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THE PERIODS OF THE PELL P-ORBITS OF POLYHEDRAL AND CENTRO-POLYHEDRAL GROUPS

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ABSTRACT. In this paper, we define the Pell p-orbit of a finitely generated group and then we obtain the lengths of the periods and the basic periods of the Pell p-orbits of the finite polyhedral groups and centro-polyhedral groups.

1. INTRODUCTION AND PRELIMINARIES

The study of recurrence sequences in groups began with the earlier work of Wall [3] where the ordinary Fibonacci sequences in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by several authors; see for example, [1, 2, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16]. In [12] extended the theory to the generalized Pell p-sequences. In this paper, we examine the behaviour of the periods and basic periods of the Pell p-orbits of the polyhedral groups $(n, 2, 2)$, $(2, n, 2)$, $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 2, 5)$ and the centro-polyhedral groups $\langle -2, n, 2\rangle$, $\langle 2, n, -2\rangle$, $\langle n, -2, 2\rangle$, $\langle n, 2, -2\rangle$, $\langle 2, -2, n\rangle$, $\langle -2, 2, n\rangle$ for $n > 2$.
In [4], the generalized Pell (p, i) numbers was defined as follows:
for p $(p = 1, 2, \cdots)$, $n > p + 1$ and $0 \leq \mathrm{i} \leq p$,

$$P_p^{(i)}(n) = 2P_p^{(i)}(n-1) + P_p^{(i)}(n-p-1), \tag{1.1}$$

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with initial conditions $P_p^{(i)}(1) = \cdots = P_p^{(i)}(i) = 0$ and $P_p^{(i)}(i+1) = \cdots = P_p^{(i)}(p+1) = 1$.

Note that if $i = 0$, the initial conditions are $P_p^{(i)}(1) = P_p^{(i)}(2) = \cdots = P_p^{(i)}(p+1) = 1$.

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, b, c, d, b, c, d, \ldots$ is periodic after the initial element a and has period 3. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, \ldots$ is simply periodic with period 4.

Reducing the generalized Pell (p,p)-sequence $\left\{P_p^{(p)}(n)\right\}$ by a modulus m, we can get repeating sequence, denoted by

$$\left\{P_p^{(p,m)}(n)\right\} = \left\{P_p^{(p,m)}(1),\ P_p^{(p,m)}(2), \ldots, P_p^{(p,m)}(p),\ P_p^{(p,m)}(p+1), \ldots, P_p^{(p,m)}(i), \ldots\right\}$$

where $P_p^{(p,m)}(i) = P_p^{(p)}(i) \pmod m$. Also, it has the same recurrence relation as in (1.1) (Deveci et al.).([12, p.3]) .

Theorem 1.1. *(Deveci et al.).([12, Theorem 2.1, p.3])* $\left\{P_p^{(p,m)}(n)\right\}$ *is a simply periodic sequence.*

The notation $h_p^p(m)$ is used for the smallest period of the sequence $\left\{P_p^{(p,m)}(n)\right\}$ (Deveci et al.).([12, p.3]) .

Let G be a finite j-generator group and let

$$X = \{(x_0, x_1, \ldots, x_{j-1}) \in \underbrace{G \times G \times \cdots \times G}_{j} \mid\ < \{x_0, x_1, \ldots, x_{j-1}\} >= G\}.$$

We call $(x_0, x_1, \ldots, x_{j-1})$ a generating j-tuple for G.

Definition 1.1. (Deveci et al.).([12, Definition 3.4, p.6]) A generalized Pell p-sequence $(p \geq 2)$ in a finite group is a sequence of group elements $x_0,\ x_1, \ldots,\ x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots,\ x_{j-1},\ (p+1 \geq j)$ each element is defined by

$$x_n = \begin{cases} x_0\left(x_{n-1}\right)^2 & \text{for } j \leq n < p+1, \\ x_{n-p-1}\left(x_{n-1}\right)^2 & \text{for } n \geq p+1. \end{cases}$$

It is required that the initial (seed) set $x_0, \ldots,\ x_{j-1}$ of the group elements sequence generates the group, thus, forcing the generalized Pell p-sequence to reflect the structure of the group.

The generalized Pell p-sequence of a group generated by $x_0, \ldots,\ x_{j-1}$ is denoted by $Q^{(p)}\left(G; x_0, x_1 \ldots, x_{j-1}\right)$.

Theorem 1.2. *(Deveci et al.).([12, Theorem 3.1, p.7]) A generalized Pell p-sequence in a finite group is simply periodic.*

In (Deveci et al.).([12, p.7]), the period of the generalized Pell p-sequence $Q^{(p)}\left(G; x_0, x_1, \ldots, x_{j-1}\right)$ had been denoted by $PerQ^{(p)}\left(G; x_0, x_1, \ldots, x_{j-1}\right)$.

Definition 1.2. (Deveci et al.).([12, Definition 3.5, p.8]) Let G be a finite j-generator groups. For a *j-tuple* $(x_0, x_1, \ldots, x_{j-1}) \in X$ the basic generalized Pell p-sequence $\overline{Q}^{(p)}\left(G; x_0, x_1, \ldots, x_{j-1}\right),\ (p \geq 2,\ p+1 \geq j)$ of the basic period m is a sequence of group elements $a_0, a_1, a_2, \ldots, a_n, \ldots$ for which, given an initial (seed) set $a_0 = x_0,\ a_1 = x_1,\ a_2 = x_2, \ldots, a_{j-1} = x_{j-1}$, each element is defined by

$$a_n = \begin{cases} a_0\left(a_{n-1}\right)^2 & \text{for } j \leq n < p+1, \\ a_{n-p-1}\left(a_{n-1}\right)^2 & \text{for } n \geq p+1 \end{cases}$$

where $m \geq 1$ is the least integer with

$$a_0 = a_m\theta,\ a_1 = a_{m+1}\theta,\ a_2 = a_{m+2}\theta, \ldots,\ a_p = a_{m+p}\theta,$$

for some $\theta \in$Aut$G$. Since G is a finite j-generator group and $a_m,\ a_{m+1}, \ldots,\ a_{m+j-1}$ generate G, it follows that θ is uniquely determined. The basic generalized Pell p-sequence $\overline{Q}^{(p)}(G; x_0, x_1, \ldots, x_{j-1})$ is finite containing m element.

Also, in (Deveci et al.).([12, p.8]), the basic period of the basic generalized Pell p-sequence $\overline{Q}^{(p)}(G; x_0, x_1, \ldots, x_{j-1})$ had been denoted by $BQ^{(p)}(G; x_0, x_1, \ldots, x_{j-1})$.

Definition 1.3. The polyhedral group (l, m, n) for $l, m, n > 1$, is defined by the presentation

$$\left\langle x, y, z : x^l = y^m = z^n = xyz = 1 \right\rangle.$$

For the generating pair (x, y), the polyhedral group (l, m, n) have the presentations

$$\left\langle x, y : x^l = y^m = (xy)^n = 1 \right\rangle$$

and

$$\left\langle x, y : x^l = y^m = (xy)^{-n} = 1 \right\rangle,$$

where $l, m, n > 1$.
The polyhedral group (l, m, n) is finite if and only if the number $k = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn+nl+lm-lmn$ is positive. Its order is $2lmn/k$ (Coxeter and Moser).([7, p. 67-68]).
In this paper, we consider polyhedral groups as 3-generator groups.

Definition 1.4. The centro-polyhedral group $\langle l, m, n\rangle$, for $l, m, n \in Z$ is defined by the presentation

$$\left\langle x, y, z : x^l = y^m = z^n = xyz \right\rangle.$$

For detailed information about these groups, see(Coxeter and Moser).([7, p. 70-71]).

2. MAIN RESULTS AND PROOFS

Definition 2.1. For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_{p+1}\}$ such that $p \geq 2$, the Pell p-orbit of G with respect to the generating set A, written $P_A^p(G)$ is the sequence $x_0 = a_1,\ x_1 = a_2, \ldots,\ x_p = a_{p+1},\ x_{n+p} = (x_{n-1})(x_{n+p-1})^2$, $n \geq 1$. The length of the period of the sequence is called the Pell p-length of G with respect to the generating set A, written $LP_A^p(G)$. Also, we denote the length of basic period of this sequence by $\overline{LP}_A^p(G)$, which is called the basic length of G with respect to the generating set A.

Firstly, we consider the Pell p-lengths and the basic Pell p-lengths of the finite polyhedral groups by the following Theorems.

Theorem 2.1. *Let G be any of the polyhedral groups $(n,2,2)$, $(2,n,2)$ and $(2,2,n)$, where $n \geq 3$. Then*

$$LP_{\{x,y,z\}}^2(G) = \begin{cases} \frac{3n}{2}, & n \text{ is even}, \\ 3n, & n \text{ is odd} \end{cases} \quad \text{and} \quad \overline{LP}_{\{x,y,z\}}^2(G) = 3.$$

Proof. Let us consider the group $(n,2,2)$. The orbit $P_{\{x,y,z\}}^2((n,2,2))$ is

$$x, y, z, x, yx^2, z, x, yx^4, z, x, yx^6, z, x, yx^8, z, x, yx^{10}, z, \ldots.$$

This sequence can be said to form layers of length three. Using the above, the sequence becomes:

$$\begin{aligned} &x_0 = x,\ x_1 = y,\ x_2 = z, \\ &x_3 = x,\ x_4 = yx^2,\ x_5 = z, \\ &x_6 = x,\ x_7 = yx^4,\ x_8 = z, \\ &x_{3i} = x,\ x_{3i+1} = yx^{2i},\ x_{3i+2} = z, \ldots. \end{aligned}$$

So, we need the smallest $i \in N$ such that $2i = nv_1$ for $v_1 \in N$.

If n is even, then $i = \frac{n}{2}$. Thus, $LP^2_{\{x,y,z\}}((n,2,2)) = \frac{3n}{2}$ and $\overline{LP}^2_{\{x,y,z\}}((n,2,2)) = 3$ since $x\theta = x$, $y\theta = yx^{-2}$ and $z\theta = z$ where θ is an outer automorphism of order $\frac{n}{2}$.

If n is odd, then $n = i$. Thus, $LP^2_{\{x,y,z\}}((n,2,2)) = 3n$ and $\overline{LP}^2_{\{x,y,z\}}((n,2,2)) = 3$ since $x\theta = x$, $y\theta = yx^{-2}$ and $z\theta = z$ where θ is an outer automorphism of order n.

The proofs for the groups $(2,n,2)$ and $(2,2,n)$ are similar to the above and are omitted. □

Theorem 2.2. ***i)*** $LP^2_{\{x,y,z\}}((2,3,3)) = \overline{LP}^2_{\{x,y,z\}}((2,3,3)) = 65.$

ii) $LP^2_{\{x,y,z\}}((2,3,4)) = \overline{LP}^2_{\{x,y,z\}}((2,3,4)) = 27.$

iii) $LP^2_{\{x,y,z\}}((2,3,5)) = \overline{LP}^2_{\{x,y,z\}}((2,3,5)) = 175.$

Proof. The orbit $P^2_{\{x,y,z\}}((2,3,3))$ is

$$x, y, z, y, 1, z, yxy, xyx, yx, xy, yxy^2, yx, y^2xy, yxy^2, yx, yxy, xy, y^2xy,$$
$$yxy, y^2, xyx, xyx, xy, yxy^2, xyx, yxy^2, yxy^2, xyx, y^2, yx, x, y^2, yxy, yx,$$
$$xyx, xyx, x, xyx, 1, x, xyx, yxy, yx, x, yxy, z, y, yx, y, 1, yx, yxy^2, 1, yx, z,$$
$$xy, y^2xy, z, z, yx, y, xyx, x, y, y^2xy, x, y, z, \ldots,$$

which has period 65. Also, $\overline{LP}^2_{\{x,y,z\}}((2,3,3)) = 65$ since $x\theta = x$, $y\theta = y$ and $z\theta = z$ where θ is identity automorphism.

The proofs of the cases ii and iii are similar to the above and are omitted. □

Now we give the Pell p-lengths and the basic Pell p-lengths of some centro polyhedral groups by the following Theorem.

Theorem 2.3. *Let G be any of the centro-polyhedral groups $\langle -2,n,2\rangle$, $\langle 2,n,-2\rangle$, $\langle n,-2,2\rangle$, $\langle n,2,-2\rangle$, $\langle 2,-2,n\rangle$ and $\langle -2,2,n\rangle$, where $n \geq 3$. Then*

$$LP^2_{\{x,y,z\}}(G) = \begin{cases} \frac{n}{2} \cdot h^2_2(4(n-1)), & n \text{ is even}, \\ n \cdot h^2_2(4(n-1)), & n \text{ is odd} \end{cases} \quad \textit{and } \overline{LP}^2_{\{x,y,z\}}(G) = h^2_2(4(n-1)),$$

where $h^2_2(4(n-1))$ denotes the smallest period of the sequence $\left\{P_2^{(2,4(n-1))}(n)\right\}$.

Proof. Let us consider the group $\langle -2, n, 2 \rangle$. It is clear that the centro polyhedral group $\langle -2, n, 2 \rangle$ is defined the presentation

$$\left\langle x, y, z : x^{-2} = y^{n} = z^{2} = xyz \right\rangle .$$

writing $x^{-2} = y^n = z^2 = xyz = s$, we find that $|s| = \frac{4n}{4n\left(\frac{1}{2}+\frac{1}{n}+\frac{1}{2}-1\right)} - 1 = n - 1$. Thus we obtain $|\langle -2, n, 2 \rangle| = 4n(n-1)$, $|x| = |z| = 4(n-1)$ and $|y| = 2n(n-1)$. Also note that z^2 is central element of the group $\langle -2, n, 2 \rangle$.

If n is a positive even integer, then the orbit $P^2_{\{x,y,z\}}(\langle -2, n, 2 \rangle)$ becomes:

$$\begin{aligned}
&x_0 = x,\ x_1 = y,\ x_2 = z,\\
&x_{h_2^2(4(n-1))} = x,\ x_{h_2^2(4(n-1))+1} = y,\ x_{h_2^2(4(n-1))+2} = zy^{k_1 \cdot 4(n-1)},\\
&x_{h_2^2(4(n-1))i} = x,\ x_{h_2^2(4(n-1))i+1} = y,\ x_{h_2^2(4(n-1))i+2} = zy^{k_1 \cdot 4(n-1)i}, \ldots,
\end{aligned}$$

where $k_1 \in N$ be such that $\left(k_1, \frac{n}{2}\right) = 1$. Since $|y| = 2n(n-1)$, we need the smallest $i \in N$ such that $k_1 \cdot 4(n-1)i = 2n(n-1)v_2$ for $v_2 \in N$. Then, we obtain $i = \frac{n}{2}$ for $v_2 = k_1$ since n is a positive even integer. Thus, $LP^2_{\{x,y,z\}}(\langle -2, n, 2 \rangle) = \frac{n}{2} \cdot h_2^2(4(n-1))$ and $\overline{LP}^2_{\{x,y,z\}}(\langle -2, n, 2 \rangle) = h_2^2(4(n-1))$ since $x\theta = x$, $y\theta = y$ and $z\theta = zy^{t_1 \cdot 4(1-n)}$ where θ is an outer automorphism of order $\frac{n}{2}$ and $t_1 \in N$ such that $\left(t_1, \frac{n}{2}\right) = 1$.

If n is a positive odd integer, then the orbit $P^2_{\{x,y,z\}}(\langle -2, n, 2 \rangle)$ becomes:

$$\begin{aligned}
&x_0 = x,\ x_1 = y,\ x_2 = z,\\
&x_{h_2^2(4(n-1))} = x,\ x_{h_2^2(4(n-1))+1} = y,\ x_{h_2^2(4(n-1))+2} = zy^{k_2 \cdot 4(n-1)},\\
&x_{h_2^2(4(n-1))i} = x,\ x_{h_2^2(4(n-1))i+1} = y,\ x_{h_2^2(4(n-1))i+2} = zy^{k_2 \cdot 4(n-1)i}, \ldots,
\end{aligned}$$

where $k_2 \in N$ be such that $(k_2, n) = 1$. Since $|y| = 2n(n-1)$, we need the smallest $i \in N$ such that $k_2 \cdot 4(n-1)i = 2n(n-1)v_3$ for $v_3 \in N$. Then, we obtain $i = n$ for $k_2 = v_3$ since n is a positive odd integer. Thus, $LP^2_{\{x,y,z\}}(\langle -2, n, 2 \rangle) = n \cdot h_2^2(4(n-1))$ and $\overline{LP}^2_{\{x,y,z\}}(\langle -2, n, 2 \rangle) = h_2^2(4(n-1))$ since $x\theta = x$, $y\theta = y$ and $z\theta = zy^{t_2 \cdot 4(1-n)}$ where θ is an outer automorphism of order n and $t_2 \in N$ such that $(t_2, n) = 1$.

The proofs for the groups $\langle 2, n, -2\rangle$, $\langle n, -2, 2\rangle$, $\langle n, 2, -2\rangle$, $\langle 2, -2, n\rangle$ and $\langle -2, 2, n\rangle$ are similar to the above and are omitted. □

All necessary calculations were carried out on the computer using the GAP computational algebra system, see (The GAP group).[17]

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