \int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|>1/r_1,|v-y|>1/r_2\}} \frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}} dudv \\
&\quad + \int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<1/r_1,|v-y|\ge 1/r_2\}} \frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}} dudv \\
&\quad + \int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|\ge 1/r_1,|v-y|<1/r_2\}} \frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}} dudv \\
&= J_1 + J_2 + J_3 + J_4,
\end{aligned}$$

we can get

$$J_1 = \int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<1/r_1,|v-y|<1/r_2\}} \frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}} dudv$$

$$
\begin{aligned}
&\le {r_1}^{n-\alpha}{r_2}^{n-\alpha}\int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<1/r_1,|v-y|<1/r_2\}}|f(u,v)|dudv\\
&\le CM_\alpha f(x,y),
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|>1/r_1,|v-y|>1/r_2\}}\frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}}dudv\\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\int_{\{(u,v)\in\mathbb{R}^{2n}:2^k/r_1<|u-x|\le 2^{k+1}/r_1,2^j/r_2<|v-y|\le 2^{j+1}/r_2\}}\\
&\qquad\qquad \times\frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}}dudv\\
&\le \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<2^{k+1}/r_1,|v-y|<2^{j+1}/r_2\}}\frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{2^{k(n-\alpha+\epsilon_1)}2^{j(n-\alpha+\epsilon_2)}}dudv\\
&\le C\sum_{k=0}^{\infty}2^{-k\epsilon_1}\sum_{j=0}^{\infty}2^{-j\epsilon_2}\frac{1}{(2^{k+1}/r_1)^{n-\alpha}}\frac{1}{(2^{j+1}/r_2)^{n-\alpha}}\\
&\qquad\qquad \times\int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<2^{k+1}/r_1,|v-y|<2^{j+1}/r_2\}}|f(u,v)|dudv\\
&\le CM_\alpha f(x,y).
\end{aligned}
$$

Next we estimate J_3,

$$
\begin{aligned}
J_3 &= \int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<1/r_1,|v-y|\ge 1/r_2\}}\frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}}dudv\\
&= \sum_{j=0}^{\infty}\int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<1/r_1,2^j/r_2\le|v-y|<2^{j+1}/r_2\}}\\
&\qquad\qquad \times\frac{{r_1}^{n-\alpha}{r_2}^{n-\alpha}|f(u,v)|}{(1+r_1|x-u|)^{n-\alpha+\epsilon_1}(1+r_2|y-v|)^{n-\alpha+\epsilon_2}}dudv\\
&= {r_1}^{n-\alpha}{r_2}^{n-\alpha}\sum_{j=0}^{\infty}\int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<1/r_1,2^j/r_2\le|v-y|<2^{j+1}/r_2\}}\frac{|f(u,v)|}{2^{j(n-\alpha+\epsilon_2)}}dudv\\
&\le {r_1}^{n-\alpha}{r_2}^{n-\alpha}\sum_{j=0}^{\infty}\int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<1/r_1,|v-y|<2^{j+1}/r_2\}}\frac{|f(u,v)|}{2^{j(n-\alpha+\epsilon_2)}}dudv\\
&\le C\sum_{j=0}^{\infty}2^{-j\epsilon_2}\frac{1}{(1/r_1)^{n-\alpha}}\frac{1}{(2^{j+1}/r_2)^{n-\alpha}}
\end{aligned}
$$

$$\times \int_{\{(u,v)\in\mathbb{R}^{2n}:|u-x|<1/r_1,|v-y|<2^{j+1}/r_2\}} |f(u,v)|dudv$$
$$\leq CM_\alpha f(x,y).$$

The estimation of J_4 is similar to J_3, so we omit the details here, we can get $J_4 \leq CM_\alpha f(x,y)$, so we finished the proof of this Lemma. □

Lemma 2.2. $1 < p,q,r < \infty$, $1/q = 1/p - \alpha/n$. *Then there exists a constant* $C > 0$ *such that*

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} (M_\alpha f_k)^r \right\}^{1/r} \right\|_{L^q} \leq C \left\| \left\{ \sum_{k\in\mathbb{Z}} |f_k|^r \right\}^{1/r} \right\|_{L^p}.$$

Proof. Let I_α be the fractional integral operator defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

and $I_\alpha f(x,y)$ be the biparameter fractional integral operator defined by

$$I_\alpha f(x,y) = \int_{\mathbb{R}^{2n}} \frac{f(u,v)}{|x-u|^{n-\alpha}|y-v|^{n-\alpha}} dudv.$$

Let

$$I_\alpha^{(1)} f = \int_{\mathbb{R}^n} \frac{f(u,v)}{|x-u|^{n-\alpha}} du,$$
$$I_\alpha^{(2)} f = \int_{\mathbb{R}^n} \frac{f(u,v)}{|y-v|^{n-\alpha}} dv.$$

then

$$I_\alpha f(x,y) = \left(I_\alpha^{(1)} \circ I_\alpha^{(2)} f\right)(x,y).$$

Since $I_\alpha^{(1)}$, $I_\alpha^{(2)}$ is bounded from L^p to L^q with

$$\|I_\alpha^{(1)} f\|_{L^q} \leq C\|f\|_{L^p},$$

$$\|I_\alpha^{(2)} f\|_{L^q} \leq C\|f\|_{L^p},$$

So we can get

$$
\begin{aligned}
&\|I_\alpha f(x,y)\|_{L^q(\mathbb{R}^{2n})}\\
&= \left\| \left\| \left(I_\alpha^{(1)} \circ I_\alpha^{(2)} f\right)(x,y) \right\|_{L_x^q(\mathbb{R}^n)} \right\|_{L_y^q(\mathbb{R}^n)}\\
&\le C \left\| \left\| \left(I_\alpha^{(2)} f\right)(x,y) \right\|_{L_x^p(\mathbb{R}^n)} \right\|_{L_y^q(\mathbb{R}^n)}\\
&\le C \left\| \left\| \left(I_\alpha^{(2)} f\right)(x,y) \right\|_{L_y^q(\mathbb{R}^n)} \right\|_{L_x^p(\mathbb{R}^n)}\\
&\le C \left\| \|f(x,y)\|_{L_y^p(\mathbb{R}^n)} \right\|_{L_x^p(\mathbb{R}^n)}\\
&= \|f(x,y)\|_{L^p(\mathbb{R}^{2n})},
\end{aligned}
$$

Since $I_\alpha f(x,y)$ is a positive operator, the above inequality has a vector-valued extension (see[6, Proposition 4.5.10]):

$$
\left\| \left\{ \sum_{k\in\mathbb{Z}} (I_\alpha |f_k|)^r \right\}^{1/r} \right\|_{L^q} \le C \left\| \left\{ \sum_{k\in\mathbb{Z}} |f_k|^r \right\}^{1/r} \right\|_{L^p}.
$$

since

$$
\begin{aligned}
&I_\alpha |f|(x,y)\\
&= \int_{\mathbb{R}^{2n}} \frac{|f(u,v)|}{|x-u|^{n-\alpha}|y-v|^{n-\alpha}} dudv\\
&\ge \int_{\{(u,v):|u-x|<r_1,|v-y|<r_2\}} \frac{|f(u,v)|}{{r_1}^{n-\alpha}{r_2}^{n-\alpha}} dudv\\
&\ge M_\alpha f(x,y),
\end{aligned}
$$

We get

$$
\left\| \left\{ \sum_{k\in\mathbb{Z}} (M_\alpha f_k)^r \right\}^{1/r} \right\|_{L^q} \le C \left\| \left\{ \sum_{k\in\mathbb{Z}} |f_k|^r \right\}^{1/r} \right\|_{L^p}.
$$

□

Lemma 2.3. *[7] Let $2 \le q < \infty$, $r > 0$ and $s \ge 0$. Then there exists a constant $C > 0$ such that*

$$\|\widehat{f}\|_{L^q(\omega_{s,q})} := \left(\int_{\mathbb{R}^{4n}} |\widehat{f}(x,y)|^q (1+x^2)^{s/2}(1+y^2)^{s/2} dxdy \right)^{1/q}$$
$$\le C\|f\|_{H^{s/q,s/q}(\mathbb{R}^{2n}\times\mathbb{R}^{2n})}.$$

for $f \in H^{s/q,s/q}(\mathbb{R}^{2n}\times\mathbb{R}^{2n})$ with $\operatorname{supp} f \subset \{\sqrt{|x|^2+|y|^2} \le r\}$.

Lemma 2.4. *[4] Let $1 < p < \infty$ and $\Psi_1 \in \mathcal{S}(\mathbb{R}^n)$, $\Psi_2 \in \mathcal{S}(\mathbb{R}^n)$ be such that* $\operatorname{supp} \Psi_1 \subset \{\xi : 1/a \le |\xi| \le a\}$ *for some $a > 1$,* $\operatorname{supp} \Psi_2 \subset \{\eta : 1/b \le |\eta| \le b\}$ *for some $b > 1$. Then for $f \in L^p(\mathbb{R}^{2n})$, there exists a constant $C > 0$ such that*

$$\left\| \left\{ \sum_{j,k\in\mathbb{Z}} \left| \Psi_1(D_1/2^j)\Psi_2(D_2/2^k)f \right|^2 \right\}^{1/2} \right\|_{L^p} \le C\|f\|_{L^p}.$$

Moreover, if $\sum_{j\in\mathbb{Z}} \Psi_i(\xi_i/2^j) = 1$ for all $\xi_i \ne 0$, $i = 1,2$. Then for $f \in L^p(\mathbb{R}^{2n})$,

$$\left\| \left\{ \sum_{j,k\in\mathbb{Z}} \left| \Psi_1(D_1/2^j)\Psi_2(D_2/2^k)f \right|^2 \right\}^{1/2} \right\|_{L^p} \approx \|f\|_{L^p}.$$

where $\left[\Psi_1(D_1/2^j)f\right](\xi_1,\xi_2) := \mathcal{F}^{-1}\left[\Psi_1(\cdot/2^j)\widehat{f}(\cdot,\cdot)\right](\xi_1,\xi_2)$, $\left[\Psi_2(D_2/2^k)f\right](\xi_1,\xi_2) := \mathcal{F}^{-1}\left[\Psi_2(\cdot/2^k)\widehat{f}(\cdot,\cdot)\right](\xi_1,\xi_2)$.

Let ϕ_1 be a C^∞-function on $[0,\infty)$ satisfying $\operatorname{supp}\phi_1 \subset [0,1/4]$ and $\phi_1(t) = 1$ on $[0,1/8]$. we set $\phi_2(t) = 1-\phi_1(t)$, and set for $\xi,\eta \in \mathbb{R}^n$,

$$\Phi_{(1)}(\xi,\eta) = \phi_1(|\xi|/|\eta|), \tag{2.1}$$

$$\Phi_{(2)}(\xi,\eta) = \phi_2(|\eta|/|\xi|), \tag{2.2}$$

$$\Phi_{(3)}(\xi,\eta) = (1-\phi_1(|\xi|/|\eta|))(1-\phi_2(|\eta|/|\xi|)). \tag{2.3}$$

Lemma 2.5. *[12] Let $\Phi_{(1)}$, $\Phi_{(2)}$, and $\Phi_{(3)}$ be defined as above. Then*

(i) $\Phi_{(1)}(\xi,\eta)+\Phi_{(2)}(\xi,\eta)+\Phi_{(3)}(\xi,\eta)=1$ *for* $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{(0,0)\}$;

(ii)$|\partial_\xi^\alpha\partial_\eta^\beta\Phi_{(i)}(\xi,\eta)|\le C_{\alpha,\beta}(|\xi|+|\eta|)^{-(|\alpha|+|\beta|)}$ *for all* $1\le i\le 3$, $\alpha,\beta\in\mathbb{Z}_+^n$ *and* $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{(0,0)\}$;

(iii) supp $\Phi_{(1)}\subset\{|\xi|\le|\eta|/2\}$, supp $\Phi_{(2)}\subset\{|\eta|\le|\xi|/2\}$ *and* supp $\Phi_{(3)}\subset\{|\xi|/8\le|\eta|\le 8|\xi|\}$.

Lemma 2.6. *[4] Let $s_1,s_2\in\mathbb{R}$, $\Psi_1,\Psi_2\in\mathcal{S}(\mathbb{R}^{2n})$ be such that* supp Ψ_1, supp Ψ_2 *are compact and none of them contains the origin. Assume that $\Phi\in C^\infty(\mathbb{R}^{2n}\backslash\{0\}\times\mathbb{R}^{2n}\backslash\{0\})$ satisfies*

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2}\Phi(\xi_1,\xi_2,\eta_1,\eta_2)|\le C_{\alpha_1\alpha_2\beta_1\beta_2}(|\xi_1|+|\eta_1|)^{-(|\alpha_1|+|\beta_1|)}(|\xi_2|+|\eta_2|)^{-(|\alpha_2|+|\beta_2|)}$$

for all $\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{Z}_+^n$. Then there exists a constant $C>0$ such that

$$\begin{aligned}&\sup_{t,s>0}\|m(t\xi_1,s\xi_2,t\eta_1,s\eta_2)\Phi(t\xi_1,s\xi_2,t\eta_1,s\eta_2)\Psi_1(\xi_1,\eta_1)\Psi_2(\xi_2,\eta_2)\|_{H^{s_1,s_2}}\\&\le C\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{H^{s_1,s_2}}.\end{aligned}$$

for all $m\in L^\infty(\mathbb{R}^{4n})$ satisfies

$$\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})}<\infty,$$

where $m_{j,k}$ is defined by (1.8).

Proof of Theorem 1.1

Let $s_1,s_2>n$, and $m\in L^\infty(\mathbb{R}^{4n})$ satisfies $\sup_{j,k\in\mathbb{Z}}\|m_{j,k,\alpha}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})}<\infty$, where $m_{j,k,\alpha}$ is defined by (1.9). Since $H^{s_1,s_2}(\mathbb{R}^{4n})\hookrightarrow H^{\min\{s_1,s_2\},\min\{s_1,s_2\}}(\mathbb{R}^{4n})$, so we should consider $H^{s,s}(\mathbb{R}^{4n})$, where $s=\min\{s_1,s_2\}>n$. We decompose m as follows:

$$m(\xi_1,\xi_2,\eta_1,\eta_2)=m(\xi_1,\xi_2,\eta_1,\eta_2)\left(\sum_{i=1}^3\Phi_{(i)}(\xi_1,\eta_1)\right)\left(\sum_{j=1}^3\Phi_{(j)}(\xi_2,\eta_2)\right)$$

$$= \sum_{i,j=1}^{3} m(\xi_1, \xi_2, \eta_1, \eta_2)\Phi_{(i)}(\xi_1, \eta_1)\Phi_{(j)}(\xi_2, \eta_2)$$
$$= \sum_{i,j=1}^{3} m_{i,j}(\xi_1, \xi_2, \eta_1, \eta_2),$$

where $\Phi_{(i)}, \Phi_{(j)}(1 \le i, j \le 3)$ are the same as in (2.1), (2.2) and (2.3). By Lemma 2.5, we divide these $m_{j,k}$ into four groups. Since the Fourier multiplier operator corresponding to every symbol $m_{j,k}$ in the group can be estimated in the same way, we just choose one to prove in each group.

The first group :

$m_{1,1}$, where supp $m_{1,1} \subset \{|\xi_1| \le |\eta_1|/2, |\xi_2| \le |\eta_2|/2\}$

$m_{2,2}$, where supp $m_{2,2} \subset \{|\eta_1| \le |\xi_1|/2, |\eta_2| \le |\xi_2|/2\}$.

The second group :

$m_{1,3}$, where supp $m_{1,3} \subset \{|\xi_1| \le |\eta_1|/2, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$

$m_{2,3}$, where supp $m_{2,3} \subset \{|\eta_1| \le |\xi_1|/2, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$

$m_{3,1}$, where supp $m_{3,1} \subset \{|\eta_1|/8 \le |\xi_1| \le 8|\eta_1|, |\xi_2| \le |\eta_2|/2\}$

$m_{3,2}$, where supp $m_{3,2} \subset \{|\eta_1|/8 \le |\xi_1| \le 8|\eta_1|, |\eta_2| \le |\xi_2|/2\}$.

The third group :

$m_{1,2}$, where supp $m_{1,2} \subset \{|\xi_1| \le |\eta_1|/2, |\eta_2| \le |\xi_2|/2\}$

$m_{2,1}$, where supp $m_{2,1} \subset \{|\eta_1| \le |\xi_1|/2, |\xi_2| \le |\eta_2|/2\}$.

The fourth group :

$m_{3,3}$, where supp $m_{3,3} \subset \{|\eta_1|/8 \le |\xi_1| \le 8|\eta_1|, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$.

Estimates for the first group .

First, we consider $m_{2,2}$, we denote it as m^1 instead of $m_{2,2}$.

Let $f, g \in \mathcal{S}(\mathbb{R}^{2n})$, since $\sum_{j\in\mathbb{Z}} \Psi_j(\xi) = 1$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$, we have

$$I_{j,k} := \Psi(D_1/2^j)\Psi(D_2/2^k)T_{m^1}(f, g)(x_1, x_2)$$
$$= \frac{1}{(2\pi)^{(4n)}} \int_{\mathbb{R}^{4n}} m^1(\xi_1, \xi_2, \eta_1, \eta_2) e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)} \Psi_j(\xi_1 + \eta_1)\widehat{f}(\xi_1, \xi_2)$$

$$
\begin{aligned}
&\times \Psi_k(\xi_2+\eta_2)\widehat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&= \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m^1(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\Psi_j(\xi_1+\eta_1)\widetilde{\Psi}_j(\xi_1)\\
&\times \widehat{f}(\xi_1,\xi_2)\Psi_k(\xi_2+\eta_2)\widetilde{\Psi}_k(\xi_2)\widehat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&= \int_{\mathbb{R}^{4n}} 2^{j(2n-\alpha)}2^{k(2n-\alpha)}\\
&\times (\mathcal{F}^{-1}m^1_{j,k,\alpha})\left(2^j(x_1-y_1),2^k(x_2-y_2),2^j(x_1-z_1),2^k(x_2-z_2)\right)\\
&\times \left(\widetilde{\Psi}_j(D_1)\widetilde{\Psi}_k(D_2)f\right)(y_1,y_2)g(z_1,z_2)dy_1dy_2dz_1dz_2,
\end{aligned}
$$

where $\Psi_k(\xi)=\Psi(\xi/2^k)$ and $\widetilde{\Psi}(\xi_1)\in\mathcal{S}(\mathbb{R}^{4n})$ such that $\widetilde{\Psi}(\xi_1)\Psi(\xi_1+\eta_1)=\Psi(\xi_1+\eta_1)$, on the $\operatorname{supp} m^1$, since $|\xi_1+\eta_1|\approx|\xi_1|$. The same is true for $\widetilde{\Psi}(\xi_2)$.

$$
m^1_{j,k,\alpha}=2^{j\alpha}2^{k\alpha}m^1(2^j\xi_1,2^k\xi_2,2^j\eta_1,2^k\eta_2)\Psi(\xi_1+\eta_1)\Psi(\xi_2+\eta_2).
$$

Take $1<t<2$ with $2n/(s+\alpha)<t<\min\{2,q_1,q_2,2n/\alpha\}$. By the Hölder inequality, Lemma 2.3 and Lemma 2.1, we have

$$
\begin{aligned}
|I_{j,k}| &\le 2^{2n(j+k)}2^{-(j+k)\alpha}\int_{\mathbb{R}^{4n}}(1+2^j|x_1-y_1|+2^j|x_1-z_1|)^s\\
&\times(1+2^k|x_2-y_2|+2^k|x_2-z_2|)^s\\
&\times\left(\mathcal{F}^{-1}m^1_{j,k,\alpha}\right)\left(2^j(x_1-y_1),2^k(x_2-y_2),2^j(x_1-z_1),2^k(x_2-z_2)\right)\\
&\times(1+2^j|x_1-y_1|+2^j|x_1-z_1|)^{-s}(1+2^k|x_2-y_2|+2^k|x_2-z_2|)^{-s}\\
&\times\left(\widetilde{\Psi}_j(D_1)\widetilde{\Psi}_k(D_2)f\right)(y_1,y_2)g(z_1,z_2)dy_1dy_2dz_1dz_2\\
&\le\Bigg(\int_{\mathbb{R}^{4n}}(1+|y_1|+|z_1|)^{t's}(1+|y_2|+|z_2|)^{t's}\\
&\times\left|\left(\mathcal{F}^{-1}m^1_{j,k,\alpha}\right)(y_1,y_2,z_1,z_2)\right|^{t'}dy_1dy_2dz_1dz_2\Bigg)^{1/t'}\\
&\times\Bigg(\int_{\mathbb{R}^{4n}}2^{(2jn+2kn)-(j+k)\alpha t}(1+2^j|x_1-y_1|+2^j|x_1-z_1|)^{-ts}
\end{aligned}
$$

$$\times (1+2^k|x_2-y_2|+2^k|x_2-z_2|)^{-ts}$$

$$\times \left|\left(\widetilde{\Psi}_j(D_1)\widetilde{\Psi}_k(D_2)f\right)(y_1,y_2)g(z_1,z_2)\right|^t dy_1dy_2dz_1dz_2\Bigg)^{1/t}$$

$$\leq C\|m^1_{j,k,\alpha}\|_{L^{t'}(w_{s,t'})}\left(\int_{\mathbb{R}^{2n}} 2^{jn+kn-((j+k)\alpha t)/2}|g(z_1,z_2)|^t(1+2^k|x_2-z_2|)^{-st/2}\right.$$

$$\left.\times (1+2^j|x_1-z_1|)^{-st/2}dz_1dz_2\right)^{1/t}\left(\int_{\mathbb{R}^{2n}} 2^{jn+kn-((j+k)\alpha t)/2}\right.$$

$$\left.\times \left|\left(\widetilde{\Psi}_j(D_1)\widetilde{\Psi}_k(D_2)f\right)(y_1,y_2)\right|^t(1+2^j|x_1-y_1|)^{-st/2}(1+2^k|x_2-y_2|)^{-st/2}dy_1dy_2\right)^{1/t}$$

$$\leq C\|m^1_{j,k,\alpha}\|_{H^{s,s}}\left(M_{\alpha t/2}\left|\widetilde{\Psi}_j(D_1)\widetilde{\Psi}_k(D_2)f\right|^t(x_1,x_2)\right)^{1/t}\left(M_{\alpha t/2}|g|^t(x_1,x_2)\right)^{1/t}.$$

Then by the Hölder inequality, Lemma 2.2 and Lemma 2.4, we have

$$\|T_{m^1}(f,g)\|_{L^q}$$

$$\leq \left\|\left\{\sum_{j,k\in\mathbb{Z}}\left|\Psi(D_1/2^j)\Psi(D_2/2^k)T_{m^1}(f,g)\right|^2\right\}^{1/2}\right\|_{L^q}$$

$$\leq C\sup_{j,k\in\mathbb{Z}}\|m^1_{j,k,\alpha}\|_{H^{s,s}}\left\|\left\{\sum_{j,k\in\mathbb{Z}}\left(M_{\alpha t/2}\left|\widetilde{\Psi}_j(D_1)\widetilde{\Psi}_k(D_2)f\right|^t\right)^{2/t}\left(M_{\alpha t/2}\left|g\right|^t\right)^{2/t}\right\}^{1/2}\right\|_{L^q}$$

$$\leq C\sup_{j,k\in\mathbb{Z}}\|m^1_{j,k,\alpha}\|_{H^{s,s}}\left\|\left\{\sum_{j,k\in\mathbb{Z}}\left(M_{\alpha t/2}\left|\widetilde{\Psi}_j(D_1)\widetilde{\Psi}_k(D_2)f\right|^t\right)^{2/t}\right\}^{1/2}\right\|_{L^{q_1}}$$

$$\times\left\|\left(\left(M_{\alpha t/2}|g|^t\right)^{2/t}\right)^{1/2}\right\|_{L^{q_2}}$$

$$=\sup_{j,k\in\mathbb{Z}}\|m^1_{j,k,\alpha}\|_{H^{s,s}}\left\|\left\{\sum_{j,k\in\mathbb{Z}}\left(M_{\alpha t/2}\left|\widetilde{\Psi}_j(D_1)\widetilde{\Psi}_k(D_2)f\right|^t\right)^{2/t}\right\}^{t/2}\right\|^{1/t}_{L^{q_1/t}}$$

$$\times\left\|\left(\left(M_{\alpha t/2}|g|^t\right)^{2/t}\right)^{t/2}\right\|^{1/t}_{L^{q_2/t}}$$

$$\leq C\sup_{j,k\in\mathbb{Z}}\|m^1_{j,k,\alpha}\|_{H^{s,s}}\left\|\left\{\sum_{j,k\in\mathbb{Z}}\left|\widetilde{\Psi}_j(D_1)\widetilde{\Psi}_k(D_2)f\right|^2\right\}^{1/2}\right\|_{L^{p_1}}\|g\|_{L^{p_2}}$$
$$\leq C\sup_{j,k\in\mathbb{Z}}\|m^1_{j,k,\alpha}\|_{H^{s,s}}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

Since supp $m^1\subset\{1/a\leq\sqrt{|\xi_1|^2+|\eta_1|^2}\leq a, 1/b\leq\sqrt{|\xi_2|^2+|\eta_2|^2}\leq b\}$ for some $a,b>1$, by Lemma 2.6 we have

$$\sup_{j,k\in\mathbb{Z}}\|m^1_{j,k,\alpha}\|_{H^{s,s}}\leq C\sup_{j,k\in\mathbb{Z}}\|m_{j,k,\alpha}\|_{H^{s,s}}$$

So

$$\|T_{m^1}\|_{L^{p_1}\times L^{p_2}\longrightarrow L^q}\leq C\sup_{j,k\in\mathbb{Z}}\|m_{j,k,\alpha}\|_{H^{s,s}}$$

Changing the roles of ξ_1,η_1 and ξ_2,η_2, we can prove

$$\|T_{m^1}\|_{L^{p_1}\times L^{p_2}\longrightarrow L^q}\leq C\sup_{j,k\in\mathbb{Z}}\|m_{j,k,\alpha}\|_{H^{s,s}}$$

where $m^1=m_{2,2}$.

Estimates for the second group :

We write m^2 instead of $m_{2,3}$. Since supp $m_{2,3}\subset\{|\eta_1|\leq|\xi_1|/2, |\eta_2|/8\leq|\xi_2|\leq 8|\eta_2|\}$, then there exists $\Psi^1\in\mathcal{S}(\mathbb{R}^n)$, such that $\Psi(\xi_2)\Psi^1(\eta_2)=\Psi(\xi_2)$ on $\{|\eta_2|/8\leq|\xi_2|\leq 8|\eta_2|\}$, where Ψ is the function which is the same as case 1.

Hence,

$$\begin{aligned}&\Psi(D_1/2^j)T_{m^2}(f,g)(x_1,x_2)\\&=\frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}}m^2(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\Psi_j(\xi_1+\eta_1)\\&\quad\times\widehat{f}(\xi_1,\xi_2)\widehat{g}(\eta_1,\eta_2)d\xi_1d\xi_2d\eta_1d\eta_2\\&=\frac{1}{(2\pi)^{(4n)}}\sum_k\int_{\mathbb{R}^{4n}}m^2(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\Psi_j(\xi_1+\eta_1)\end{aligned}$$

$$\times \widetilde{\Psi}_j(\xi_1)\Psi_k(\xi_2)\widehat{f}(\xi_1,\xi_2)\Psi_k^1(\eta_2)\widehat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

$$= \frac{1}{(2\pi)^{(4n)}}\sum_k \int_{\mathbb{R}^{4n}} m^2(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\Psi_j(\xi_1+\eta_1)$$

$$\times \widetilde{\Psi}_j(\xi_1)\Psi_k(\xi_2)\Psi_k^2(\xi_2)\widehat{f}(\xi_1,\xi_2)\Psi_k^1(\eta_2)\widehat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

$$= \sum_k \int_{\mathbb{R}^{4n}} 2^{j(2n-\alpha)}2^{k(2n-\alpha)}$$

$$\times \left(\mathcal{F}^{-1}m^2_{j,k,\alpha}\right)\left(2^j(x_1-y_1),2^k(x_2-y_2),2^j(x_1-z_1),2^k(x_2-z_2)\right)$$

$$\times \left(\widetilde{\Psi}_j(D_1)\Psi_k^2(D_2)f\right)(y_1,y_2)\left(\Psi_k^1(D_2)g\right)(z_1,z_2)dy_1dy_2dz_1dz_2$$

$$:= \sum_k II_{j,k},$$

where $\widetilde{\Psi}$ is the same as the first group and $\Psi(\xi_2)\Psi^2(\xi_2) = \Psi(\xi_2)$.

$$m^2_{j,k,\alpha} = 2^{j\alpha}2^{k\alpha}m^2(2^j\xi_1,2^k\eta_1,2^j\xi_2,2^k\eta_2)\Psi(\xi_1+\eta_1)\Psi(\xi_2).$$

Take $1<t<2$ satisfying $2n/(s+\alpha)<t<\min\{2,q_1,q_2,2n/\alpha\}$. Arguing in the same way as in the first group, we can prove

$$|II_{j,k}| \le C\|m^2_{j,k,\alpha}\|_{H^{s,s}}\left(M_{\alpha t/2}\left|\widetilde{\Psi}_j(D_1)\Psi_k^2(D_2)f\right|^t(x_1,x_2)\right)^{1/t}$$

$$\times\left(M_{\alpha t/2}\left|\Psi_k^1(D_2)g\right|^t(x_1,x_2)\right)^{1/t}.$$

By the Hölder inequality, Lemma 2.2 and Lemma 2.4, we can obtain

$$\|T_{m^2}(f,g)(x_1,x_2)\|_{L^q}$$

$$\le \left\|\left\{\sum_{j\in\mathbb{Z}}\left|\Psi(D_1/2^j)T_{m^2}(f,g)\right|^2\right\}^{1/2}\right\|_{L^q}$$

$$\le C\sup_{j,k\in\mathbb{Z}}\|m^2_{j,k,\alpha}\|_{H^{s,s}}\left\|\left\{\sum_{j,k\in\mathbb{Z}}\left(M_{\alpha t/2}\left|\widetilde{\Psi}_j(D_1)\Psi_k^2(D_2)f\right|^t\right)^{2/t}\right.\right.$$

$$\left.\left.\times\left(M_{\alpha t/2}\left|\Psi_k^1(D_2)g\right|^t\right)^{2/t}\right\}^{1/2}\right\|_{L^q}$$

$$\leq C \sup_{j,k\in\mathbb{Z}} \|m^2_{j,k,\alpha}\|_{H^{s,s}} \left\| \left\{ \sum_{j,k\in\mathbb{Z}} \left(M_{\alpha t/2} \left| \widetilde{\Psi}_j(D_1)\Psi^2_k(D_2)f \right|^t \right)^{2/t} \right\}^{1/2} \right\|_{L^{q_1}}$$

$$\times \left\| \left\{ \sum_{k\in\mathbb{Z}} \left(M_{\alpha t/2} \left| \Psi^1_k(D_2)g \right|^t \right)^{2/t} \right\}^{1/2} \right\|_{L^{q_2}}$$

$$= \sup_{j,k\in\mathbb{Z}} \|m^2_{j,k,\alpha}\|_{H^{s,s}} \left\| \left\{ \sum_{j,k\in\mathbb{Z}} \left(M_{\alpha t/2} \left| \widetilde{\Psi}_j(D_1)\Psi^2_k(D_2)f \right|^t \right)^{2/t} \right\}^{t/2} \right\|^{1/t}_{L^{q_1/t}}$$

$$\times \left\| \left\{ \sum_{k\in\mathbb{Z}} \left(M_{\alpha t/2} \left| \Psi^1_k(D_2)g \right|^t \right)^{2/t} \right\}^{t/2} \right\|^{1/t}_{L^{q_2/t}}$$

$$\leq C \sup_{j,k\in\mathbb{Z}} \|m^2_{j,k,\alpha}\|_{H^{s,s}} \left\| \left\{ \sum_{j,k\in\mathbb{Z}} \left| \widetilde{\Psi}_j(D_1)\Psi^2_k(D_2)f \right|^2 \right\}^{1/2} \right\|_{L^{p_1}} \left\| \left\{ \sum_{k\in\mathbb{Z}} \left| \Psi^1_k(D_2)g \right|^2 \right\}^{1/2} \right\|_{L^{p_2}}$$

$$\leq C \sup_{j,k\in\mathbb{Z}} \left\|m^2_{j,k,\alpha}\right\|_{H^{s,s}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}},$$

Since supp $m^1 \subset \{1/a \leq \sqrt{|\xi_1|^2 + |\eta_1|^2} \leq a, 1/b \leq \sqrt{|\xi_2|^2 + |\eta_2|^2} \leq b\}$ for some $a, b > 1$, by Lemma 2.6 we have

$$\sup_{j,k\in\mathbb{Z}} \|m^2_{j,k,\alpha}\|_{H^{s,s}} \leq C \sup_{j,k\in\mathbb{Z}} \|m_{j,k,\alpha}\|_{H^{s,s}}$$

So

$$\|T_{m^2}\|_{L^{p_1}\times L^{p_2}\longrightarrow L^q} \leq C \sup_{j,k\in\mathbb{Z}} \|m_{j,k,\alpha}\|_{H^{s,s}}$$

By interchanging the roles of ξ_1, η_1 and ξ_2, η_2, we can prove other situations in the second group.

Estimates for the third group:

We write m^3 instead of $m_{1,2}$, the proof is similar to case 1.

$$\begin{aligned} III_{j,k} &= \Psi(D_1/2^j)\Psi(D_2/2^k)T_{m^3}(f,g)(x_1,x_2) \\ &= \frac{1}{(2\pi)^{(4n)}} \int_{\mathbb{R}^{4n}} m^3(\xi_1,\xi_2,\eta_1,\eta_2) e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)} \Psi_j(\xi_1+\eta_1)\widehat{f}(\xi_1,\xi_2) \end{aligned}$$

$$
\begin{aligned}
&\times \Psi_k(\xi_2+\eta_2)\widehat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&= \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m^3(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\Psi_j(\xi_1+\eta_1)\widetilde{\Psi}_j(\eta_1)\\
&\times \widehat{f}(\xi_1,\xi_2)\Psi_k(\xi_2+\eta_2)\widetilde{\Psi}_k(\xi_2)\widehat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&= \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m^3(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\Psi_k(\xi_2+\eta_2)\widetilde{\Psi}_k(\xi_2)\\
&\times \widehat{f}(\xi_1,\xi_2)\Psi_j(\xi_1+\eta_1)\widetilde{\Psi}_j(\eta_1)\widehat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&= \int_{\mathbb{R}^{4n}} 2^{j(2n-\alpha)}2^{k(2n-\alpha)}\\
&\times \left(\mathcal{F}^{-1}m^3_{j,k,\alpha}\right)\left(2^j(x_1-y_1),2^k(x_2-y_2),2^j(x_1-z_1),2^k(x_2-z_2)\right)\\
&\times \left(\widetilde{\Psi}_k(D_2)f\right)(y_1,y_2)\left(\widetilde{\Psi}_j(D_1)g\right)(z_1,z_2)dy_1dy_2dz_1dz_2,
\end{aligned}
$$

where $\Psi,\widetilde{\Psi}$ are defined the same way in the first group and

$$m^3_{j,k,\alpha} = 2^{j\alpha}2^{k\alpha}m^3(2^j\xi_1,2^k\xi_2,2^j\eta_1,2^k\eta_2)\Psi(\xi_1+\eta_1)\Psi(\xi_2+\eta_2).$$

As we did in the first group, we can prove

$$|III_{j,k}| \le C\left\|m^3_{j,k,\alpha}\right\|_{H^{s,s}}\left(M_{\alpha t/2}\left|\widetilde{\Psi}_k(D_2)f\right|^t(x_1,x_2)\right)^{1/t}\left(M_{\alpha t/2}\left|\widetilde{\Psi}_j(D_1)g\right|^t(x_1,x_2)\right)^{1/t},$$

where $\max\{1,2n/(s+\alpha)\} < t < 2$.

In the similar way of case 1, we can obtain

$$\|T_{m^3}\|_{L^{p_1}\times L^{p_2}\longrightarrow L^q} \le C\sup_{j,k\in\mathbb{Z}}\|m^3_{j,k,\alpha}\|_{H^{s,s}} \le C\sup_{j,k\in\mathbb{Z}}\|m_{j,k,\alpha}\|_{H^{s,s}}.$$

By interchanging the roles of ξ_1,η_1 and ξ_2,η_2, we can prove the same conclusion for $m_{2,1}$ in the third group.

Estimates for the fourth group:

We write m^4 instead of $m_{3,3}$, the proof is similar to case 2, we omit the details here.

First, we can obtain

$$
\begin{aligned}
&|T_{m^4}(f,g)(x_1,x_2)| \\
&\le C \sup_{j,k\in\mathbb{Z}} \left\|m^4_{j,k,\alpha}\right\|_{H^{s,s}} \left\{\sum_{j,k\in\mathbb{Z}} \left(M_{\alpha t/2}\left(\left|\widetilde{\widetilde{\Psi}}_j(D_1)\widetilde{\widetilde{\Psi}}_k(D_2)f\right|^t\right)(x_1,x_2)\right)^{2/t}\right\}^{1/2} \\
&\quad\times \left\{\sum_{j,k\in\mathbb{Z}} \left(M_{\alpha t/2}\left(\left|\widetilde{\widetilde{\Psi}}_j(D_1)\widetilde{\widetilde{\Psi}}_k(D_2)g\right|^t\right)(x_1,x_2)\right)^{2/t}\right\}^{1/2},
\end{aligned}
$$

where $\max\{1, 2n/(s+\alpha)\} < t < 2$.

$$
m^4_{j,k,\alpha} = 2^{j\alpha}2^{k\alpha}m^4(2^j\xi_1, 2^k\eta_1, 2^j\xi_2, 2^k\eta_2)\Psi(\xi_1+\eta_1)\widetilde{\Psi}(\xi_1)\Psi(\xi_2+\eta_2)\widetilde{\Psi}(\xi_2).
$$

Since $q_1/t, q_2/t, 2/t > 1$, by the Hölder inequality, Lemma 2.2 and Lemma 2.4, we can obtain

$$
\begin{aligned}
&\|T_{m^4}(f,g)(x_1,x_2)\|_{L^q} \\
&\le C \sup_{j,k\in\mathbb{Z}} \left\|m^4_{j,k,\alpha}\right\|_{H^{s,s}} \left\|\left\{\sum_{j,k\in\mathbb{Z}} \left(M_{\alpha t/2}\left|\widetilde{\widetilde{\Psi}}_j(D_1)\widetilde{\widetilde{\Psi}}_k(D_2)f\right|^t\right)^{2/t}\right\}^{1/2}\right\|_{L^{q_1}} \\
&\quad\times \left\|\left\{\sum_{j,k\in\mathbb{Z}} \left(M_{\alpha t/2}\left|\widetilde{\widetilde{\Psi}}_j(D_1)\widetilde{\widetilde{\Psi}}_k(D_2)g\right|^t\right)^{2/t}\right\}^{1/2}\right\|_{L^{q_2}} \\
&= \sup_{j,k\in\mathbb{Z}} \left\|m^4_{j,k,\alpha}\right\|_{H^{s,s}} \left\|\left\{\sum_{j,k\in\mathbb{Z}} \left(M_{\alpha t/2}\left|\widetilde{\widetilde{\Psi}}_j(D_1)\widetilde{\widetilde{\Psi}}_k(D_2)f\right|^t\right)^{2/t}\right\}^{t/2}\right\|_{L^{q_1/t}}^{1/t} \\
&\quad\times \left\|\left\{\sum_{j,k\in\mathbb{Z}} \left(M_{\alpha t/2}\left|\widetilde{\widetilde{\Psi}}_j(D_1)\widetilde{\widetilde{\Psi}}_k(D_2)g\right|^t\right)^{2/t}\right\}^{t/2}\right\|_{L^{q_2/t}}^{1/t} \\
&\le C \sup_{j,k\in\mathbb{Z}} \left\|m^4_{j,k,\alpha}\right\|_{H^{s,s}} \left\|\left\{\sum_{j,k\in\mathbb{Z}} \left|\widetilde{\widetilde{\Psi}}_j(D_1)\widetilde{\widetilde{\Psi}}_k(D_2)f\right|^2\right\}^{1/2}\right\|_{L^{p_1}}
\end{aligned}
$$

$$\times \left\| \left\{ \sum_{j,k\in\mathbb{Z}} \left| \tilde{\tilde{\Psi}}_j(D_1)\tilde{\tilde{\Psi}}_k(D_2)g \right|^2 \right\}^{1/2} \right\|_{L^{p_2}}$$
$$\leq C \sup_{j,k\in\mathbb{Z}} \left\| m^4_{j,k,\alpha} \right\|_{H^{s,s}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} .$$

Since supp $m^1 \subset \{1/a \leq \sqrt{|\xi_1|^2 + |\eta_1|^2} \leq a, 1/b \leq \sqrt{|\xi_2|^2 + |\eta_2|^2} \leq b\}$ for some $a, b > 1$, by Lemma 2.6 we have

$$\|T_{m^4}\|_{L^{p_1}\times L^{p_2}\longrightarrow L^q} \leq C \sup_{j,k\in\mathbb{Z}} \|m^4_{j,k,\alpha}\|_{H^{s,s}} \leq C \sup_{j,k\in\mathbb{Z}} \|m_{j,k,\alpha}\|_{H^{s,s}}.$$

Therefore, we have finished the proof of Theorem 1.1. □

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