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UNIQUENESS RESULTS RELATED TO POLYNOMIAL AND DIFFERENTIAL POLYNOMIAL

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Abstract. In this paper we study the problem of uniqueness of meromorphic functions concerning polynomial and differential polynomial with finite weight. Our result generalizes the results proved by Q. Zhang in 2005 and also results obtained by Harina P. Waghamore and Husna V. in 2016.

1. Introduction and Definitions

In this paper, by meromorphic functions we will always mean meromoprphic functions in complex plane. Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a, f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity k is counted k times in the set. If these zeros points are only counted once, then we denote the set by $\overline{E}(a, f)$. Let f and g be two nonconstant meromorphic functions. If $E(a, f) = E(a, g)$, then we say that f and g share the value a CM, if $\overline{E}(a, f) = \overline{E}(a, g)$, then we say that f and g share the value a IM.

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Definition 1.1[5]**.** Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k. We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$ possibly outside a set r of finite linear measure. A meromorphic function a is said to be a small function of f, if $T(r, a) = S(r, f)$ as $r \longrightarrow \infty$. Also it is known to us that the hyper order of f, denoted by $\rho_1(f)$, is defined by

$$\rho_1(f) = \limsup_{r \to \infty} \frac{loglogT(r, f)}{logr}.$$

For any constant a, we denote by $N_{k)}(r, \frac{1}{f-a})$ the counting function for zeros of $f - a$ with multiplicity no more than k, and by $\overline{N}_{k)}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, \frac{1}{f-a})$ be the counting function for zeros of $f - a$ with multiplicity at least k and $\overline{N}_{(k}(r, \frac{1}{f-a})$ be the corresponding one for which multiplicity is not counted. Set
$N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}(r, \frac{1}{f-a}) + ... + \overline{N}_{(k}(r, \frac{1}{f-a})$.
We define

$$\Theta(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)} \quad , \quad \delta(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)},$$

We further define

$$\delta_k(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N_k(r, \frac{1}{f-a})}{T(r, f)}$$

Clearly,

$$0 \leq \delta(a,f) \leq \delta_k(a,f) \leq \delta_{k-1}(a,f)... \leq \delta_2(a,f) \leq \delta_1(a,f) = \Theta(a,f).$$

Definition 1.2 [4] Let $n_{0j}, n_{1j}, ..., n_{kj}$ be nonnegative integers. The expression $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}}...(f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$. The sum

$$P[f] = \sum_{j=1}^{t} b_j M_j[f],$$

is called a differential polynomial generated by f of degree $\bar{d}(P) = max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_p = max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, where $T(r,b_j) = S(r,f)$ for $j = 1,2,...t$.

The number $\underline{d}(P) = min\{d(M_j) : 1 \leq j \leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\bar{d}(P) = \underline{d}(P)$. Moreover, $P[f]$ is called a linear differential polynomial genetrated by f if $\bar{d}(P) = 1$. Otherwise, $P[f]$ is called a non-linear differential polynomial.

We denote by $Q = max\{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = max\{n_{1j} + 2n_{2j} + ... + kn_{kj} : 1 \leq j \leq t\}$.

R. Brück [3] first considered the uniqueness problems of an entire function sharing one value with its derivative and proved the following result.

Theorem 1.1. [3] *Let f be an entire function which is not constant. If f and f' share the value 1 CM and if $N(r, \frac{1}{f'}) = S(r,f)$, then $\frac{f'-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.*

R. Brück [3] further posed the following conjecture.

Conjecture A. *Let f be a entire function which is not constant, $\rho_1(f)$ be the first iterated order of f. If $\rho_1(f) < +\infty$ and $\rho_1(f)$ is not a positive integer and if f and f' share one value a CM, then $\frac{f'-a}{f-a} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.*

Yang [8] proved that the conjecture is true if f is an entire function of finite order. Zhang [12] extended Theorem 1.1 meromorphic functions. Yu [11] recently considered the problem of an entire or meromorphic function sharing one small function with its derivative and proved the following two theorems.

Theorem 1.2. [11] *Let f be a non-constant entire function and $a \equiv a(z)$ be a meromorphic function such that $a \not\equiv 0, \infty$ and $T(r,a) = o(T(r,f))$as $r \to \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0,f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.*

Theorem 1.3. [11] *Let f be a non-constant, non-entire meromorphic function and $a \equiv a(z)$ be a meromorphic function such that $a \not\equiv 0, \infty$ and $T(r,a) = o(T(r,f))$ as $r \to +\infty$. If*

(i) f and a have no common poles,

(ii) $f - a$ and $f^{(k)} - a$ share the value 0 CM,

(iii) $4\delta(0,f) + 2\Theta(\infty,f) > 19 + 2k$,

then $f \equiv f^{(k)}$, where k is a positive integer.

Nowadays, the idea of weighted sharing is being used immensely for further investigation of the Bruck's result(see[6],[11])[12]). Lahiri [6] obtained the following result which is an improvement of Theorem 1.3.

Theorem 1.4. [6] *Let* f *be a non-constant meromorphic function and* k *be a positive integer. Also let* $a \equiv a(z)(\not\equiv 0, \infty)$ *be a meromorphic function such that* $T(r,a) = S(r,f)$. *If*

(i) a *has no zero(pole) which is also a zero(pole) of* f *or* $f^{(k)}$ *with the same multiplicity.*

(ii) $f - a$ *and* $f^{(k)} - a$ *share (0,2) CM,*

(iii) $2\delta_{2+k}(0,f) + (4+k)\Theta(\infty, f) > 5 + k$,

then $f \equiv f^{(k)}$.

In 2005, Zhang [13] improved the above result and proved the following theorem.

Theorem 1.5. [13] *Let* f *be a non-constant meromorphic function and* $k(\geq 1), l(\geq 0)$ *be integers. Also let* $a \equiv a(z)(\not\equiv 0, \infty)$ *be a meromorphic function such that* $T(r,a) = S(r,f)$. *Suppose that* $f - a$ *and* $f^{(k)} - a$ *share* $(0, l)$. *If* $l \geq 2$ *and*

$$(3+k)\Theta(\infty, f) + 2\delta_{2+k}(0,f) > k + 4,$$

or $l = 1$ *and*

$$(4+k)\Theta(\infty, f) + 3\delta_{2+k}(0,f) > k + 6,$$

or $l = 0$, *i.e.,* $f - a$ *and* $f^{(k)} - a$ *share the value 0 IM and*

$$(6+2k)\Theta(\infty, f) + 5\delta_{2+k}(0,f) > 2k + 10,$$

then $f \equiv f^{(k)}$.

Recently Harina P.W. and Husna V.[7] extend the result of Q. Zhang[13] and proved the following results.

Theorem 1.6. [7] *Let f be a non-constant meromorphic function, $k(\geq 1)$, $n(\geq 1)$, $m(\geq 2)$, $l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$. If*
$l \geq 2$ and

$$(2k+3)\Theta(\infty, f) + \delta_2(0, f) + 2\delta_{1+k}(0, f) > 2k + 6 - n,$$

$l = 1$ and

$$(2k + \frac{7}{2})\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \delta_2(0, f) + 2\delta_{1+k}(0, f) > 2k + 7 - n,$$

or $l = 0$ and

$$(3k+6)\Theta(\infty, f) + 2\Theta(0, f) + \delta_2(0, f) + 3\delta_{1+k}(0, f) > 3k + 12 - n.$$

Then $f^n \equiv (f^{(k)})^m$.

Motivated by such uniqueness investigations, it is natural to consider the problem in a more general setting: Let f be a non-constant meromorphic function, $P[f]$ be a non-constant differential polynomial of f, $p(z)$ be a polynomial of degree $n \geq 1$, and $a(\not\equiv 0, \infty)$ be a meromorphic function satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$. If $p(f)$ and $P[f]$ share (a, l), $l \geq 0$, then is it true that $p(f) \equiv P[f]$?

Generally this is not true, but under certain essential conditions, we prove the following result:

2. Main results

Theorem 2.1. *Let f be a non-constant meromorphic function, and $a(\not\equiv 0, \infty)$ be a meromorphic function satisfying $T(r, a) = S(r, f)$, as $r \to \infty$, and $p(z)$ be a polynomial of degree $n \geq 1$ with $p(0) = 0$. Let $P[f]$ be a nonconstant differential polynomial* *of f. Suppose $p(f)$ and $P[f]$ share (a,l), with one of the following conditions :*

(i) $l\geq 2$ *and*

(2.1) $(2Q+3)\Theta(\infty,f)+2n\Theta(0,p(f))+2\overline{d}(p)\delta(0,f)>2Q+3+4\overline{d}(p)-2\underline{d}(p)+n,$

(ii) $l=1$ *and*

(2.2) $(2Q+4)\Theta(\infty,f)+4n\Theta(0,p(f))+2\overline{d}(p)\delta(0,f)>2Q+4+4\overline{d}(p)-2\underline{d}(p)+3n,$

(iii) $l=0$ *and*

(2.3) $(4Q+6)\Theta(\infty,f)+6n\Theta(0,p(f))+4\overline{d}(p)\delta(0,f)>4Q+6+8\overline{d}(p)-4\underline{d}(p)+5n,$

then $p(f)\equiv P[f]$.

Example 2.1. *Consider the function $f(z)=sin\alpha z+1-\frac{1}{\alpha^4}$, where $\alpha\neq 0,\pm1,\pm i$ and $p(z)=z$. Then $p(f)$ and $P[f]\equiv f^{(iv)}$ share $(1,l)$, $l\geq 0$ and none of the inequalities* (2.1) (2.2) *and* (2.3) *is satisfied, and $p(f)\neq P[f]$. Thus the conditions in Theorem 2.1 cannot be removed.*

Remark 2.1. *Theorem 2.1 generalizes Theorem 1.5 and Theorem 1.6.*

3. Lemmas

Lemma 3.1. [4] *Let f be a non-constant meromorphic function, k be a positive integer,then*

$$N_p(r,\frac{1}{f^{(k)}})\leq N_{p+k}(r,\frac{1}{f})+k\overline{N}(r,f)+S(r,f).$$

Lemma 3.2. [11] *Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial of f. Then*

$$m\left(r, \frac{P[f]}{f^{\overline{d}(p)}}\right) \leq \left(\overline{d}(p) - \underline{d}(p)\right) m\left(r, \frac{1}{f}\right) + S(r, f), \tag{3.1}$$

$$N\left(r, \frac{P[f]}{f^{\overline{d}(p)}}\right) \leq \left(\overline{d}(p) - \underline{d}(p)\right) N\left(r, \frac{1}{f}\right) + Q\left[\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + S(r, f), \tag{3.2}$$

$$N\left(r, \frac{1}{P[f]}\right) \leq Q\overline{N}(r, f) + \left(\overline{d}(p) - \underline{d}(p)\right) m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{\overline{d}(p)}}\right) + S(r, f), \tag{3.3}$$

where $Q = max_{1\leq i\leq m}\{n_{i0} + n_{i1} + 2n_{i2} + ... + kn_{ik}\}$.

Lemma 3.3. [1] *Let f and g be two non-constant meromorphic functions.*

(i) If f and g share $(1, 0)$, then

$$\overline{N}_L\left(r, \frac{1}{f-1}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r), \tag{3.4}$$

where $S(r) = o(T(r))$ as $r \to \infty$ with $T(r) = max\{T(r, f); T(r, g)\}$.

(ii) If f and g share $(1, 1)$, then

$$\begin{aligned} 2\overline{N}_L\left(r, \frac{1}{f-1}\right) + 2\overline{N}_L\left(r, \frac{1}{g-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{f-1}\right) - \overline{N}_{f>2}\left(r, \frac{1}{g-1}\right) \\ \leq N(r, \frac{1}{g-1}) - \overline{N}(r, \frac{1}{g-1}). \end{aligned} \tag{3.5}$$

4. **Proof of Theorem 2.1**

Proof. Let $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + ... + a_1 z$, where $a_1, a_2, ..., a_{n-1}$ are constants, $F = \frac{p(f)}{a}$ and $G = \frac{P[f]}{a}$. Then

$$F - 1 = \frac{p(f) - a}{a} \text{ and } G - 1 = \frac{P[f] - a}{a} \tag{4.1}$$

Since $p(f)$ and $P[f]$ share (a,l), it follows that F and G share $(1,l)$ except at the zeros and poles of $a(z)$. Also note that

$$\overline{N}(r,F)=\overline{N}(r,f)+S(r,f) \quad and \quad \overline{N}(r,G)=\overline{N}(r,f)+S(r,f). \tag{4.2}$$

Define

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{4.3}$$

Case 1. $H\not\equiv 0$, From (4.3) it is easy to see that $m(r,H)=S(r,f)$.

Subcase 1.1. $l\geq 1$. From (4.3) we have

$$\begin{aligned} N(r,H)\leq{}& \overline{N}(r,F)+\overline{N}_{(l+1}(r,\frac{1}{F-1})+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G}) \\ &+\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+\overline{N}(r,a)+\overline{N}(r,\frac{1}{a}), \end{aligned} \tag{4.4}$$

where $N_0(r,\frac{1}{F'})$ denotes the counting function of the zeros of F' which are not the zeros of F and $F-1$, and $\overline{N}_0(r,\frac{1}{F'})$ denotes its reduced form. In the same way, we can define $N_0(r,\frac{1}{G'})$ and $\overline{N}_0(r,\frac{1}{G'})$. Let z_0 be a simple zero of $F-1$ but $a(z_0)\neq 0,\infty$, then z_0 is also the simple zero of $G-1$. By calculating z_0 is the zero of H, So

$$N_{1)}(r,\frac{1}{F-1})\leq N(r,\frac{1}{H})+N(r,a)+N(r,\frac{1}{a})\leq N(r,H)+S(r,f) \tag{4.5}$$

Noticing that $N_{1)}(r,\frac{1}{G})=N_{1)}(r,\frac{1}{F})+S(r,f)$.

$$\begin{aligned} \overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1})&=N_{1)}(r,\frac{1}{F-1})+\overline{N}_{(2}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1}) \\ &\leq \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_{(l+1}(r,\frac{1}{G-1}) \\ &+\overline{N}_{(2}(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})+\overline{N}_0(r,\frac{1}{F'}) \\ &+\overline{N}_0(r,\frac{1}{G'})+S(r,f) \end{aligned}$$

While $l \geq 2$,

$$\overline{N}_{(l+1}(r, \frac{1}{G-1}) + \overline{N}_{(2}(r, \frac{1}{G-1}) + \overline{N}(r, \frac{1}{G-1}) \leq N(r, \frac{1}{G-1}) \leq T(r, G) + O(1),$$

so,

$$\begin{aligned}\overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) &\leq \overline{N}(r, F) + \overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{G}) \\ &+ \overline{N}_0(r, \frac{1}{F'}) + \overline{N}_0(r, \frac{1}{G'}) + T(r, G) + S(r, f).\end{aligned}$$

By the second fundamental theorem and from (3.3) we have

$$\begin{aligned}T(r, F) + T(r, G) &\leq \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{F-1}) \\ &+ \overline{N}(r, \frac{1}{G-1}) - N_0(r, \frac{1}{F'}) - N_0(r, \frac{1}{G'}) + S(r, F) + S(r, G) \\ &\leq 3\overline{N}(r, F) + N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + T(r, G) + S(r, f),\end{aligned}$$

$$\begin{aligned}&so, \;\; T(r, F) \leq 3\overline{N}(r, F) + N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + S(r, f), \\ &i.e, \;\; T(r, f) \leq 3\overline{N}(r, f) + N_2(r, \frac{1}{p(f)}) + N_2(r, \frac{1}{P[f]}) + S(r, f)\end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned}T(r, f) &\leq 3\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{p(f)}) + 2\{Q\overline{N}(r, f) + (\overline{d}(P) - \underline{d}(P))m(r, \frac{1}{f}) + \overline{d}(P)N(r, \frac{1}{f})\} \\ &+ S(r, f) \\ &\leq (2Q+3)\{1 - \Theta(\infty, f)\} + 2n\{1 - \Theta(0, p(f))\} + 2\overline{d}(P)\{1 - \delta(0, f)\}T(r, f) \\ &+ 2(\overline{d}(P) - \underline{d}(P))T(r, f) + S(r, f).\end{aligned}$$

That is,

$$nT(r,f) = T(r,F) + S(r,f)$$
$$\leq (2Q+3)\{1-\Theta(\infty,f)\} + 2n\{1-\Theta(0,p(f))\} + 2\overline{d}(P)\{1-\delta(0,f)\}T(r,f)$$
$$+ 2(\overline{d}(P) - \underline{d}(P))T(r,f) + S(r,f)$$

which yields that

$$[\{(2Q+3)\Theta(\infty,f) + 2n\Theta(0,p(f)) + 2\overline{d}(P)\delta(0,f)\} - \{2Q+3+n+4\overline{d}(P) - 2\underline{d}(P)\}]T(r,f)$$
$$\leq S(r,f)$$

That is,

$$[(2Q+3)\Theta(\infty,f) + 2n\Theta(0,p(f)) + 2\overline{d}(P)\delta(0,f)] \leq 2Q+3+4\overline{d}(P) - 2\underline{d}(P) + n$$

which contradicts (2.1).

While $l = 1$,

$$\overline{N}_{(l+1}(r,\frac{1}{G-1}) + \overline{N}(r,\frac{1}{G-1}) \leq N(r,\frac{1}{G-1}) \leq T(r,G) + O(1),$$

so by Lemma 3.1 we have

$$\overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) \leq \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_{(2}(r,\frac{1}{F-1}) + \overline{N}_0(r,\frac{1}{F'})$$
$$+ \overline{N}_0(r,\frac{1}{G'}) + T(r,G) + S(r,f).$$
$$\leq \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{F'}) + \overline{N}_0(r,\frac{1}{G'}) + T(r,G) + S(r,f)$$
$$\leq 2\overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{G}) + N_2(r,\frac{1}{F}) + \overline{N}_0(r,\frac{1}{G'}) + T(r,G) + S(r,f)$$

By the second fundamental theorem and from (3.3) we have

$$T(r,F) + T(r,G) \leq 4\overline{N}(r,F) + 2N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + T(r,G) + S(r,f),$$

so

$$T(r,F) \le 4\overline{N}(r,F) + 2N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + S(r,f),$$

i.e.,

$$\begin{aligned}T(r,F) &\le 4\overline{N}(r,f) + 2N_2(r,\frac{1}{p(f)}) + N_2(r,\frac{1}{P[f]}) + S(r,f),\\ &\le 4\overline{N}(r,f) + 4\overline{N}(r,\frac{1}{p(f)}) + 2\overline{N}(r,\frac{1}{P[f]}) + S(r,f)\\ &\le 4\overline{N}(r,f) + 4\overline{N}(r,\frac{1}{p(f)}) + 2\{Q\overline{N}(r,f) + (\overline{d}(P) - \underline{d}(P))m(r,\frac{1}{f})\\ &+ \overline{d}(P)N(r,\frac{1}{f})\} + S(r,f)\\ &\le (2Q+4)\{1-\Theta(\infty,f)\} + 4n\{1-\Theta(0,p(f))\} + 2\overline{d}(P)\{1-\delta(0,f)\}T(r,f)\\ &+ 2(\overline{d}(P) - \underline{d}(P))T(r,f) + S(r,f).\end{aligned}$$

That is

$$nT(r,f) = T(r,F) + S(r,f)$$

$$\begin{aligned}nT(r,f) &\le (2Q+4)\{1-\Theta(\infty,f)\} + 4n\{1-\Theta(0,p(f))\} + 2\overline{d}(P)\{1-\delta(0,f)\}T(r,f)\\ &+ 2(\overline{d}(P) - \underline{d}(P))T(r,f) + S(r,f)\end{aligned}$$

which implies that

$$\{(2Q+4)\Theta(\infty,f) + 4n\Theta(0,p(f)) + 2\overline{d}(P)\delta(0,f)\} - \{2Q+4+4\overline{d}(P) - 2\underline{d}(P)) + 3n\}$$
$$T(r,f) \le S(r,f)$$

That is,

$$\{(2Q+4)\Theta(\infty,f) + 4n\Theta(0,p(f)) + 2\overline{d}(P)\delta(0,f)\} \le 2Q+4+4\overline{d}(P) - 2\underline{d}(P)) + 3n,$$

this contradicts (2.2).

Subcase 1.2. $l = 0$. In this case, F and G share 1 IM except the zeros and poles of $a(z)$. Let z_0 be the zero of $F-1$ with multiplicity p and the zero of $G-1$ with multiplicity q.

We denote by $N_E^{1)}(r,\frac{1}{F})$ the counting function of the zeros of $F-1$ where $p=q=1$; by $\overline{N}_E^{(2}(r,\frac{1}{F})$ the counting function of the zeros of $F-1$ where $p=q\geq 2$; by $\overline{N}_L(r,\frac{1}{F})$ the counting function of the zeros of $F-1$ where $p>q\geq 1$, each point in these counting functions is counted only once. In the same way, we can define $N_E^{1)}(r,\frac{1}{G})$,$\overline{N}_E^{(2}(r,\frac{1}{G})$ and $\overline{N}_L(r,\frac{1}{G})$. It is easy to see that

$$N_E^{1)}(r,\frac{1}{F-1})=N_E^{1)}(r,\frac{1}{G-1})+S(r,f),$$

$$\overline{N}_E^{(2}(r,\frac{1}{F-1})=\overline{N}_E^{(2}(r,\frac{1}{G-1})+S(r,f),$$

$$\begin{aligned}\overline{N}(r,\frac{1}{F-1})&=\overline{N}(r,\frac{1}{G-1})+S(r,f)\\ &=N_E^{1)}(r,\frac{1}{F-1})+\overline{N}_E^{(2}(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{F-1})\\ &\quad+\overline{N}_L(r,\frac{1}{G-1})+S(r,f).\end{aligned}\tag{4.6}$$

From (4.3) we have now

$$\begin{aligned}N(r,H)\leq{}&\overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_L(r,\frac{1}{F-1})\\ &+\overline{N}_L(r,\frac{1}{G-1})+\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+S(r,f).\end{aligned}\tag{4.7}$$

In this case, (4.5) is replaced by

$$N_E^{1)}(r,\frac{1}{F-1})\leq N(r,H)+S(r,f).\tag{4.8}$$

From (4.6),(4.7) and (4.8) and Lemma 3.1 for $p=1$, $k=1$, noticing

$$N_E^{(2}(r,\frac{1}{G-1})+\overline{N}_L(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})\leq N(r,\frac{1}{G-1})\leq T(r,G)+S(r,f),$$

then

$$\begin{aligned}\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1}) &= N_E^{1)}(r,\frac{1}{F-1})+\overline{N}_E^{(2}(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{G-1})\\ &+\overline{N}(r,\frac{1}{G-1})\\ &\leq \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+2\overline{N}_L(r,\frac{1}{F-1})\\ &+\overline{N}_L(r,\frac{1}{G-1})+\overline{N}_E^{(2}(r,\frac{1}{G-1})+\overline{N}_L(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})\\ &+\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+S(r,f)\\ &\leq \overline{N}(r,F)+2\overline{N}(r,\frac{1}{F'})+\overline{N}(r,\frac{1}{G'})+T(r,G)+S(r,f)\\ &\leq 4\overline{N}(r,F)+2N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+T(r,G)+S(r,f).\end{aligned}$$

By the second fundamental theorem and from (3.3) we can obtain

$$T(r,F)+T(r,G)\leq 6\overline{N}(r,F)+3N_2(r,\frac{1}{F})+2N_2(r,\frac{1}{G})+T(r,G)+S(r,f),$$

so

$$\begin{aligned}T(r,F) &\leq 6\overline{N}(r,F)+3N_2(r,\frac{1}{F})+2N_2(r,\frac{1}{G})+S(r,f),\\ &\leq 6\overline{N}(r,f)+6\overline{N}(r,\frac{1}{p(f)})+4N(r,\frac{1}{P[f]})+S(r,f)\\ &\leq 6\overline{N}(r,f)+6\overline{N}(r,\frac{1}{p(f)})+4\{Q\overline{N}(r,f)+(\overline{d}(P)-\underline{d}(P))m(r,\frac{1}{f})+N(r,\frac{1}{f^{\overline{d}(P)}})\}\\ &+S(r,f)\\ &\leq (4Q+6)\{1-\Theta(\infty,f)\}+6n\{1-\Theta(0,f)\}+4(\overline{d}(P)-\underline{d}(P))T(r,f)\\ &+4\overline{d}(P)\{1-\delta(0,f)\}+S(r,f).\end{aligned}$$

That is,

$$\begin{aligned} nT(r,f) &= T(r,F) + S(r,f) \\ &\le (4Q+6)\{1-\Theta(\infty,f)\} + 6n\{1-\Theta(0,p(f))\} + 4(\overline{d}(P) - \underline{d}(P))T(r,f) \\ &+ 4\overline{d}(P)(1-\delta(0,f)) + S(r,f) \end{aligned}$$

which implies that

$$\{(4Q+6)\Theta(\infty,f) + 6n\Theta(0,p(f)) + 4\overline{d}(P)\delta(0,f)\} - \{4Q+6+8\overline{d}(P) - 4\underline{d}(P)) + 5n\} \le S(r,f).$$

That is,

$$(4Q+6)\Theta(\infty,f) + 6n\Theta(0,p(f)) + 4\overline{d}(P)\delta(0,f) \le 4Q+6+8\overline{d}(P) - 4\underline{d}(P)) + 5n$$

which contradicts (2.3).

Case 2. Let $H \equiv 0$.

On integration we get from (4.3)

$$\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \tag{4.9}$$

where C, D are constants and $C \neq 0$. Here, the following two cases arise:

Subcase 2.1. Suppose $D \neq 0, -1$. Rewriting (4.9) as

$$\frac{G-1}{C} = \frac{F-1}{D+1-DF}$$

we have

$$\overline{N}(r,G) = \overline{N}(r, \frac{1}{F - \frac{(D+1)}{D}}).$$

In this case, the second fundamental theorem of Nevanlinna yields

$$\begin{aligned} nT(r,f) &= T(r,F) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-\frac{(D+1)}{D}}) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,G) + S(r,f) \\ &\leq 2\overline{N}(r,f) + \overline{N}(r,\frac{1}{p(f)}) + S(r,f) \\ &= [2\{1-\Theta(\infty,f)\} + n\{1-\Theta(0,p(f))\}]T(r,f) + S(r,f). \end{aligned}$$

$$\Longrightarrow \qquad 2\Theta(\infty,f) + n\Theta(0,p(f)) \leq 2,$$

which contradicts (2.1),(2.2) and (2.3).

Subcase 2.2. When $D = 0$. Then from (4.9) we have

$$G = CF - (C-1). \tag{4.10}$$

Therefore, if $C \neq 1$, then

$$\overline{N}(r,\frac{1}{G}) = \overline{N}(r,\frac{1}{F-\frac{(C-1)}{C}}).$$

By the second fundamental theorem and (3.3) gives

$$
\begin{aligned}
nT(r,f) &= T(r,F) + S(r,f)\\
&\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-\frac{(C-1)}{C}}) + S(r,f)\\
&\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + S(r,f)\\
&\leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{p(f)}) + \overline{N}(r,\frac{1}{P[f]}) + S(r,f)\\
&\leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{p(f)}) + Q\overline{N}(r,f) + (\overline{d}(P)-\underline{d}(P))m(r,\frac{1}{f}) + N(r,\frac{1}{f^{\overline{d}(P)}}) + S(r,f)\\
&\leq (Q+1)\overline{N}(r,f) + \overline{N}(r,\frac{1}{p(f)}) + (\overline{d}(P)-\underline{d}(P))T(r,f) + \overline{d}(P)N(r,\frac{1}{f}) + S(r,f)\\
&\leq [(Q+1)\{1-\Theta(\infty,f)\} + n\{1-\Theta(0,p(f))\} + \overline{d}(P)\{1-\delta(0,f)\}]T(r,f)\\
&+ (\overline{d}(P)-\underline{d}(P))T(r,f) + S(r,f).
\end{aligned}
$$

That is,

$$[\{(Q+1)\Theta(\infty,f)+n\Theta(0,p(f))+\overline{d}(P)\delta(0,f)\}-\{Q+1+2\overline{d}(P)-\underline{d}(P)\}]T(r,f) \leq S(r,f),$$

which implies that

$$(Q+1)\Theta(\infty,f) + n\Theta(0,p(f)) + \overline{d}(P)\delta(0,f) \leq Q+1+2\overline{d}(P)-\underline{d}(P),$$

which contradicts (2.1),(2.2) and (2.3).

Thus, $C=1$ and so in this case from (4.10) we obtain $F \equiv G$ and so

$$p(f) \equiv P[f].$$

□

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