

$B_{\alpha,\beta}$ -OPERATOR AND FITZPATRICK FUNCTIONS

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ABSTRACT. In this paper, we introduce $B_{\alpha,\beta}$ - operator as a new technique to generalize Fitzpatrick functions. We study the properties of this new operator, such as convexity, weak *-closedness, locally boundedness and Fenchel duality. Furthermore, we present new concept that is C_n^α -monotone bifunction. By the application of the new operator, we prove that

$$\psi_{\beta,n}(x, x^*) = \langle x^*, x \rangle \quad \forall (x, x^*) \in G(B_{\alpha,\beta}),$$

for each $n \geq 2$. This equation is a generalization for a known result which is studied by many researchers.

1. INTRODUCTION

In recent years, Fitzpatrick function became a major tool for the connection between maximal monotone theory and convex analysis. The use of monotone and maximal monotone operators have been found to provide very powerful techniques for studying problems in various branches of applied mathematics, such as optimization, variational inequality, Nash equilibria, and partial differential equation. [6, 9, 13, 14, 16, 22, 23] and [24].

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Fitzpatrick function was also used to study the *n-cyclically monotone operator*, as seen in [5], to be an extension of the monotone operator introduced by Fitzpatrick in [16]. It is believed that this approach has an robust role in this area.

In this work, we suppose that E is a real Banach space with dual E^* . The a multivalued mapping $T : E \rightrightarrows E^*$ is said to be a *monotone operator*, provided that $\langle x^* - y^*, x - y \rangle \geq 0$ for each $x, y \in \text{dom}T$, $x^* \in T(x)$ and $y^* \in T(y)$, where $\text{dom}T = \{x \in E : Tx \neq \emptyset\}$ its *domain*, and the *graph* of the operator T is $G(T) = \{(x, x^*) \in E \times E^* : x^* \in T(x)\}$. We suppose that $\alpha, \beta : C \times C \rightarrow \mathbb{R} \cup \{+\infty\}$ are two bifunctions, where C is an arbitrary subset of E .

Monotone bifunctions were introduced in the seminal paper by Blum and Oettli [8], while it is studied and generalized by several authors [1, 3, 7, 12, 17, 19] and [18]. In fact β is monotone bifunction if

$$\beta(x, y) + \beta(y, x) \leq 0 \quad (\forall x, y \in C).$$

It is worth mentioning that Monotone bifunctions were mainly studied in conjunction with the so-called equilibrium problem, which includes variational inequalities as special cases. For example, find $x_1 \in C$ such that

$$\beta(x_1, x) \geq 0 \quad (\forall x \in C).$$

This paper is divided into four sections. In addition to the introduction, in Section 2, we refer to some definitions that will assist us in the study. Section 3 introduces the new operator $B_{\alpha, \beta}$ and analyse their properties. In Section 4, we introduce the application of the new type of operator generalizing some results of a Fitzpatrick functions. In addition, the concepts of the class C_n^α -monotone bifunction are introduced.

The aim of the work is to present various new results concerning $B_{\alpha, \beta}$ -operator, as well as to establish a new technique of generalizing the result of Fitzpatrick function.

2. EXPLAINING NOTIONS AND DEFINITIONS

It is important that we recall some relevant notions to our study by referring to some definitions. These definitions help us to find out the main results of the study.

Definition 2.1. ([16, Definition 3.1, p. 61]) Let $A : E \multimap E^*$ be multivalued map. The Fitzpatrick function of A is $\beta_A(x, x^*) : E \times E^* \rightarrow]-\infty, +\infty]$

$$(2.1) \quad (x, x^*) \mapsto \sup_{(y, y^*) \in G(A)} (\langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle).$$

The above function is obviously convex and lower semicontinuous, where topology of $E \times E^*$ is *weak* \times *weak*^{*} [16]. Moreover, the Fitzpatrick function has been essentially a key note tool in the study of monotone bifunctions in the resent years [3, 5] and [10].

Definition 2.2. ([4, Definition 5, p.7]) Suppose that E is a Banach space. Then, a map $T : E \multimap E^*$ is said locally bounded at $x_0 \in E$, if there exist $\epsilon > 0$ and $m > 0$, such that $\|x^*\| \leq m$, $\forall x^* \in T(x)$ and $x \in B(x_0, \epsilon)$.

Definition 2.3. ([17, Section 5, p.13]) A bifunction β is called *cyclically monotone*, if

$$(2.2) \quad \sum_{i=1}^n \beta(x_i, x_{i+1}) \leq 0 (\forall x_1, \dots, x_{n+1} \in C),$$

where $x_{n+1} = x_1$.

Definition 2.4. Assume that E is a Banach space and that $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is a function. Then

i) The Fenchel-Moreau conjugate $f^* : E^* \rightarrow \overline{\mathbb{R}}$ of f is defined by

$$f^*(x^*) = \sup_{x \in E} \langle x^*, x \rangle - f(x).$$

ii) The (Fenchel) subdifferential $\partial f(x) : E \multimap E^*$ of f is defined by

$$\partial f(x) = \begin{cases} x^* \in E^* : f(y) - f(x) \geq \langle x^*, y - x \rangle, \forall y \in E, & \text{if } f(x) \in \mathbb{R} \\ \emptyset & \text{if } f(x) \notin \mathbb{R}. \end{cases}$$

iii) For $\varepsilon \geq 0$, the ε - subdifferential $\partial f_\varepsilon(x) : E \multimap E^*$ of f , which is introduced, in [20] is defined as

$$\partial f_\varepsilon(x) = \begin{cases} x^* \in E^* : f(y) - f(x) \geq \langle x^*, y - x \rangle - \varepsilon, \forall y \in E, & \text{if } f(x) \in \mathbb{R} \\ \emptyset & \text{if } f(x) \notin \mathbb{R}. \end{cases}.$$

Here, we shall recall a new type of monotone operators introduced in [2] is called it α -monotone,

Definition 2.5. Assume that $T : E \multimap E^*$ and $\alpha : \text{dom}T \times \text{dom}T \rightarrow \mathbb{R}$ are two maps. T is considered to be α - monotone, if for each $x, y \in \text{dom}T$, $x^* \in T(x)$ and $y^* \in T(y)$ such that

$$(2.3) \quad \langle x^* - y^*, x - y \rangle \geq \alpha(x, y).$$

In 2016 Hashoosh et.al. introduced a new class of monotone bifunctions. It is defined as follows:

Definition 2.6. ([2, Definition 1, p.1]) A bifunction $\beta : C \times C \rightarrow \mathbb{R}$ is called α -monotone, if

$$(2.4) \quad \beta(x, y) + \beta(y, x) + \alpha(x, y) \leq 0 \quad (\forall x, y \in C).$$

Example 2.1. Let $E = \mathbb{R}$, $K = \mathbb{R}$ and let $\beta : K \times K \rightarrow \mathbb{R}$ be bifunction defined by

$$\beta(u, v) = \cos(u - v)^2 + (u - v)^2,$$

for all $u, v \in K$. Then

$$\beta(u, v) + \beta(v, u) = 2 \cos(u - v)^2 + 2(u - v)^2 \not\leq 0,$$

where $u \neq v$. Therefore β is not monotone bifunction.

But, it easy to see that β is α -monotone bifunction with $\alpha(u, v) = -5(u - v)^2$. In fact,

$$\begin{aligned} \beta(u, v) + \beta(v, u) &= 2 \cos(u - v)^2 + 2(u - v)^2 \\ &\leq 5(u - v)^2 \\ &= -\alpha(u, v). \end{aligned}$$

The original definition of monotone bifunction is introduced in [9] and generalized by several authors [10, 11] and [13].

Proposition 2.1. ([17, Proposition 5.1, p.13]) Assume that E is a vector space, $\phi \neq C \subseteq E$, and $G: C \times C \rightarrow \mathbb{R}$ is a bifunction. Then, G is a cyclically monotone, iff there exists a function $f : C \rightarrow \mathbb{R}$, such that

$$(2.5) \quad G(x, y) \leq f(y) - f(x) \quad \forall x, y \in C.$$

Definition 2.7. Let E be a Banach space. A mapping $\Lambda : X \rightarrow \mathbb{R}$ is said to be *lower semicontinuous* (for short *,(l.s.c)*) at $x_0 \in E$, if

$$\Lambda(x_0) \leq \liminf_n \Lambda(x_n),$$

for any sequence x_n of E such that $x_n \rightarrow x_0$.

3. $B_{\alpha,\beta}$ -operator and Fenchel duality

In this section, we introduce the class of $B_{\alpha,\beta}$ -operators with their properties, such as convexity, weak*-closedness and locally boundedness of $B_{\alpha,\beta}$. In addition, we are going to prove two results by using Fenchel duality.

Definition 3.1. Let $\alpha, \beta : C \times C \rightarrow \mathbb{R}$ be two bifunctions, where C is an arbitrary subset of a Banach space E . The map $B_{\alpha,\beta} : E \rightarrow E^*$ is given by

$$(3.1) \quad B_{\alpha,\beta}(x) = \begin{cases} x^* \in E^* & \text{if } \beta(y, x) + \langle x^*, x \rangle \geq \langle x^*, y \rangle + \frac{\alpha(y, x)}{2} \quad (\forall y \in C) \\ \emptyset & \text{if } x \notin C. \end{cases}$$

Theorem 3.1. Assume that $\beta : C \times C \rightarrow \mathbb{R}$ is α -monotone bifunction. Then $B_{\alpha,\beta}$ is α -monotone, where $\alpha(x, y) + \alpha(y, x) = 0$.

Proof. For $x^* \in B_{\alpha,\beta}(x), y^* \in B_{\alpha,\beta}(y)$,

$$\begin{aligned} \beta(y, x) + \langle x^*, x - y \rangle &\geq \frac{\alpha(y, x)}{2} \\ \beta(x, y) + \langle y^*, y - x \rangle &\geq \frac{\alpha(x, y)}{2} = -\frac{\alpha(y, x)}{2}. \end{aligned}$$

Then,

$$\begin{aligned} \beta(y, x) + \beta(x, y) + \langle x^* - y^*, x - y \rangle &\geq 0 \\ \langle x^* - y^*, x - y \rangle &\geq -\beta(y, x) - \beta(x, y) \geq \alpha(y, x). \end{aligned}$$

Therefore, $B_{\alpha,\beta}$ is α - monotone operator. □

Remark 1. In special case, one can easily check

i) $B_{\alpha,\beta}$ is monotone if $\alpha \equiv 0$ or β is monotone bifunction.

ii) $B_{\alpha,\beta} = \partial f(x)$ if substituing $\beta(x, y) - \frac{\alpha(y, x)}{2} = f(y) - f(x)$.

Here, we recall a new type of a subdifferential concept is called α - subdifferential, denoted by $\partial_\alpha f$.

Definition 3.2. ([20, Definition 7, p.2]) Assume that E is a Banach space, and $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper function. One can say that $x^* \in E^*$ is a α -subdifferential of f in $x \in \text{dom} f = \{x : f(x) < \infty\}$, if $f(y) - \frac{\alpha(y, x)}{2} \geq f(x) + \langle x^*, y - x \rangle \quad (\forall y \in E)$.

Remark 2. From definition 3.2, one can check $\partial_\alpha f(x) = B_{\alpha,\beta}(x)$ if $\beta(y, x) = f(y) - f(x)$.

It is known that the subdifferential of any function is monotone. However, this fact for $\partial_\alpha f(x)$ is extended.

Corollary 3.1. *Let f be a function and α be symmetric; (i.e; $\alpha(x, y) = \alpha(y, x) \forall x, y \in E$). Then $\partial_\alpha f(x) : E \rightarrow E^*$ is α -monotone.*

Proof. Let $x^* \in \partial_\alpha f(x)$ and $y^* \in \partial_\alpha f(y)$. From Definition 3.2

$$(3.2) \quad x^* \in \partial_\alpha f(x) \iff f(y) - \frac{\alpha(y, x)}{2} \geq f(x) + \langle x^*, y - x \rangle$$

$$(3.3) \quad y^* \in \partial_\alpha f(y) \iff f(x) - \frac{\alpha(x, y)}{2} \geq f(y) + \langle y^*, x - y \rangle$$

In adding (3.2) to (3.3), we get

$$-\frac{1}{2}\alpha(y, x) - \frac{1}{2}\alpha(x, y) \geq -\langle x^* - y^*, x - y \rangle$$

so,

$$\langle x^* - y^*, x - y \rangle \geq \alpha(x, y).$$

□

From what has been given, one can easily check that if α is antisymmetric, then $\partial_\alpha f(x)$ is monotone operator.

In what follows, we are going to study the weak *-closedness and the convexity of $B_{\alpha,\beta}$.

Theorem 3.2. Assume that β is a monotone bifunction. Then, $B_{\alpha,\beta}$ is convex and weak *-closed for all $x \in C$, where C is a convex subset of E .

Proof. Let $z^* = \lambda x_1^* + (1 - \lambda)x_2^*$, $x_1^*, x_2^* \in B_{\alpha, \beta}(x)$, $\lambda \in [0, 1]$.

Then for any $y \in C$,

$$\begin{aligned} \langle z^*, y - x \rangle &= \langle \lambda x_1^* + (1 - \lambda)x_2^*, y - x \rangle \\ &= \lambda \langle x_1^*, y - x \rangle + (1 - \lambda) \langle x_2^*, y - x \rangle \\ &\leq \lambda \left(\beta(y, x) - \frac{\alpha(y, x)}{2} \right) + (1 - \lambda) \left(\beta(y, x) - \frac{\alpha(y, x)}{2} \right) \\ &= \beta(y, x) - \frac{\alpha(y, x)}{2}. \end{aligned}$$

This means that $z^* \in B_{\alpha, \beta}(x)$. Therefore, $B_{\alpha, \beta}(x)$ is convex $\forall x \in C$.

To prove that $B_{\alpha, \beta}$ is weak *-closed, assume that $y^* \in (B_{\alpha, \beta}(x))^c$; (i.e, y^* in the complement of $B_{\alpha, \beta}(x)$). There is $y_0 \in C$, such that

$$\beta(y_0, x) + \langle y^*, x - y_0 \rangle < \frac{1}{2} \alpha(y, x).$$

Choose t_0 in which $\beta(y_0, x) + \langle y^*, x - y_0 \rangle < t_0 < \frac{1}{2} \alpha(y_0, x)$. Suppose that $\langle y^*, x - y_0 \rangle < t_0 - \beta(y_0, x) = -t$.

Suppose that $U := \{x^* \in E^* : \langle x^*, x - y_0 \rangle < -t\}$ is w^* -open [21]. If $z^* \in U$, then $\langle z^*, x - y_0 \rangle < -t = t_0 - \beta(y_0, x)$. Therefore,

$$\beta(y_0, x) + \langle z^*, x - y_0 \rangle < t_0 < \frac{1}{2} \alpha(y_0, x)$$

It means that $z^* \in (B_{\alpha, \beta}(x))^c$, so $U \subset (B_{\alpha, \beta}(x))^c$. Therefore, $(B_{\alpha, \beta}(x))^c$ is weak *-open, so $B_{\alpha, \beta}(x)$ is a weak *-closed. \square

Theorem 3.3. Let β be a bifunction, and $t, s \geq 0$ and $t + s = 1$. Then,

$$(tB_{\alpha, \beta_1} + sB_{\alpha, \beta_2})(x) \subseteq B_{\alpha, t\beta_1 + s\beta_2}(x).$$

Proof. Let $x^* \in (tB_{\alpha, \beta_1} + sB_{\alpha, \beta_2})(x)$, $x^* = tx_1^* + sx_2^*$, where $x_1^* \in B_{\alpha, \beta_1}(x)$ and $x_2^* \in B_{\alpha, \beta_2}(x)$.

$$(3.4) \quad \beta_1(y, x) + \langle x_1^*, x - y \rangle \geq \frac{1}{2} \alpha(y, x)$$

$$(3.5) \quad \beta_2(y, x) + \langle x_2^*, x - y \rangle \geq \frac{1}{2}\alpha(y, x)$$

when (3.4) is multiplied by t and (3.5) by s , and they are added together, we have

$$(t\beta_1 + s\beta_2)(y, x) + \langle tx_1^* + sx_2^*, x - y \rangle \geq \frac{(t+s)}{2}\alpha(y, x),$$

so,

$$x^* \in B_{\alpha, t\beta_1 + s\beta_2}(x).$$

It means that

$$(tB_{\alpha, \beta_1} + sB_{\alpha, \beta_2})(x) \subseteq B_{\alpha, t\beta_1 + s\beta_2}(x).$$

□

Theorem 3.4. Let $\alpha, \beta : C \times C \rightarrow \mathbb{R}$ be two bifunctions on $intC \cap B(x_0, \epsilon) \neq \emptyset$. If β is bounded from above to a constant m and α is bounded below to $\frac{m}{2}$. Then $B_{\alpha, \beta}(x)$ is locally bounded at x_0 .

Proof. Assume that $\epsilon > 0, m \in \mathbb{R}$ such that $B(x_0, \epsilon) \subset C$ and $\beta(y, x) \leq m$ and $\alpha(y, x) \geq \frac{m}{2} \forall x, y \in B(x_0, \epsilon)$. Suppose that $x^* \in B_{\alpha, \beta}(x)$, $\|x - x_0\| \leq \frac{\epsilon}{2}$ and $\|w\| \leq 1$.

Then,

$$\left\| x + \frac{\epsilon}{2}w - x_0 \right\| \leq \|x - x_0\| + \frac{\epsilon}{2}\|w\| \leq \epsilon$$

Put $y = x + \frac{\epsilon}{2}w$. Therefore,

$$\frac{\epsilon}{2}\langle x^*, w \rangle = \langle x^*, y - x \rangle \leq \beta(y, x) - \frac{\alpha(y, x)}{2} \leq m - \frac{m}{2} = \frac{m}{2}.$$

Then

$\|x^*\| \leq \frac{m}{\epsilon}$. Hence, $B_{\alpha, \beta}$ is a locally bounded at x_0 . □

The following example shows that $B_{\alpha, \beta}$ may be unbounded.

Example 3.1. Let $x \in C$ and Let $\alpha, \beta : C \times C \rightarrow \mathbb{R}$ be two bifunctions by

$$\alpha(y, x) = 4 \|x - y\| \text{ and } \beta(y, x) = \|x - y\|.$$

If $x^* \in B_{\alpha, \beta}(x)$, then

$$\langle x^*, x - y \rangle \geq \frac{1}{2} \alpha(y, x) - \beta(y, x) = \|x - y\|,$$

and

$$\langle nx^*, x - y \rangle \geq \|x - y\| \quad \forall n \geq 1.$$

Hence, $nx^* \in B_{\alpha, \beta}(x)$. Therefore, $B_{\alpha, \beta}(x)$ is unbounded.

For bifunctions β_1, β_2 and $\beta : C \times C \rightarrow \mathbb{R}$, $\beta_1 + \beta_2$ is defined as follows:

$$(\beta_1 + \beta_2)(y, x) = \beta_1(y, x) + \beta_2(y, x).$$

Theorem 3.5. If α, β_1 and $\beta_2 : C \times C \rightarrow \mathbb{R}$ are bifunctions, such that $\alpha(y, x) \geq 0 \quad \forall (y, x) \in C \times C$. Then $B_{\alpha, \beta_1}(x) + B_{\alpha, \beta_2}(x) \subset B_{\alpha, \beta_1 + \beta_2}(x) (\forall x \in E)$.

Proof. Assume that $x \in C, x^* \in B_{\alpha, \beta_1}(x) + B_{\alpha, \beta_2}(x)$, so $x^* = x_1^* + x_2^*$, for some $x_1^* \in B_{\alpha, \beta_1}(x)$ and $x_2^* \in B_{\alpha, \beta_2}(x)$. Then

$$(3.6) \quad \beta_1(y, x) + \langle x_1^*, x - y \rangle \geq \frac{\alpha(y, x)}{2}.$$

$$(3.7) \quad \beta_2(y, x) + \langle x_2^*, x - y \rangle \geq \frac{\alpha(y, x)}{2}.$$

By adding (3.6) to (3.7), we obtain

$$(\beta_1 + \beta_2)(y, x) + \langle x^*, x - y \rangle \geq \frac{\alpha(y, x)}{2} (\forall y \in C).$$

Hence, $x^* \in B_{\alpha, \beta_1 + \beta_2}(x)$. □

The following properties of $B_{\alpha, \beta}(x)$ are studied via Fenchel duality.

Theorem 3.6. Let $\alpha, \beta : C \times C \rightarrow \mathbb{R}$ be two bifunctions, if

(i) $\alpha(x, x) \geq 0, \forall x, y \in C,$

(ii) $\beta(y, x) = f(y) - f(x),$

then

$$x^* \in B_{\alpha,\beta}(x) \Leftrightarrow \langle x^*, x \rangle = f_{\alpha}^*(x^*, x) + f(x),$$

where, $f_{\alpha}^* : E^* \times E \rightarrow \mathbb{R}$ define as follows

$$(3.8) \quad f_{\alpha}^*(x^*, x) = \sup_{y \in C} \left[\langle x^*, y \rangle - f(y) + \frac{\alpha(y, x)}{2} \right].$$

Proof. Let $x^* \in B_{\alpha,\beta}(x)$, from (3.1) and (ii), then

$$\begin{aligned} f(y) - f(x) &\geq \langle x^*, y \rangle - \langle x^*, x \rangle + \frac{\alpha(y, x)}{2}. \\ \langle x^*, x \rangle &\geq \langle x^*, y \rangle + \frac{\alpha(y, x)}{2} - f(y) + f(x). \end{aligned}$$

Therefore,

$$\langle x^*, x \rangle \geq \sup_{y \in C} \left[\langle x^*, y \rangle + \frac{\alpha(y, x)}{2} - f(y) \right] + f(x),$$

so,

$$(3.9) \quad \langle x^*, x \rangle \geq f_{\alpha}^*(x^*, x) + f(x).$$

On the other hand, replacing $y = x$ in (3.8), and from condition (i),

$$(3.10) \quad \langle x^*, x \rangle \leq f_{\alpha}^*(x^*, x) + f(x).$$

From (3.9) and (3.10) the equality concludes.

Conversely, Let $\langle x^*, x \rangle = f_{\alpha}^*(x^*, x) + f(x).$

$$\begin{aligned} \langle x^*, x \rangle &= \sup_{y \in C} \left[\langle x^*, y \rangle - f(y) + \frac{\alpha(y, x)}{2} \right] + f(x) \\ &\geq \left[\langle x^*, y \rangle - f(y) + \frac{\alpha(y, x)}{2} \right] + f(x) \\ &= \left[\langle x^*, y \rangle - \beta(y, x) + \frac{\alpha(y, x)}{2} \right] \end{aligned}$$

so, $x^* \in B_{\alpha,\beta}(x)$. □

Corollary 3.2. One can check from Remark 2 and Theorem 3.6,

$$x^* \in \partial_\alpha f(x) \Leftrightarrow f(x) + f_\alpha^*(x^*, x) = \langle x, x^* \rangle.$$

The next result was first established in [15] by Censor, Iusem and Zenios for sub-differential operator. Now we extend and develop the result for $B_{\alpha,\beta}$ -operator.

Proposition 3.1. Let $\alpha, \beta : C \times C \rightarrow \mathbb{R}$ be two bifunctions, and the assumptions (i – ii) in Theorem 3.6 hold. In addition, assume that

iii) $B_{\alpha,\beta}$ is a para-monotone i.e., if $x^* \in B_{\alpha,\beta}(x), y^* \in B_{\alpha,\beta}(y)$, then

$$\langle x^* - y^*, x - y \rangle = 0.$$

iv) $f_\alpha^*(x^*, x) \leq \langle x^*, y \rangle - f(y), \forall x, y \in C$ and $\forall x^*, y^* \in E^*$.

Then $x^* \in B_{\alpha,\beta}(y)$ and $y^* \in B_{\alpha,\beta}(x)$.

Proof. Using (i), (iv) and (3.8),

$$(3.11) \quad f_\alpha^*(x^*, x) = \langle x^*, y \rangle - f(y).$$

From Theorem 3.6 for $(x^*, x), (y^*, y) \in G(B_{\alpha,\beta})$, one can get

$$(3.12) \quad \langle x^*, x \rangle = f_\alpha^*(x^*, x) + f(x).$$

$$(3.13) \quad \langle y^*, y \rangle = f_\alpha^*(y^*, y) + f(y).$$

By adding (3.12) and (3.13) and by condition (iii), one can have

$$\langle x^*, y \rangle + \langle y^*, x \rangle = f_\alpha^*(x^*, x) + f(x) + f_\alpha^*(y^*, y) + f(y).$$

So,

$$[f_\alpha^*(x^*, x) + f(y) - \langle x^*, y \rangle] + [f_\alpha^*(y^*, y) + f(x) - \langle y^*, x \rangle] = 0.$$

From (3.11) $f_\alpha^*(x^*, x) + f(y) = \langle x^*, y \rangle$, $f_\alpha^*(y^*, y) + f(x) = \langle y^*, x \rangle$.

This means that $x^* \in B_{\alpha,\beta}(y)$ and $y^* \in B_{\alpha,\beta}(x)$. □

4. General Fitzpatrick transform

In this section, we generalize some results of a Fitzpatrick functions by using the operator $B_{\alpha,\beta}$. In addition, the concepts of the class C_n^α -monotone are introduced and studied.

Definition 4.1. Assume that $\alpha, \beta : C \times C \rightarrow \mathbb{R} \cup \{+\infty\}$ are two bifunctions. The function $\psi_\beta : E \times E^* \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined as follows:

$$(4.1) \quad \psi_\beta(x^*, x) = \sup_{y \in E} \left[\langle x^*, y \rangle + \beta(x, y) + \frac{\alpha(x, y)}{2} \right] \quad \forall (x, x^*) \in E \times E^*.$$

Remark 3. If $\alpha(\cdot, y), \beta(\cdot, y)$ are *l.s.c* and convex $\forall x, y \in \text{dom}\beta \cap \text{dom}\alpha$, one can easily check that ψ_β is also *l.s.c* and convex.

Lemma 4.1. Assume that $\alpha, \beta : C \times C \rightarrow \mathbb{R} \cup \{+\infty\}$ are two bifunctions, and that $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is a function in which

- (i) $\text{dom}f = C = \{x : f(x) < \infty\} \neq \emptyset$,
- (ii) $\beta(x, y) \geq f(x) - f(y)$,
- (iii) α is symmetric,

then

$$\psi_\beta(x, x^*) \geq f_\alpha^*(x^*, x) + f(x).$$

Proof. From (i), there is $x \in C$. Then

$$\begin{aligned} \psi_\beta(x, x^*) &= \sup_{y \in E} \left[\langle x^*, y \rangle + \beta(x, y) + \frac{\alpha(x, y)}{2} \right] \\ &\geq \sup_{y \in E} \left[\langle x^*, y \rangle - f(y) + \frac{\alpha(x, y)}{2} + f(x) \right] \\ &= f_\alpha^*(x^*, x) + f(x). \end{aligned}$$

Moreover, the equality holds in (3.5), if the equality holds in the condition (ii). \square

Next to what we have introduced above, we will give the concept of $n\alpha$ -cyclically monotone

Definition 4.2. Let $\alpha, \beta : C \times C \rightarrow \mathbb{R} \cup \{+\infty\}$ be two bifunctions. Then β is called $n\alpha$ -cyclically monotone (for short C_n^α -monotone), if

$$(4.2) \quad 2 \sum_{i=1}^{i=n} \beta(x_{i+1}, x_i) + \sum_{i=1}^{i=n} \alpha(x_{i+1}, x_i) \leq 0.$$

for any cyclic $x_1, x_2, \dots, x_{n+1} = x_1$.

β is called C^α -monotone if β is C_n^α -monotone for each $n \geq 2$.

Remark 4. If α is symmetric, one can check that

(1) If β is C_2^α -monotone bifunction, then β is α -monotone bifunction.

(2) Define $\frac{\alpha(y, x)}{2} = \beta_1(y, x) - \beta(y, x)$, then

(a) β is α -monotone bifunction $\iff \beta_1$ is monotone bifunction.

(b) β is C^α -monotone bifunction $\iff \beta_1$ is cyclically monotone bifunction.

What follows is a theorem that is considered extension of Proposition 2.1.

Theorem 4.1. Assume that $\alpha, \beta : C \times C \rightarrow \mathbb{R}$ are two bifunctions, where α is symmetric. Then β is C^α -monotone iff there exists a function $f : C \rightarrow \mathbb{R}$ such that

$$(4.3) \quad \beta(y, x) + f(y) + \frac{1}{2}\alpha(y, x) \leq f(x) (\forall x, y \in C).$$

Proof. Assume that $\frac{\alpha(y, x)}{2} = \beta_1(y, x) - \beta(y, x)$. According to Remark 4, β is C^α -monotone, iff β_1 is a cyclically monotone. Hence, from Proposition 2.1, there exists a function $f : C \rightarrow \mathbb{R}$, such that $\beta_1(y, x) \leq f(x) - f(y) \forall x, y \in C$. Then

$$(4.4) \quad \beta(y, x) + f(y) + \frac{1}{2}\alpha(y, x) \leq f(x).$$

Conversely, if 4.4 holds, also by the assumption, then

$$\beta_1(y, x) \leq f(x) - f(y).$$

Proposition 2.1 and Remark 4 imply that β is C^α -monotone bifunction. \square

What we are going to do in the next part of our study is to generalize the recursion formula for Fitzpatrick transform of order n by using two bifunctions.

Definition 4.3. Let $\alpha, \beta : C \times C \rightarrow \mathbb{R} \cup +\{\infty\}$ be two bifunctions. The transform of β and α of order $n \in \{2, 3, \dots\}$ at $(x, x^*) \in E \times E^*$ is defined by recursion formula as follows

$$(4.5) \quad \psi_{\beta,n}(x, x^*) = \sup_{y \in E} \left[\psi_{\beta,n-1}(y, x^*) + \beta(x, y) + \frac{\alpha(x, y)}{2} \right]$$

such that

$$\psi_{\beta,1}(x, x^*) = \langle x^*, x \rangle.$$

The Fitzpatrick function of infinite order is defined by

$$(4.6) \quad \psi_{\beta,\infty}(x, x^*) = \sup_{n \geq 2} \psi_{\beta,n}.$$

Here, we shall generalize the recursion formula in 4.5.

$$(4.7) \quad \begin{aligned} \psi_{\beta,2}(x, x^*) &= \sup_{y \in E} \left[\psi_{\beta,1}(y, x^*) + \beta(x, y) + \frac{1}{2}\alpha(x, y) \right] \\ &= \sup_{y \in E} \left[\langle x^*, y \rangle + \beta(x, y) + \frac{1}{2}\alpha(x, y) \right]. \end{aligned}$$

$$(4.8) \quad \psi_{\beta,3}(x, x^*) = \sup_{y \in E} \left[\psi_{\beta,2}(y, x^*) + \beta(x, y) + \frac{1}{2}\alpha(x, y) \right].$$

Substituting in (4.8) y by z , and x by y , then

$$(4.9) \quad \psi_{\beta,3}(x, x^*) = \sup_{y \in E, z \in E} \left[\langle x^*, z \rangle + \beta(y, z) + \frac{1}{2}\alpha(y, z) + \beta(x, y) + \frac{1}{2}\alpha(x, y) \right].$$

Substituting in (4.9) if z by x_1 and y by x_2 , then

$$(4.10) \quad \psi_{\beta,3}(x, x^*) = \sup_{x_1, x_2 \in E} \left[\langle x^*, x_1 \rangle + \sum_{i=1}^{i=2} \beta(x_{i+1}, x_i) + \frac{1}{2} \sum_{i=1}^{i=2} \alpha(x_{i+1}, x_i) \right].$$

In continuing with the transformation of Fitzpatrick by using the same way above and by using the mathematical induction, we can do the Fitzpatrick transformation of α and β of order n as:

$$(4.11) \quad \psi_{\beta,n}(x, x^*) = \sup_{x_1, \dots, x_n \in E} \left[\langle x^*, x_1 \rangle + \sum_{i=1}^{n-1} \beta(x_{i+1}, x_i) + \frac{1}{2} \sum_{i=1}^{n-1} \alpha(x_{i+1}, x_i) \right].$$

From what has been presented above, the main result of this section is given below.

Theorem 4.2. Let $\alpha, \beta : E \times E \rightarrow \mathbb{R} \cup \{+\infty\}$ be two bifunctions, where $\alpha(x_1, x) \geq 0$ for each $x_1 \in C$. Then

$$\psi_{\beta,n}(x, x^*) = \langle x^*, x \rangle \quad \forall (x, x^*) \in G(B_{\alpha,\beta}).$$

If β is C_n^α -monotone for each $n \geq 2$.

Proof. Assume that $(x, x^*) \in G(B_{\alpha,\beta})$, and β be C_n^α -monotone. Then $\forall x_1, \dots, x_{n-1} \in \text{dom}\beta, x \in \text{dom}(B_{\alpha,\beta})$

$$(4.12) \quad \sum_{i=1}^{n-1} \beta(x_{i+1}, x_i) + \beta(x_1, x) + \frac{1}{2} \sum_{i=1}^n \alpha(x_{i+1}, x_i) \leq 0.$$

Since $(x, x^*) \in G(B_{\alpha,\beta})$,

$$\beta(x_1, x) + \langle x^*, x \rangle \geq \langle x^*, x_1 \rangle + \frac{1}{2} \alpha(x_1, x).$$

So,

$$(4.13) \quad \langle x^*, x \rangle \geq \langle x^*, x_1 \rangle - \beta(x_1, x).$$

Adding (4.12) to (4.13), will result in

$$(4.14) \quad \sum_{i=1}^{n-1} \beta(x_{i+1}, x_i) + \langle x^*, x_1 \rangle + \frac{1}{2} \sum_{i=1}^{n-1} \alpha(x_{i+1}, x_i) + \frac{1}{2} \alpha(x_1, x) \leq \langle x^*, x \rangle.$$

By taking supremum on $x_1, \dots, x_{n-1} \in \text{dom}\beta$ in (4.14), one can obtain

$$\psi_{\beta,n}(x, x^*) \leq \langle x^*, x \rangle \quad (\forall (x, x^*) \in G(B_{\alpha,\beta})).$$

Conversely, by $\alpha(x_1, x) \geq 0$, and from (3.1) one can get

$$(4.15) \quad \beta(x, x) \geq \frac{1}{2}\alpha(x, x) \geq 0.$$

By applied (4.15) in (4.5) one can obtain

$$\psi_{\beta,n}(x, x^*) \geq \psi_{\beta,n_1}(x, x^*) \geq \cdots \geq \psi_{\beta,1}(x, x^*) = \langle x^*, x \rangle.$$

□

Corollary 4.1. The converse of Theorem 4.2 holds, if

$$\beta(x_1, x) \leq \langle x^*, x_1 - x \rangle \quad (\forall x_1 \in C).$$

Proof.

$$\begin{aligned} & \sum_{i=1}^n \beta(x_{i+1}, x_i) + \frac{1}{2} \sum_{i=1}^n \alpha(x_{i+1}, x_i) \\ = & \sum_{i=1}^{n-1} \beta(x_{i+1}, x_i) + \beta(x_1, x) + \frac{1}{2} \sum_{i=1}^{n-1} \alpha(x_{i+1}, x_i) + \frac{1}{2} \alpha(x_1, x) \\ \leq & \sum_{i=1}^{n-1} \beta(x_{i+1}, x_i) + \langle x^*, x_1 \rangle + \frac{1}{2} \sum_{i=1}^{n-1} \alpha(x_{i+1}, x_i) - \langle x^*, x \rangle \\ \leq & \psi_{\beta,n}(x, x^*) - \langle x^*, x \rangle \\ = & 0. \end{aligned}$$

Hence, β is C_n^α - monotone bifunction. □

Corollary 4.2. Let $\alpha, \beta : E \times E \rightarrow \mathbb{R} \cup \{+\infty\}$ are two bifunctions, and $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function. If the following hold

$$(i) \quad \alpha(x, y) = \alpha(y, x) \geq 0,$$

$$(ii) \quad \beta(x, y) \geq f(x) - f(y),$$

then

$$\psi_{\beta,n}(x, x^*) \geq f_{\alpha^*}(x^*, x) + f(x).$$

Proof. We will use the recursion formula in Definition 4.3 and induction on n . In case $n = 2$ is proved in Lemma 4.1. We suppose that the result satisfies when $n = m$, i.e., for all $(x, x^*) \in E \times E^*$

$$\psi_{\beta,m}(x, x^*) \geq f_{\alpha}^*(x^*, x) + f(x).$$

To prove for $n = m + 1$, from Definition 4.3, one can have

$$\begin{aligned} \psi_{\beta,m+1}(x, x^*) &\geq \psi_{\beta,m}(y, x^*) + \beta(x, y) + \frac{1}{2}\alpha(x, y) \\ &\geq f(y) + f_{\alpha}^*(x^*, y) + f(x) - f(y) + \frac{1}{2}\alpha(x, y) \\ &\geq f_{\alpha}^*(x^*, y) + f(x) \quad \forall y \in C. \end{aligned}$$

Then

$$\psi_{\beta,n}(x, x^*) \geq f_{\alpha}^*(x^*, x) + f(x).$$

□

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