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SOME RESULTS CONCERNING RIESZ BASES AND FRAMES IN BANACH SPACES

KHOLE TIMOTHY POUMAI [(1)] AND S. K. KAUSHIK [(2)]

Abstract. In this paper, we give characterizations of Riesz bases and near Riesz bases in Banach spaces. The notion of atomic system is defined and a characterization of atomic system has been given. Also results exhibiting relationship between frames, atomic systems and Riesz bases have been proved. Further, we show that every atomic system is a projection of a Riesz basis in Banach spaces. Finally, we give some duality results of an atomic system for Banach spaces.

1. Introduction and Preliminaries

Frames were introduced in 1952 by Duffin and Schaeffer [5]. They infact abstracted Gabor's [9] method to define frames for Hilbert space. Let $\mathcal{H}$ be a real (or complex) separable Hilbert space with inner product $\langle .,\rangle$. A countable sequence $\{f_n\} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist numbers $A,\ B > 0$ such that

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|_{\mathcal{H}}^2, \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The scalars A and B are called the *lower* and *upper frame bounds* of the frame, respectively. They are not unique. The inequality in (1.1) is called the *frame inequality* of the frame. For more details related to frames and Riesz bases in Hilbert spaces, one may refer to [4, 11]. Feichtinger and Gröcheing [7] extended the notion of frames

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to Banach space and defined the notion of atomic decomposition. Gröcheing [10] introduced a more general concept for Banach spaces called Banach frame. Casazza, Christensen and Stoeva [2] studied E_d-frame and E_d-Bessel sequence. Recall that a BK-space is, by definition, a Banach (scalar) sequence space in which the coordinate functionals are continuous.

Definition 1.1. [2] Let E be a Banach space and E_d be a BK-space, a sequence $\{f_n\}_{n=1}^{\infty} \subseteq E^*$ is called an E_d*-frame* for E if

(1) $\{f_n(x)\} \in E_d$, for all $x \in E$,

(2) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A \parallel x \parallel_E \leq \parallel \{f_n(x)\} \parallel_{E_d} \leq B \parallel x \parallel_E, \text{ for all } x \in E. \tag{1.2}$$

A and B are called E_d*-frame* bounds. If atleast (1) and the upper bound condition in (1.2) are satisfied, then $\{f_n\}$ is called an E_d*-Bessel sequence* for E. If $\{f_n\}$ is an E_d*-Bessel sequence* for E, then $U : E \to E_d$ given by

$$U(x) = \{f_n(x)\}, \text{ for } x \in E$$

is a bounded linear operator and U is called the *analysis operator* associated to E_d-*Bessel sequence* $\{f_n\}$. If $\{f_n\}$ is an E_d*-frame* and there exists a sequence $\{x_n\} \subseteq E$ such that $x = \sum\limits_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$. Then, a pair (x_n, f_n) is called an *atomic decomposition* for E with respect E_d. Further, if $\{f_n\}$ is an E_d*-frame* for E and there exists a bounded linear operator $S : E_d \longrightarrow E$ such that $S(\{f_n(x)\}) = x$ for all $x \in$E, then $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d.

In [15], Stoeva defined and studied E_d-Riesz basis.

Definition 1.2. [15, 3] For a Banach space E and a BK-space E_d, an E_d*-Riesz basis* for E is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq E$, which is complete in E and there exist constants

$0 < A \leq B < \infty$ such that

$$(1.3)\; A\|\{c_n\}_{n=1}^{\infty}\|_{E_d} \leq \|\sum_{n=1}^{\infty} c_n x_n\|_E \leq B\|\{c_n\}_{n=1}^{\infty}\|_{E_d} \text{ for every } \{c_n\}_{n=1}^{\infty} \in E_d.$$

A number A (resp. B) in (1.3) is called a lower (resp. upper) E_d-Riesz basis bound.

We give few results in the form of lemmas which will be used in the subsequent work.

Lemma 1.1 (Cassaza–Christensen–Stoeva)**.** ([2, Lemma 3.1]) *Let E_d be a BK-space for which the canonical unit vectors $\{e_n\}$ form a Schauder basis. Then the space $Y_d = \{\{h(e_n)\} : h \in E_d^*\}$ with norm $\|\{h(e_n)\}\|_{Y_d} = \|h\|_{E_d^*}$ is a BK-space isometrically isomorphic to E_d^*. Also, every continuous linear functional Φ on E_d has the form $\Phi\{c_n\} = \sum_{n=1}^{\infty} c_n d_n$, where $\{d_n\} \in Y_d$ is uniquely determined by $d_n = \Phi(e_n)$, and $\|\Phi\| = \|\{\Phi(e_n)\}\|_{Y_d}$.*

Lemma 1.2 (Taylor–Lay)**.** ([16, Theorem 12.9, p. 251]) *Let X, Y be Banach spaces and $S : X \to Y$ be a bounded linear operator from X into Y. Then the following are equivalent.*

(1) *S has a pseudoinverse operator $S^{\dagger}$.*

(2) *There exist closed subspaces W, Z of X, Y such that*

$$X = kerS \oplus W,\; Y = S(X) \oplus Z.$$

Lemma 1.3 (Stoeva)**.** ([15, Propositions 3.3 and 3.4]) *Let E_d be BK-space which has a sequence of canonical unit vectors as Schauder basis and $\{x_n\}_{n=1}^{\infty} \subseteq E$ be a sequence. Then*

(1) *every E_d-Riesz basis $\{x_n\}$ is a Schauder basis for E.*

(2) *$\{x_n\}$ is a Riesz basis if and only if the operator T, given by $T\{\alpha_n\}_{n=1}^{\infty} = \sum_{n=1}^{\infty} \alpha_n x_n$ is an isomorphism of E_d onto E.*

Throughout this paper, E will denote a Banach space over the scalar field $\mathbb{K}$ (which is $\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, $\widetilde{[f_n]}$ the closed linear span of $\{f_n\} \subseteq E^*$ in the $\sigma(E^*, E)$-toplogy. Further, E_d denotes a BK-space which has a sequence of canonical unit vectors $\{e_n\}_{n=1}^{\infty}$ as Schauder basis, E_d^* the conjugate space of E_d and $Y_d = \{\{h(e_n)\} : h \in E_d^*\}$ denotes a BK-space which is defined in Lemma 1.1. $\pi : E \longrightarrow E^{**}$ is the natural canonical projection from E onto E^{**}. This paper is devoted to the study of frames, E_d-Riesz bases, near E_d-Riesz bases, atomic systems in Banach spaces and the extremality property of atomic systems. In Section 2, we give the necessary and sufficient conditions for the existence of frames in Banach spaces. We also study the relationship between E_d-Riesz bases and frames. A necessary and sufficient condition for a frame to be an E_d-Riesz basis is obtained. Also, we give a necessary and sufficient condition for a frame to be a near E_d-Riesz basis. Section 3 is devoted to atomic systems in Banach spaces. We prove that every atomic system is a frame in Banach spaces. We also show that E_d-Riesz bases and near E_d-Riesz bases are atomic systems in Banach spaces. Also, various properties of atomic systems which are similar to that of the properties of frames in Hilbert spaces have been given. Further, we show that atomic systems are compression of E_d-Riesz bases in Banach spaces. Finally, we discuss the extremality property of atomic systems in the conjugate of a Banach space.

2. Frames and E_d-Riesz bases

P. A. Terekhin [17, 18] introduced and studied the notion of frames for Banach spaces.

Definition 2.1. [17, 18] Let E be a Banach space and E_d be a BK-space which has a sequence of canonical unit vectors $\{e_n\}$ as Schauder basis. A sequence $\{x_n\}_{n=1}^{\infty} \backslash \{0\} \subseteq E$ is called a *frame* for E with respect to E_d if

(1) $\{f(x_n)\} \in Y_d$ for all $f \in E^*$,

(2) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{Y_d} \leq B\|f\|_{E^*}, \text{ for all } f \in E^*. \tag{2.1}$$

We refer (2.1) as the *frame inequalities.* If atleast (1) and the upper bound condition in (2.1) are satisfied, then $\{x_n\}$ is called *Bessel sequence* for E with respect to E_d.

In the following result, we give the necessary and sufficient conditions for the existence of frames in Banach spaces.

Theorem 2.1. *$\{x_n\}_{n=1}^{\infty} \subseteq E$ is a frame for E with respect to E_d if and only if there exists a bounded linear operator $\mathcal{T} : E_d \to E$ from E_d onto E for which $\mathcal{T}(e_n) = x_n$, for all $n \in I\!N$.*

Proof. Let $f \in E^*$, B be upper bound of the frame $\{x_n\}$. By Lemma 1.1, $\{f(x_n)\}$=$\{\Phi_f(e_n)\}$ for some $\Phi_f \in E_d^*$ and $\|\{f(x_n)\}\|_{Y_d} = \|\Phi_f\|_{E_d^*}$. Let $n, m \in I\!N$ with $n \leq m$ and $\{c_n\} \in E_d$, then

$$\begin{aligned}
\|\sum_{k=n}^{m} c_k x_k\|_E &= \sup_{f \in E^*, \|f\|=1} |\sum_{k=n}^{m} c_k f(x_k)| \\
&= \sup_{f \in E^*, \|f\|=1} |\sum_{k=n}^{m} c_k \Phi_f(e_k)| \\
&= \sup_{f \in E^*, \|f\|=1} |\Phi_f(\sum_{k=n}^{m} c_k e_k)| \\
&\leq \sup_{f \in E^*, \|f\|=1} \|\Phi_f\| \|\sum_{k=n}^{m} c_k e_k\|_{E_d} \\
&= \sup_{f \in E^*, \|f\|=1} \|\{f(x_n)\}\|_{Y_d} \|\sum_{k=n}^{m} c_k e_k\|_{E_d} \\
&\leq B\|\sum_{k=n}^{m} c_k e_k\|_{E_d}.
\end{aligned}$$

Hence, $\mathcal{T} : E_d \to E$ given by $\mathcal{T}\{c_n\} = \sum\limits_{n=1}^{\infty} c_n x_n$, $\{c_n\} \in E_d$ is well defined bounded linear operator from E_d into E. Moreover, $\mathcal{T}(e_n) = x_n$, for all $n \in \mathbb{N}$. By Lemma 1.1 and for $f \in E^*$ we have

$$\|\{f(x_n)\}\|_{Y_d} = \|\{f(\mathcal{T}(e_n))\}\|_{Y_d} = \|\{\mathcal{T}^*(f)(e_n)\}\|_{Y_d} = \|\mathcal{T}^* f\|_{E_d^*}$$

and from the frame inequalities we have $\mathcal{T}^*$ is one-one and $\mathcal{T}^*(E^*)$ is closed. Thus by [13, Theorem 4.15, p. 103], $\mathcal{T}$ is onto.

Conversely, let $\mathcal{T} : E_d \to E$ be well defined bounded linear operator from E_d onto E with $\mathcal{T}(e_n) = x_n$, for all $n \in \mathbb{N}$. So $\mathcal{T}^*$ is one-one and $\mathcal{T}^*(E^*)$ is closed by [13, Theorem 4.15, p. 103]. Again, by [6, Lemma 1, p. 487], there exists constant $C > 0$ such that $\|f\| \leq C\|\mathcal{T}^*(f)\|$ for all $f \in E^*$. Let $f \in E^*$. Also

$$\{f(x_n)\} = \{f(\mathcal{T}(e_n))\} = \{T^* f(e_n)\} \in Y_d.$$

Then, by using Lemma 1.1 and for $f \in E^*$ we have

$$\|f\|_{E^*} \leq C\|\mathcal{T}^*(f)\|_{E_d^*} = \|\{\mathcal{T}^* f(e_n)\}\|_{Y_d} = \|\{f(\mathcal{T}(e_n))\}\|_{Y_d} = \|\{f(x_n)\}\|_{Y_d}.$$

To show the upper inequality,

$$\|\{f(x_n)\}\|_{Y_d} = \|\{\mathcal{T}^* f(e_n)\}\|_{Y_d} = \|\mathcal{T}^*(f)\|_{E_d^*} \leq \|\mathcal{T}\|\|f\|_{E^*}, \text{ for all } f \in E^*.$$

Hence, $\{x_n\}$ is a frame for E. $\square$

Note that, $\{x_n\}$ is a *Bessel sequence* for E if and only if $\mathcal{T} : E_d \to E$ is a well defined bounded linear operator from E_d into E for which $\mathcal{T}(e_n) = x_n$, for all $n \in \mathbb{N}$. The operator $\mathcal{T}$ is called the *synthesis operator* associated to *Bessel sequence* $\{x_n\}$ and bounded linear operator $\mathcal{R} : E^* \to Y_d$ given by

$$\mathcal{R}(f) = \{f(x_n)\}, \text{ for } f \in E^*.$$

is called the *analysis operator* associated to *Bessel sequence* $\{x_n\}$. As in Lemma 1.1, Y_d is isometrically isomorphic to E_d^*. So, let $J_d : Y_d \to E_d^*$ be isometrically isomorphism from Y_d onto E_d^*. Therefore, $\mathcal{T}^* = J_d \circ \mathcal{R}$.

Remark 1. Every E_d-Riesz basis is a frame for E with respect to E_d. This is very clear from Lemma 1.3 and Theorem 2.1.

Next, we give the equivalent conditions for a sequence $\{x_n\}_{n=1}^{\infty} \subseteq E$ to be an E_d-Riesz basis in Banach spaces.

Proposition 2.1. *Let $\{x_n\}_{n=1}^{\infty} \subseteq E$. Then, the following conditions are equivalent.*

(1) *$\{x_n\}$ is an E_d-Riesz basis for E.*

(2) *There exists an isomorphism T from E_d onto E for which $T(e_n) = x_n$, for all $n \in I\!N$.*

(3) *$\{x_n\}$ is a Schauder basis for E and $\sum_n a_n x_n$ converges if and only if $\{a_n\} \in E_d$.*

(4) *$\{x_n\}$ is complete in E and there exist constants $0 < A \leq B < \infty$ such that for every finite sequence of scalars $c_1, c_2, ..., c_n$, we have*

$$A\|\{c_k\}_{k=1}^{n}\|_{E_d} \leq \|\sum_{k=1}^{n} c_k x_k\|_E \leq B\|\{c_k\}_{k=1}^{n}\|_{E_d}$$

(5) *$\{x_n\}$ is a complete Bessel sequence in E and possesses a biorthogonal system $\{f_n\} \subseteq E^*$ which is also an E_d-Bessel sequence and total over E i.e. $\widetilde{[f_n]} = E^*$.*

Proof. (1) $\Leftrightarrow$ (2) Follows from Lemma 1.3.

(2) $\Rightarrow$ (5) By given hypothesis, $\{x_n\}$ is a frame for E and $\{x_n\}$ is complete. Let $\{l_n\}$ be sequence of coordinate functionals on E_d and take $f_n = (T^{-1})^* l_n$, $n \in \mathbb{N}$. Now for $x \in E$, we have $f_n(x) = l_n(T^{-1}(x))$, $n \in \mathbb{N}$. So, $\{f_n(x)\} = T^{-1}(x) \in E_d$, for all $x \in E$. Moreover,

$$\|\{f_n(x)\}\|_{E_d} = \|T^{-1}(x)\|_{E_d} \leq \|T^{-1}\|\|x\|_E, \text{ for all } x \in E.$$

To show biorthogonality, let $i, j \in \mathbb{N}$ we have

$$f_i(x_j) = l_i T^{-1}(x_j) = l_i(T^{-1}T(e_j) = l_i(e_j) = \delta_{ij}.$$

Finally, let $x \in E$ and $f_n(x) = 0$ for all $n \in \mathbb{N}$. Then, $T^{-1}(x) = 0$ which implies that $x = 0$. Hence, $\widetilde{[f_n]} = E^*$.

(5) $\Rightarrow$ (2) Since, $\{x_n\}$ is a complete Bessel sequence and its synthesis operator $\mathcal{T}$ given by $\mathcal{T}(\{a_n\}) = \sum\limits_{n=1}^{\infty} a_n x_n$ is well defined bounded linear operator from E_d into E. Also $\{f_n(x)\} \in E_d$, for all $x \in E$. Let $x \in E$ and let $z = \sum\limits_{n=1}^{\infty} f_n(x)x_n$. So, for any $j \in \mathbb{N}$ we have $f_j(z) = \sum\limits_{n=1}^{\infty} f_n(x)f_j(x_n) = f_j(x)$. Therefore $f_j(z - x) = 0$ for all $j \in \mathbb{N}$. Since, $\{f_n\}$ is total over E, so $x = z$. Thus,

$$x = \sum_{n=1}^{\infty} f_n(x)x_n = \mathcal{T}(\{f_n(x)\}), \text{ for all } x \in E.$$

That shows $\mathcal{T}$ is onto. To show one-one, let $\mathcal{T}(\{a_n\}) = \sum\limits_{n=1}^{\infty} a_n x_n = 0$. Take any $j \in \mathbb{N}$. Then, $f_j(\sum\limits_{n=1}^{\infty} a_n x_n) = 0$. From here $a_j = 0$, for all $j \in \mathbb{N}$. Hence, $\mathcal{T}$ is an isomorphism from E_d onto E.

(4) $\Rightarrow$ (3) Take $c_n = 1$ and $c_j = 0$ if $j \neq n$. So, $0 < A \leq \|x_n\|$ which shows x_n is non zero for all $n \in \mathbb{N}$. Take any finite sequence of scalars $c_1, c_2, ..., c_n$. Let $N, M \in \mathbb{N}$ with $N \leq M$, then

$$\|\sum_{k=1}^{N} c_k x_k\|_E \leq B\|\sum_{k=1}^{N} c_k e_k\|_{E_d} \leq B\|\sum_{k=1}^{M} c_k e_k\|_{E_d} \leq \frac{B}{A}\|\sum_{k=1}^{M} c_k x_k\|_E.$$

Also, $\overline{span}\{x_n\}_{n=1}^{\infty} = E$. Thus, $\{x_n\}$ is a Shauder basis for E. It remains to show that $\sum\limits_{n} a_n x_n$ converges if and only if $\{a_n\} \in E_d$. For this, take $N, M \in \mathbb{N}$ with $N \leq M$

$$A\|\sum_{k=N}^{M} a_k e_k\|_{E_d} \leq \|\sum_{k=N}^{M} a_k x_k\|_E \leq B\|\sum_{k=N}^{M} a_k e_k\|_{E_d}.$$

From here it clearly shows that $\sum\limits_n a_n x_n$ converges if and only if $\{a_n\} \in E_d$.
(3) $\Rightarrow$ (2) By the given conditions, Schauder basis $\{x_n\}$ is equivalent to Schauder basis $\{e_n\}$. So there exists an isomorphism T from E_d onto E such that $T(e_n) = x_n$, for all $n \in \mathbb{N}$.
(2) $\Rightarrow$ (4) As $\{T(e_n) = x_n\}$ is a Schauder basis for E, so $\{x_n\}$ is complete in E. Let $c_1, c_2, ..., c_n$ be any n scalars. Then

$$\|\{c_k\}_{k=1}^n\|_{E_d} = \|T^{-1}T(\{c_k\}_{k=1}^n)\|_{E_d} \leq \|T^{-1}\|\|\sum_{k=1}^n c_k x_k\|_E,$$

$$\|\sum_{k=1}^n c_k x_k\|_E = \|T(\{c_k\}_{k=1}^n)\|_E \leq \|T\|\|(\{c_k\}_{k=1}^n)\|_{E_d}.$$

□

In the following theorem, we give the characterization of Riesz bases from frames in Banach spaces.

Theorem 2.2. *Let $\{x_n\}$ be a frame for E with respect to E_d and $\mathcal{T}$ be its synthesis operator. Then, the following conditions are equivalent.*

(1) *$\{x_n\}$ is an E_d-Riesz basis for E.*
(2) *$\mathcal{T}$ is one-one.*
(3) *$\{x_n\}$ is a Schauder basis for E.*
(4) *$\{x_n\}$ has unique biorthogonal system $\{f_n\} \subseteq E^*$.*
(5) *If $\sum\limits_{n=1}^{\infty} d_n x_n = 0$ for some $\{d_n\}_{n=1}^{\infty} \in E_d$, then $d_n = 0$ for all $n \in \mathbb{N}$.*
(6) *$\{x_n\}$ is minimal, that is $x_j \notin \overline{span}\{x_n\}_{n \neq j}$.*

Proof. (1) $\Leftrightarrow$ (2) Obvious.
(2) $\Rightarrow$ (3) Since, $\{e_n\}$ is a Schauder basis for E_d and $\mathcal{T}$ is an isomorphism from E_d onto E. So, $\{x_n\} = \{\mathcal{T}(e_n)\}$ is a Schauder basis for E.

(2) $\Rightarrow$ (4) Let $\{l_n\} \subseteq E_d^*$ be sequence of coordinate functionals on E_d. Take $f_n = (\mathcal{T}^{-1})^* l_n \in E^*$, for $n \in \mathbb{N}$. So that

$$f_n(x_j) = (\mathcal{T}^{-1})^* l_n(x_j) = l_n \mathcal{T}^{-1}(x_j) = l_n \mathcal{T}^{-1}\mathcal{T}(e_j) = l_n(e_j) = \delta_{nj}.$$

(4) $\Rightarrow$ (3) Let $\{f_n\}_{n=1}^{\infty} \subset E^*$ be a sequence which is unique biorthogonal system to $\{x_n\}$. But for any $x \in E$, $x = \sum\limits_{n=1}^{\infty} \alpha_n x_n$, $\{\alpha_n\} \in E_d$. Therefore, $f_n(x) = \alpha_n$, for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ is a Schauder basis for E.

(3) $\Rightarrow$ (4) Obvious.

(2) $\Leftrightarrow$ (5) Obvious.

(4) $\Leftrightarrow$ (6). Follows from [11, Lemma 5.4, p. 155]. □

Definition 2.2. A frame $\{x_n\} \subseteq E$ is called a *near E_d-Riesz basis* for E if there exists a finite subset σ of $\mathbb{N}$ for which $\{x_n\}_{n \in \mathbb{N} \setminus \sigma}$ is an E_d-Riesz basis for E.

Next, we give the following characterization of near E_d-Riesz bases in Banach spaces. This result generalizes the result due to Holub [12, Theorem 2.4], let $\{x_n\}_{n=1}^{\infty}$ be a frame in Hilbert space $\mathcal{H}$ and $Q : l^2 \to \mathcal{H}$ be the associated preframe operator. Then, ker$Q$ is finite dimensional $\Leftrightarrow$ $\{x_n\}_{n=1}^{\infty}$ is a near-Riesz basis for $\mathcal{H}$.

Theorem 2.3. *Let $\{x_n\}$ be a frame for E with respect to E_d and $\mathcal{T}$ be the associated synthesis operator. Then, ker$\mathcal{T}$ is finite dimensional if and only if $\{x_n\}$ is a near E_d-Riesz basis for E.*

Proof. Let ker$\mathcal{T}$ be finite dimensional subspace of E_d. So, there exists a complemented subspace M of E_d such that $E_d = M \oplus ker\mathcal{T}$. And let Q be projection from E_d onto M and $I_{E_d} - Q$ is a projection from E_d onto ker$\mathcal{T}$. Define $\Gamma : E_d \to E \oplus kerT$ as

$$\Gamma(\alpha) = (\mathcal{T}(\alpha), (I_{E_d} - Q)(\alpha)), \text{ for } \alpha \in E_d.$$

Then, Γ is an isomorphism from E_d onto $E \oplus ker\mathcal{T}$ by [18, Theorem 2]. Let $\{e_n\}$ be sequence of canonical unit vectors as Schauder basis of E_d and take $y_n = \Gamma(e_n)$, for all $n \in \mathbb{N}$. Then, $\{y_n\}$ is a Schauder basis of $E \oplus ker\mathcal{T}$. Let P be projection from $E \oplus ker\mathcal{T}$ onto E. By the construction of Γ, we have $\mathcal{T} = P\Gamma$. Thus $x_n = \mathcal{T}(e_n) = P(\Gamma(e_n)) = P(y_n)$, $n \in \mathbb{N}$. Let dim(ker$\mathcal{T}$)=N, by [1, Theorem 2] there exists a finite subset $\sigma = \{k_1, k_2, k_3, ..., k_N\}$ of $\mathbb{N}$, where $k_i \neq k_j$ at $i \neq j$ for which $\{P(y_n)\}_{n\in\mathbb{N}\backslash\sigma}$ is a Schauder basis for E. Thus, by Theorem 2.2, $\{x_n\}_{n\in\mathbb{N}\backslash\sigma}$ is a E_d-Riesz basis for E. Hence, $\{x_n\}_{n=1}^{\infty}$ is near E_d-Riesz basis for E.

For the converse, let us assume that Ker$\mathcal{T}$ is infinitely dimensional subspace of E_d and let $\{u_n\}_{n=1}^{\infty}$ be the basis of ker$\mathcal{T}$. By given hypothesis, there exists a finite subset σ of $\mathbb{N}$ such that $\{x_n\}_{n\in\mathbb{N}\backslash\sigma}$ is a Riesz basis. So, by Theorem 2.2, $\mathcal{T}|_{[e_n]_{n\in\mathbb{N}\backslash\sigma}}$ is an isomorphism from $[e_n]_{n\in\mathbb{N}\backslash\sigma}$ onto $[x_n]_{n\in\mathbb{N}\backslash\sigma} = E$. But codim($[e_n]_{n\in\mathbb{N}\backslash\sigma}$)=card($\sigma$)=k$<\infty$. By [14, Lemma 4.1, p. 268], there exists a non zero element $u \in [e_n]_{n\in\mathbb{N}\backslash\sigma} \cap [u_n]_{n=1}^{k+1}$. But, $[u_n]_{n=1}^{k+1} \subset ker\mathcal{T}$. Thus, $\mathcal{T}|_{[e_n]_{n\in\mathbb{N}\backslash\sigma}}(u) = \mathcal{T}(u) = 0$. That gives us $u = 0$, which is a contradiction. Hence, ker$\mathcal{T}$ is finite dimensional subspace of E_d. $\square$

Corollary 2.1. *Let $\{x_n\}$ be a frame for E with respect to E_d for which $\{x_n\}_{n\in\mathbb{N}\backslash\sigma}$ is an E_d-Riesz basis for some finite subset σ of $\mathbb{N}$ and $\mathcal{T}$ be the synthesis operator of $\{x_n\}$. Then, card(σ)=dim(ker$\mathcal{T}$).*

Proof. From Theorem 2.3, $\{y_n\}$ is a Schauder basis of $E \oplus ker\mathcal{T}$ and $\{x_n\}_{n\in\mathbb{N}\backslash\sigma}$ is a Schauder basis for E. Thus, $[y_n] = [x_n]_{n\in\mathbb{N}\backslash\sigma} \oplus ker\mathcal{T}$. Therefore, $card(\sigma) = dim(ker\mathcal{T})$. $\square$

Remark 2. Let $\{x_n\}$ be near E_d-Riesz basis and $\mathcal{T}$ be its associated synthesis operator, then $\mathcal{T}$ is a Fredholm operator. By Theorem 2.3, ker$\mathcal{T}$ is finitely dimensional. But, $ker\mathcal{R} = kerT^*$ and from *frame inequalities* $ker\mathcal{T}^* = \{0\}$. Also $\mathcal{T}$ is bounded linear operator from E_d onto E. Hence, $\mathcal{T}$ is Fredholm operator.

3. Atomic systems

The notion of *a family of local atoms for a closed subspace* $\mathcal{H}_0$ of a Hilbert space $\mathcal{H}$ is introduced in [8]. We generalize this notion and define atomic system for Banach spaces.

Definition 3.1. Let E be a Banach Space and E_d be a BK-space which has a sequence of canonical unit vectors $\{e_n\}$ as Schauder basis. A Bessel sequence $\{x_n\}_{n=1}^{\infty}\backslash\{0\} \subseteq E$ for E with respect to E_d is called an *atomic system* for E with respect to E_d if there exists an E_d-Bessel sequence $\{f_n\} \subseteq E^*$ for E such that for every $x \in E$, we have $x = \sum\limits_{n=1}^{\infty} f_n(x)x_n$. We shall call $\{f_n\}$ as the *associated* E_d*-Bessel sequence* for atomic system $\{x_n\}$.

Remark 3. An atomic system for E with respect to a BK-space E_d is a frame for E with respect to E_d. By definition, there exists an E_d-Bessel sequence $\{f_n\} \subseteq E^*$ with bound D and satisfying $x = \sum\limits_{n=1}^{\infty} f_n(x)x_n,$ for all $x \in E$. So, for $f \in E^*$ we have

$$\begin{aligned}\|f\|_{E^*} &= \sup_{x\in E,\|x\|=1} |f(\sum_{n=1}^{\infty} f_n(x)x_n)| \leq \sup_{x\in E,\|x\|=1} D\|x\|_E\|\{f(x_n)\}\|_{Y_d} \\ &= D\|\{f(x_n)\}\|_{Y_d}.\end{aligned}$$

Thus, $\{x_n\}$ is a frame for E with respect to E_d.

Remark 4. It is observed that every frame $\{x_n\}$ for Hilbert space $\mathcal{H}$ is an atomic system. However, every frame for Banach need not be atomic system, that we will show later.

In the following, we shall show that every E_d-Riesz basis for E is an atomic system for E.

Theorem 3.1. *Every E_d-Riesz basis $\{x_n\}$ for E is an atomic system.*

Proof. Let A, B be bounds of an E_d-Riesz basis $\{x_n\}$. Obviously, $\{x_n\}$ is a frame and its synthesis operator $\mathcal{T}$ is an isomorphism from E_d onto E. Take $f_n = (\mathcal{T}^{-1})^* l_n$, $n \in \mathbb{N}$ and $\{l_n\}_{n=1}^{\infty}$ be sequence of coordinate functionals on E_d. For $x \in E$ we have

$$f_n(x) = (\mathcal{T}^{-1})^* l_n(x) = l_n(\mathcal{T}^{-1}(x)), \text{ for all } n \in \mathbb{N}.$$

So, $\{f_n(x)\} = \mathcal{T}^{-1}(x) \in E_d$, for all $x \in E$. Moreover,

$$\|\{f_n(x)\}\|_{E_d} = \|\mathcal{T}^{-1}(x)\|_{E_d} \le \|\mathcal{T}^{-1}\|\|x\|_E \le A^{-1}\|x\|_E, \text{ for all } x \in E.$$

Also, $x = \mathcal{T}\mathcal{T}^{-1}(x) = \mathcal{T}(\{f_n(x)\}) = \sum\limits_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$. □

Next, we give the following characterization of atomic systems.

Theorem 3.2. *Let $\{x_n\}$ be frame for E with respect to E_d and $\mathcal{T}$ be its associated synthesis operator, then the following are equivalent.*

1. *$\{x_n\}$ is an atomic system.*
2. *$\mathcal{T}$ has pseudoinverse $\mathcal{T}^\dagger$.*
3. *$ker\mathcal{T}$ is complemented subspace of E_d.*
4. *$\mathcal{T}^*(E^*)$ is complemented subspace of E_d^*.*
5. *$\mathcal{R}(E^*)$ is complemented subspace of $\mathcal{Y}_d$*
6. *$\mathcal{T}^*$ has pseudoinverse $\mathcal{T}^{*\dagger}$.*
7. *There exists a complemented subspace M of E_d with $\mathcal{T}Q(E_d) = E$, where Q is a projection from E_d onto M and positive constants $0 < A \le B < \infty$ such that*

$$A\|\{\alpha_n\}\|_{E_d} \le \|\sum_n \alpha_n x_n\|_E \le B\|\{\alpha_n\}\|_{E_d}, \textit{ for all } \{\alpha_n\} \in M.$$

Proof. (1) $\Rightarrow$ (2) By given hypothesis, there exists an E_d-Bessel sequence $\{f_n\} \subseteq E^*$ such that

$$x = \sum_{n=1}^{\infty} f_n(x)x_n, \text{ for all } x \in E.$$

Let $U : E \to E_d$ be the associated analysis operator of E_d-Bessel sequence $\{f_n\}$ given by $U(x) = \{f_n(x)\}$, $x \in E$. Then, $\mathcal{T}U = I_E$ and $\mathcal{T}U\mathcal{T} = \mathcal{T}$. Hence, $\mathcal{T}$ has pseudoinverse.

(2) $\Rightarrow$ (1) $\mathcal{T}\mathcal{T}^\dagger$ is a projection from E onto $\mathcal{T}(E_d) = E$. So $\mathcal{T}\mathcal{T}^\dagger = I_E$. Take $f_n = (\mathcal{T}^\dagger)^*(l_n)$, $n \in \mathbb{N}$, where $\{l_n\} \subseteq E_d^*$ is a sequence of coordinate functionals on E_d. So, for $x \in E$, we have

$$f_n(x) = (\mathcal{T}^\dagger)^*(l_n(x)) = l_n(\mathcal{T}^\dagger(x)).$$

This gives $\{f_n(x)\} = \mathcal{T}^\dagger(x) \in E_d$, for all $x \in E$. Further

$$\|\{f_n(x)\}\|_{E_d} \leq \|\mathcal{T}^\dagger\|\|x\|_E, \text{ for all } x \in E.$$

Thus, $\{f_n\}$ is an E_d-Bessel sequence for E. Also, for $x \in E$, we have

$$x = \mathcal{T}\mathcal{T}^\dagger(x) = \mathcal{T}(\{f_n(x)\}) = \sum_{n=1}^{\infty} f_n(x)x_n.$$

Hence, $\{x_n\}$ is an atomic system for E.

(2) $\Leftrightarrow$ (6) Straight forward.

(4) $\Leftrightarrow$ (5) Straight forward.

(2) $\Leftrightarrow$ (3) Since $\mathcal{T}(E_d) = E$, by Lemma 1.2, the equivalence follows.

(4) $\Leftrightarrow$ (6) Since $ker\mathcal{T}^* = \{0\}$, by Lemma 1.2, the equivalence follows.

(3) $\Rightarrow$ (7) By given hypothesis we have $E_d = M \oplus ker\mathcal{T}$, where M is a closed subspace of E_d. Let $T_1 : M \to E$ is restriction of $\mathcal{T}$ to M. By [16, Theorem 6.3, p. 29], T_1 is

an isomorphism from M onto E. So, for $\{\alpha_n\} \in M$ we have

$$\|\{\alpha_n\}\|_{E_d} = \|T_1^{-1}T_1(\{\alpha_n\})\|_{E_d} \leq \|T_1^{-1}\|\|T_1(\{\alpha_n\})\|_E = \|T_1^{-1}\|\|\sum_n \alpha_n x_n\|_E.$$

Also,

$$\|\sum_n \alpha_n x_n\|_E = \|T_1(\{\alpha_n\})\|_E \leq \|T_1\|\|\{\alpha_n\}\|_{E_d}, \text{ for all } \{\alpha_n\} \in M.$$

(7) $\Rightarrow$ (2) Let $T_1 : M \to E$ as restriction of $\mathcal{T}$ to M. Then, $T_1(\{\alpha_n\}) = \sum_n \alpha_n x_n$, for $\{\alpha_n\} \in M$. From the given condition, T_1 is invertible and $\mathcal{T} = T_1 Q$. Thus,

$$\mathcal{T}T_1^{-1}\mathcal{T}(\alpha) = T_1 Q T_1^{-1} T_1 Q(\alpha) = T_1 Q^2(\alpha) = \mathcal{T}(\alpha), \text{ for all } \alpha \in E_d.$$

Hence, $\mathcal{T}$ has pseudoinverse. □

Remark 5. A near E_d-Riesz basis for a Banach space E is an atomic system.

A Banach space X is said to be *primary* if each of its infinite-dimensional complemented subspace is isomorphic to X.

Theorem 3.3. *Let E_d be primary BK-space. Let E be not isomorphic to E_d. Then, none of the frames $\{x_n\}$ for E with respect to E_d is an atomic system.*

Proof. On the contrary, let $\{x_n\}$ be an atomic system. So by Theorem 3.2, $E_d = ker\mathcal{T} \oplus M$, where M is a closed subspace of E_d. Moreover, from the proof of Theorem 3.2, the restriction of $\mathcal{T}$ on M, $\mathcal{T} : M \to E$ is an isomorphism from M onto E. But M is isomorphic to E_d. Thus, E is isomorphic to E_d which is a contradiction. Hence, $\{x_n\}$ is not an atomic system. □

Now, we will show that a frame need not be an atomic system. We give the following example.

Example 3.1. *Let $E = L[0,1]$ be the Lebesgue space. We extend each function $\phi \in L[0,1]$ beyond the unit interval by zero. Define*

$$\phi_n(t) = \phi_{j,k}(t) = 2^j \phi(2^j t - k), \ n \in I\!N,$$

where j=0,1,2,... and $k = 0, 1, ..., 2^j - 1$ are such that $n = 2^j + k$. In [18, Example 1], *it is shown that system of functions $\{\phi_n(t) : t \in [0,1]\}$ is a frame for E with respect to $E_d = l_1$. But l_1 is a primary Banach space which is not isomorphic to $L[0,1]$. Hence, by* Theorem 3.3, *this system of functions $\{\phi_n(t) : t \in [0,1]\}$ is not an atomic system for E with respect to l_1.*

In the following, we shall show that an atomic system for E is a projection of an E_d-Riesz basis of an ambient Banach space containing E.

Theorem 3.4. *Let $\{x_n\}_{n=1}^{\infty} \subseteq E$. Then $\{x_n\}$ is an atomic system for E with respect to E_d if and only if there exist a Banach space $\mathcal{Z}$ with E as its complemented subspace and Riesz basis $\{y_n\}$ for $\mathcal{Z}$ such that $P(y_n) = x_n$ for all $n \in I\!N$, where P is a projection from $\mathcal{Z}$ onto E.*

Proof. Let $\{x_n\}$ be an atomic system for E and $\mathcal{T}$ be its synthesis operator. By Theorem 3.2, ker$\mathcal{T}$ is a complemented subspace of E_d. So, $E_d = M \oplus ker\mathcal{T}$, where M is a closed subspace of E_d and let Q be a projection from E_d onto M. Take $\mathcal{Z} = E \oplus ker\mathcal{T}$. From Theorem 2.3, a map $\Gamma : E_d \to E \oplus ker\mathcal{T}$ defined by $\Gamma(\alpha) = (\mathcal{T}(\alpha), (I_{E_d} - Q)(\alpha))$ for $\alpha \in E_d$ is an isomorphism from E_d onto $\mathcal{Z}$. Take $y_n = \Gamma(e_n)$, for $n \in \mathbb{N}$. So, $\{y_n\}$ is a Schauder basis of $\mathcal{Z}$ and $\overline{span}\{y_n\}_{n=1}^{\infty} = \mathcal{Z}$. Let $\alpha = \{\alpha_n\} \in E_d$ and we have

$$\|\alpha\|_{E_d} = \|\Gamma^{-1}\Gamma(\alpha)\|_{E_d} \leq \|\Gamma^{-1}\|\|\Gamma(\alpha)\|_{\mathcal{Z}} = \|\Gamma^{-1}\|\|\sum_{n=1}^{\infty} \alpha_n y_n\|_{\mathcal{Z}}.$$

Also,

$$\|\sum_{n=1}^{\infty}\alpha_n y_n\|_{\mathcal{Z}} = \|\Gamma(\alpha)\|_{\mathcal{Z}} \leq \|\Gamma\|\|\alpha\|_{E_d}, \text{ for all } \alpha \in E_d.$$

Thus, $\{y_n\}$ is a E_d-Riesz basis for E. Let P be projection from $\mathcal{Z}$ onto E. We know that $\mathcal{T}(e_n) = x_n$, for all $n \in \mathbb{N}$. From the definition of Γ, we have $T = P\Gamma$. Hence, $x_n = \mathcal{T}(e_n) = P\Gamma(e_n) = P(y_n)$, for all $n \in \mathbb{N}$.

Conversely, let $\mathcal{Z} = E \oplus W$, where W is a closed subspace of $\mathcal{Z}$. Let $\{y_n\}$ be E_d-Riesz basis for $\mathcal{Z}$ with bounds A, B and $P(y_n) = x_n$, for all $n \in \mathbb{N}$. Then, by Proposition 2.1 there is an isomorphism $\Gamma : E_d \to \mathcal{Z}$ from E_d onto $\mathcal{Z}$ such that $\Gamma(e_n) = y_n$, for all $n \in \mathbb{N}$. Define $T : E_d \to E$ as $T = P\Gamma$. So, T is surjective and $T(e_n) = P\Gamma(e_n) = P(y_n) = x_n$, for all $n \in \mathbb{N}$. By Theorem 2.1, $\{x_n\}$ is a frame for E and T is the synthesis operator. Let $S : E \to E_d$ be the restriction of Γ^{-1} to E. Then,

$$TST = P\Gamma\Gamma^{-1}P\Gamma = P^2\Gamma = P\Gamma = T.$$

Thus, S is the pseudoinverse of T. By Theorem 3.2, $\{x_n\}$ is an atomic system. □

Next, we discuss dual atomic system of a given atomic system $\{x_n\}$ analogue to that of the dual frame in Hilbert spaces. That is, if $\{x_n\}$ is an atomic system for E with $\{f_n\}$ as its associated E_d-Bessel sequence and E_d^* has a sequence of canonical unit vectors $\{e_n^*\}$ as Schauder basis. Then, $\{f_n\}$ is an atomic system for E^* with respect to Y_d and $(f_n, \pi(x_n))$ is an atomic decomposition for E^* with respect to Y_d. In the following, we give duality results of a given atomic system $\{x_n\}$.

Theorem 3.5. *Let E_d^* has a sequence of canonical unit vectors $\{e_n^*\}$ as Schauder basis and $\{x_n\} \subseteq E$. Then,*

(1) *if $\{x_n\}$ is an atomic system for E with respect to E_d and $\{f_n\} \subseteq E^*$ be its associated E_d-Bessel sequence, then $\{f_n\}$ is an atomic system for E^* with respect to Y_d.*

(2) $\{x_n\}$ *be an atomic system for* E *with respect to* E_d *if and only if there exists an* E_d*-Bessel sequence* $\{f_n\} \subseteq E^*$ *such that* $(f_n, \pi(x_n))$ *is an atomic decomposition for* E^* *with respect to* Y_d.

Proof. (1) By given hypotheses, $\{x_n\}$ is a Bessel sequence for E with respect to E_d and

$$x = \sum_{n=1}^{\infty} f_n(x)x_n, \text{ for all } x \in E.$$

Let $\mathcal{T}$ be the associated synthesis operator of $\{x_n\}$ and U be the associated analysis operator of $\{f_n\}$. Then, $I = \mathcal{T}U$. So $I^* = U^*\mathcal{T}^*$ and $f = U^*\mathcal{T}^*(f)$, for all $f \in E^*$. Therefore, $U^* : E_d^* \to E^*$ is surjective. Define $V : Y_d \to E^*$ as $V = U^*J_d$. Obviously, $\{J_d^{-1}(e_n^*)\}_{n=1}^{\infty}$ is basis of Y_d and V is also sujective. Also, for $n \in \mathbb{N}$ we have

$$U^*e_n^*(x) = e_n^*(U(x)) = e_n^*(\{f_n(x)\}) = f_n(x), \text{ for all } x \in E.$$

Thus, $f_n = U^*e_n^*$ and $V(J_d^{-1}e_n^*) = f_n$, for all $n \in \mathbb{N}$. Since V is surjective, so for any $f \in E^*$ there exists $\alpha = \{\alpha_n\} \in Y_d$ such that $f = V(\alpha)$ and

$$f = V(\alpha) = V(\sum_{n=1}^{\infty} \alpha_n J_d^{-1}(e_n^*)) = \sum_{n=1}^{\infty} \alpha_n U^*J_dJ_d^{-1}(e_n^*) = \sum_{n=1}^{\infty} \alpha_n f_n. \tag{3.1}$$

By Theorem 2.1, $\{f_n\}$ is a frame for E^* with V as its synthesis operator. Moreover,

$$VJ_d^{-1}\mathcal{T}^*V = U^*J_dJ_d^{-1}\mathcal{T}^*U^*J_d = U^*J_d = V$$

Thus, V has pseudoinverse $V^\dagger$ and by Theorem 3.2, $\{f_n\}$ is an atomic system for E^* with respect to Y_d.

(2) Since, $\{x_n\}$ is an atomic system, so $\{x_n\}$ is a frame for E. Let $\{f_n\}$ be its associated E_d-Bessel sequence with bound C. As in Proof (1), we have $I_{E^*}^* = U^*\mathcal{T}^* = U^*J_d\mathcal{R}$ and $\{J_d^{-1}(e_n^*)\}$ is a basis of Y_d. Now, for any $f \in E^*$ we have

$$f = U^*\mathcal{T}^*(f) = U^*J_d\mathcal{R}(f) = U^*J_d(\{f(x_n)\}(f) = \sum_{n=1}^{\infty} f(x_n)U^*J_dJ_d^{-1}(e_n^*).$$

Hence,

$$f = \sum_{n=1}^{\infty} f(x_n) f_n, \text{ for all } f \in E^*. \tag{3.2}$$

Conversely, since $(f_n, \pi(x_n))$ is an atomic decomposition of E^* with respect to Y_d, so there exist constants $0 < A \leq B < 0$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{Y_d} \leq B\|f\|_{E^*}, \text{ for all } f \in E^*,$$

and

$$f = \sum_{n=1}^{\infty} f(x_n) f_n, \text{ for all } f \in E^*.$$

Also, $\{f_n(x)\} \in E_d$, for all $x \in E$. Let $x \in E$ and $N \in \mathbb{N}$. Then

$$\begin{aligned} \|x - \sum_{k=1}^{N} f_k(x) x_k\|_E &= \sup_{f \in E^*, \|f\|=1} |\sum_{k=N+1}^{\infty} f_k(x) f(x_k)| \\ &\leq B\|\sum_{k=N+1}^{\infty} f_k(x) e_k\|_{E_d} \to 0 \text{ as } N \to \infty \end{aligned}$$

Hence, $x = \sum\limits_{n=1}^{\infty} f_n(x) x_n$, for all $x \in E$. □

In [18], P. A. Terekhin discussed an analogue of the extremality property of frame expansion in E. We shall discuss such similar property in the conjugate space E^* from a given atomic system. A closed subspace F in a Banach space E is said to be 1-*complemented* or *constrained*, if F is the range of a norm one projection on E. In the following result, we characterize 1-complemented subspace of Y_d in term of an atomic system for E.

Theorem 3.6. *Let E_d^* has a sequence of canonical unit vectors $\{e_n^*\}$ as Schauder basis. Let $\{x_n\}$ be frame for E. Then, the following conditions are equivalent.*

(1) *$\mathcal{R}(E^*)$ is 1-complemented subspace of $\mathcal{Y}_d$.*

(2) $\{x_n\}$ *is an atomic system for* E *such that among all* $\{\alpha_n\} \in \mathcal{Y}_d$ *for which the representation (3.1) holds, the sequence* $\{f(x_n)\}$ *of the coefficients of expansion (3.2) has minimum* $\mathcal{Y}_d$ *norm,*

$$\|\{f(x_n)\}\|_{\mathcal{Y}_d} \leq \|\{\alpha_n\}\|_{\mathcal{Y}_d}. \tag{3.3}$$

Proof. (2) $\Rightarrow$ (1) By Theorem 3.2, $\mathcal{R}(E^*)$ is a complemented subspace of Y_d. Let $\{f_n\}$ be its associated E_d-Bessel sequence. As proved in Theorem 3.5, we have

$$I_{E^*} = U^*\mathcal{T}^* = U^*J_dJ_d^{-1}\mathcal{T}^* = V\mathcal{R}$$

So, $V = V\mathcal{R}V$ and $\mathcal{R} = \mathcal{R}V\mathcal{R}$. Indeed, $Q = \mathcal{R}V$ is a projection from Y_d onto $\mathcal{R}(E^*)$ and $Y_d = \mathcal{R}(E^*) \oplus ker\mathcal{R}V$. It is obvious that $kerV \subseteq ker\mathcal{R}V$. Let $\alpha \in ker\mathcal{R}V$, then $\mathcal{R}V(\alpha) = 0$. So, $V\mathcal{R}V(\alpha) = 0$ and $V(\alpha) = 0$. Thus, $Y_d = kerV \oplus \mathcal{R}(E^*)$. Let $\alpha \in Y_d$, then $\alpha = \alpha_0 + Q(\alpha)$, where $\alpha_0 \in kerV$ and $Q(\alpha) \in \mathcal{R}(E^*)$. Moreover, $Q(\alpha) = \mathcal{R}(f)$ for some $f \in E^*$ and

$$f = V\mathcal{R}(f) = VQ(\alpha) = V(\alpha - \alpha_0) = V(\alpha),$$

which satisfies (3.1). Thus, $\|Q(\alpha)\|_{Y_d} = \|\mathcal{R}(f)\|_{Y_d} \leq \|\alpha\|_{Y_d}$. Therefore, $\|Q\| = 1$. Hence $\mathcal{R}(E^*)$ is 1-complemented subspace of Y_d.

(1) $\Rightarrow$ (2) By Theorem 3.2, $\{x_n\}$ is an atomic system for E. Also, there exists a projection Q from Y_d onto $\mathcal{R}(E^*)$ and $\|Q\| = 1$. Take an arbitrary element $\alpha = \{\alpha_n\} \in Y_d$ which satisfies the representation (3.1). Then, $f = V(\alpha)$, for $f \in E^*$. As shown above that $E_d^* = kerV \oplus \mathcal{R}(E^*)$. Let $\alpha = \alpha_0 + Q(\alpha)$, for $\alpha_0 \in kerV$ and $Q(\alpha) \in \mathcal{R}(E^*)$. Then, $Q(\alpha) = f^1$, for some $f^1 \in E^*$. Also, as shown above

$$f^1 = V\mathcal{R}(f^1) = V(Q(\alpha)) = V(\alpha) = f.$$

Thus, $Q(\alpha) = \mathcal{R}(f)$. Finally, we have

$$\|\{f(x_n)\}\|_{Y_d} = \|\mathcal{R}(f)\|_{Y_d} = \|Q(\alpha)\| \leq \|Q\|\|\alpha\|_{Y_d} = \|\alpha\|_{Y_d}.$$

□

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(1) Department of Mathematics, Motilal Nehru College, University of Delhi, Delhi, India, 110021

E-mail address: kholetim@yahoo.co.in

(2) Department of Mathematics, Kirori Mal College, University of Delhi, Delhi, India, 110007

E-mail address: shikk2003@yahoo.co.in