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ON PERTURBATION OF FRAMES IN LOCALLY CONVEX SPACES

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ABSTRACT. We present some Paley-Wiener type perturbation results for frames in a real (or complex) complete locally convex separable topological vector space.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a real (or complex) Hilbert space with inner product $\langle .,. \rangle$. A countable sequence $\{f_k\}_{k \in I} \subset \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist numbers $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{k \in I} |\langle f, f_k \rangle|^2 \leqslant B\|f\|^2 \text{ for all } f \in \mathcal{H}.$$

The operator $S : \mathcal{H} \to \mathcal{H}$ given by

$$Sf = \sum_{k \in I} \langle f, f_k \rangle f_k$$

is called the *frame operator* of the frame $\{f_k\}_{k \in I}$. The frame operator S is a positive and invertible operator on $\mathcal{H}$. This gives the *reconstruction formula* for all $f \in \mathcal{H}$:

$$f = SS^{-1}f = \sum_{k \in I} \langle S^{-1}f, f_k \rangle f_k = \sum_{k \in I} \langle f, S^{-1}f_k \rangle f_k. \tag{1.1}$$

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The scalars $\{\langle f, f_k\rangle\}$ are called *frame coefficients* of the vector $f \in \mathcal{H}$. Thus, a frame for a Hilbert space $\mathcal{H}$ allows each element in the space $\mathcal{H}$ to be written as a linear combination of the elements in the frame (not necessarily unique as in case of bases). Duffin and Schaeffer [11] in 1952, introduced frames for Hilbert spaces in the context of non-harmonic Fourier series. Daubechies, Grossmann and Meyer revived frames in [10].

Recently many mathematicians and engineers generalized the frame theory in various directions, see [2, 3, 6, 7, 8] and references therein. The theory of topological algebras itself has undergone considerable development since the appearance of Gelfand's paper [14] on normed algebras. Frames which give series representation of each vector (which is similar to the reconstruction formula in equation (1.1)) in topological algebras were introduced in [24] and further studied in [13].

The perturbation theory is a very important tool in various areas of both pure and applied mathematics. The basic of Paley and Wiener was that, a system that is sufficiently close to an orthonormal system (basis) in a Hilbert space also forms an orthonormal system (basis). Since then, a number of variations and generalizations of perturbations to the atomic decompositions, Hilbert frames, and Banach frames have been studied, see [4, 5, 9, 12, 15, 18]. In this paper, we generalize some Paley-Wiener type perturbation results given in [9, 12, 15, 16, 18, 19] to frames in a sequentially complete locally convex topological vector space over the real or complex field $\mathbb{K}$.

In the rest part of this section, we recall some basic notations, definitions and results about frames in topological vector spaces. The pair $(\mathcal{A}, \tau)$ is said to be a *topological vector space* if $\mathcal{A}$ is a vector space over the complex or the real field $\mathbb{K}$ with a compatible topology τ. A topological vector space with a base of convex neighbourhoods of the origin is called a *locally convex topological vector space.*

Definition 1.1. [24] A countable sequence $\mathcal{F} \equiv \{x_n\} \subset \mathcal{A}$ is a *τ-frame* for $(\mathcal{A}, \tau)$ if there exists a sequence $\{f_n\} \subset \mathcal{A}'$, such that for each $x \in \mathcal{A}$

$$x = \tau\text{-} \lim_{n \to \infty} \sum_{i=1}^{n} f_i(x) x_i \;= \tau\text{-} \sum_{i=1}^{\infty} f_i(x) x_i,$$

where the sequence $\{\sum_{i=1}^{n} f_i(x) x_i\}$ converges in the topology τ of $\mathcal{A}$.

For fundamental properties regarding convergence of sequences in topological vector spaces, we refer [20, 22, 25].

Remark 1. The sequence $\{f_n\} \subset \mathcal{A}'$ is called an *associated sequence of functionals*, which need not be unique. The associated functionals f_n $(n \in \mathbb{N})$ need not be continuous.

Definition 1.2. [24] A τ-frame $\mathcal{F} \equiv \{x_n\}$ for $(\mathcal{A}, \tau)$ is a *τ-Schauder frame* for $\mathcal{A}$ if all associated functionals f_n $(n \in \mathbb{N})$ are τ-continuous.

Remark 2. Bonet et al. [1] studied shrinking and boundedly complete Schauder frames in Fréchet spaces. Frames in Fréchet spaces further studied in [21].

The following example provides existence of a τ-frame in $\mathcal{A}$ which is not a Schauder frame for $\mathcal{A}$. This motivates the study of frames and related properties in locally convex spaces.

Example 1.1. *Let* $\mathcal{A} = \Big\{ \{\xi_j\} \subset \mathbb{C} : \xi_j = 0$ *except for finitely many* $j \Big\}$ *and let* τ *be the topology induced by the metric* $d(x, y) = \sup_{1 \le j < \infty} |\xi_j - \eta_j|,\ x = \{\xi_i\}, y = \{\eta_i\} \in \mathcal{A}$. *Then,* $(\mathcal{A}, \tau)$ *is a locally convex separable topological vector space. Define* $\{x_n\} \subset \mathcal{A}$ *by*

$$x_1 = \chi_1, \quad x_2 = \chi_2 \quad \textit{and} \quad x_n = \chi_2 + \frac{\chi_n}{n},\ n \ge 3,$$

where $\{\chi_n\} \subset \mathcal{A}$ *is sequence of canonical unit vectors , i.e.,* $\chi_n = \delta_{n,m}$ $(n, m \in \mathbb{N})$. *Choose* $\{f_n\} \subset \mathcal{A}'$ *as follows.*

$$f_1(x) = \xi_1,\ f_2(x) = \xi_2 - \sum_{j=3}^{\infty} j\xi_j,\ f_n(x) = n\xi_n,\ n \geq 3\ (x = \{\xi_j\} \in \mathcal{A}).$$

Then, $x = \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i$ *for all* $x \in \mathcal{A}$. *Hence* $\mathcal{F} \equiv \{x_n\}$ *is a* τ*-frame for* $\mathcal{A}$. *One may observe that* $\mathcal{F}$ *is not a Schauder frame for* $\mathcal{A}$.

Lemma 1.1. [23, p. 108] *Let* X *and* Y *be topological vector spaces, where* Y *is Hausdorff. Let* $f, g : X \to Y$ *be continuous maps which agree on a dense subset of* X. *Then* $f \equiv g$.

The following key-lemma gives an isomorphism on a sequentially complete locally convex topological vector space and can be found in [17].

Lemma 1.2. [17, p. 328] *Let* $(\mathcal{A}, \tau)$ *be a sequentially complete locally convex topological vector space and* $S : \mathcal{A} \to \mathcal{A}$ *a linear operator. Let one of the conditions be satisfied, namely,*

(1) *for each* $p \in D_\tau$, *there exists* λ_p, $0 < \lambda_p < 1$ *such that* $p(Sx) \leqslant \lambda_p p(x)$ *for all* $x \in \mathcal{A}$,

or

(2) *there exists* $p_0 \in D_\tau$ *and* δ, $0 < \delta < 1$ *such that* $p_0(Sx) \leqslant \delta p_0(x)$ *for all* $x \in \mathcal{A}$ *and for each* $p \in D_\tau$ *there exists* $k_p > 0$ *with* $p(Sx) \leqslant k_p p_0(x)$ *for all* $x \in \mathcal{A}$.

Then, $R = I - S$ *is a topological isomorphism from* $(\mathcal{A}, \tau)$ *onto itself, where* I *is the identity map of* $\mathcal{A}$.

Throughout this paper, $(\mathcal{A}, \tau)$ denotes a sequentially complete locally convex separable topological vector space, where the topology τ of $\mathcal{A}$ is considered to be Hausdorff. All sequence considered in the paper are countable and indexed by the set of natural numbers $\mathbb{N}$. D_τ denotes the family of all semi-norms generating the topology

τ of $\mathcal{A}$. By $\mathcal{A}^*$ and $\mathcal{A}'$, we denote the topological dual and the algebraic dual of $\mathcal{A}$, respectively. For a set $Z \subset \mathcal{A}$, $[Z]^\tau$ shall denote the τ-closure of the span of Z in $\mathcal{A}$.

2. The Main Results

We start with one of the Paley-Wiener type perturbation result for a τ-frame for $\mathcal{A}$.

Theorem 2.1. *Let $\{x_n\}$ be a finitely linearly independent τ-frame for $(\mathcal{A}, \tau)$ and let $\{y_n\} \subset \mathcal{A}$. For $p \in D_\tau$ and $0 < \lambda_p < 1$, if*

$$p\Big(\sum_{i=1}^{m} \alpha_i(x_i - y_i)\Big) \leqslant \lambda_p p\Big(\sum_{i=1}^{m} \alpha_i x_i\Big), \tag{2.1}$$

for all finite sequences $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of scalars, then $\{y_n\}$ is a τ-frame for $(\mathcal{A}, \tau)$.

Proof. By (2.1), we have

$$(1 - \lambda_p)p\Big(\sum_{i=1}^{m} \alpha_i x_i\Big) \leqslant p\Big(\sum_{i=1}^{m} \alpha_i y_i\Big) \leqslant (1 + \lambda_p)p\Big(\sum_{i=1}^{m} \alpha_i x_i\Big). \tag{2.2}$$

Define a linear operator $R_0 : \operatorname{span}\{x_n\} \to [y_n]$ by

$$R_0\Big(\sum_{i=1}^{m} \beta_i x_i\Big) = \sum_{i=1}^{m} \beta_i y_i.$$

Then, by using (2.2), we have

$$\begin{aligned} p(R_0(x)) &= p\Big(\sum_{i=1}^{m} \alpha_i y_i\Big) \\ &\leqslant (1 + \lambda_p)p\Big(\sum_{i=1}^{m} \alpha_i x_i\Big) \\ &= (1 + \lambda_p)p(x), \ x = \sum_{i=1}^{m} \alpha_i x_i \in \operatorname{span}\{x_j\}. \end{aligned}$$

This gives $p(R_0(x)) \leqslant (1 + \lambda_p)p(x)$ for all $x \in \operatorname{span}\{x_j\}$. Therefore, R_0 is continuous, and by the Hahn-Banach extension theorem (see [22]), R_0 can be continuously extended as a linear map $R : [x_n]^\tau = \mathcal{A} \to [y_n]^\tau \subset \mathcal{A}$ with $Rx_n = y_n$, $n \geqslant 1$. Since $\{x_n\}$ is a τ-frame for $(\mathcal{A}, \tau)$, there exists a sequence $\{f_n\} \subset \mathcal{A}'$ such that

$$x = \tau\text{-} \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i \text{ for each } x \in \mathcal{A}.$$

Let I be the identity map on $\mathcal{A}$. Then, by using (2.1), we compute

$$\begin{aligned} p((I-R)x) &= p(x - Rx) \\ &= p\Big(\tau\text{-} \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i - \tau\text{-} \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)Rx_i\Big) \\ &= p\Big(\tau\text{-} \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)(x_i - y_i)\Big) \\ &\leqslant \lambda_p p(x) \text{ for all } x \in \mathcal{A}. \end{aligned}$$

Therefore, by Lemma 1.2, R is a topological isomorphism from $(\mathcal{A}, \tau)$ onto itself. Thus for all $x \in \mathcal{A}$, we have

$$Rx = \tau\text{-} \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)Rx_i = \tau\text{-} \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)y_i.$$

Hence $\{y_n\}$ is a τ-frame for $T(\mathcal{A}) = \mathcal{A}$. □

Example 2.1. *Let* $\mathcal{A} = \Big\{\{\xi_j\} \subset \mathbb{C} : \sum_{i=1}^{\infty} |\xi_i| < \infty\Big\}$ *and* τ *be the topology induced by the metric* $d(x, y) = \sum_{j=1}^{\infty} |\xi_j - \eta_j|$, $x = \{\xi_i\}, y = \{\eta_i\} \in \mathcal{A}$. *Then* $(\mathcal{A}, \tau)$ *is a separable complete locally convex space. Define a finitely linearly independent sequence* $\{x_n\} \subset \mathcal{A}$ *by*

$$x_1 = \chi_1 \text{ and } x_n = (-1)^{n+1}\chi_1 + \chi_n,\ n \geq 2,$$

where χ_n *denote the canonical unit vector. Let* $\{f_n\} \subset \mathcal{A}'$ *be sequence given by*

$$f_1(x) = \xi_1 + \xi_2 - \chi_3 + \xi_4 - \xi_5 + \ldots. \quad \text{and} \quad f_n(x) = \xi_n,\ x = \{\xi_j\} \in \mathcal{A}\ (n \geq 2).$$

Then, $x = \tau\text{-}\lim_{n\to\infty}\sum_{i=1}^{n} f_i(x)x_i$ *for each* $x \in \mathcal{A}$*. Hence* $\mathcal{F} \equiv \{x_n\}$ *is* τ*-frame for* $\mathcal{A}$*. Let* $\{y_n\} \subset \mathcal{A}$ *be a sequence given by*

$$y_1 = \chi_1,\ y_n = \chi_n - \chi_{n+1}, \quad n \geq 2.$$

It can be easily computed that for all finite sequences $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ *of scalars and for* $0 < \lambda < 1$*, we get*

$$p\Big(\sum_{i=1}^{m} \alpha_i(x_i - y_i)\Big) \leqslant \lambda p\Big(\sum_{i=1}^{m} \alpha_i x_i\Big),$$

where $p \in D_\tau$*. Hence by Theorem 2.1,* $\{y_n\}$ *is a* τ*-frame for* $\mathcal{A}$*.*

The following theorem gives a perturbation result, where the perturbed sequence is the image of a given frame sequence under a continuous linear operator.

Theorem 2.2. *Let* $\{x_n\} \subset \mathcal{A}$ *be a* τ*-frame for* $(\mathcal{A}, \tau)$ *with associated sequence of functionals* $\{f_n\} \subset \mathcal{A}'$*. Suppose* $\{y_n\} \subset \mathcal{A}$ *is a sequence such that*

$$p\Big(\tau\text{-}\lim_{n\to\infty}\sum_{i=1}^{n} f_i(x)(x_i - y_i)\Big) \leqslant Mp(x) \text{ for all } x \in \mathcal{A},$$

where M *is an element of the scalar field* $\mathbb{K}$ *and* p *is a member of the family of all semi-norms generating the topology* τ*. If there exists a continuous linear operator* T *on* $\mathcal{A}$ *such that* $Tx_n = y_n$ *for all* $n \in \mathbb{N}$*, then* $\{y_n\}$ *is also a* τ*-frame for* $\mathcal{A}$*.*

Proof. Since $\{x_n\} \subset \mathcal{A}$ is a τ-frame for $(\mathcal{A}, \tau)$ with $\{f_n\} \subset \mathcal{A}'$ as an associated sequence of functionals, we have

$$x = \tau\text{-}\lim_{n\to\infty}\sum_{i=1}^{n} f_i(x)x_i \text{ for all } x \in \mathcal{A}.$$

Define an operator $L : \mathcal{A} \to \mathcal{A}$ by

$$Lx = \tau\text{-}\lim_{n\to\infty}\sum_{i=1}^{n} f_i(x)y_i, \text{ for all } x \in \mathcal{A}.$$

Then, L is well defined. Indeed, for $n > m$, we compute

$$\begin{aligned} p\Big(\sum_{i=1}^{n} f_i(x)y_i - \sum_{j=1}^{m} f_j(x)y_j\Big) =& p\Big(\sum_{i=1}^{n} f_i(x)Tx_i - \sum_{j=1}^{m} f_j(x)Tx_j\Big) \\ =& p\Big(T\Big(\sum_{i=1}^{n} f_i(x)x_i - \sum_{j=1}^{m} f_j(x)x_j\Big)\Big) \\ & \longrightarrow 0 \quad \text{as } n, m \to \infty. \end{aligned}$$

Therefore, $\tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)y_i$ is a Cauchy sequence in $\mathcal{A}$ and hence convergent for all $x \in \mathcal{A}$.

Let I be the identity map on $\mathcal{A}$. Then, for a fixed (but arbitrary) $x \in \mathcal{A}$, we have

$$\begin{aligned} p((I - L)x) &= p(x - Lx) \\ &= p\Big(\tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i - \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)y_i\Big) \\ &= p\Big(\tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)(x_i - y_i)\Big) \\ &\leqslant Mp(x). \end{aligned}$$

Therefore, by Lemma 1.2, L is a topological isomorphism from $\mathcal{A}$ onto itself. Define a sequence of functionals $\{g_n\} \subset \mathcal{A}'$ by $g_i = f_i L^{-1}$ for all $i \in \mathbb{N}$. Then, for all $x \in \mathcal{A}$, we have

$$\begin{aligned} \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} g_i(x)y_i &= \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} (f_i L^{-1})(x)y_i \\ &= \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(L^{-1}x)y_i \\ &= L(L^{-1}x) \\ &= x. \end{aligned}$$

Hence $\{y_n\}$ is a τ-frame for $\mathcal{A}$. $\square$

Example 2.2. *Let* $\mathcal{A} = \Big\{\{\xi_j\} \subset \mathbb{C} : \tau\text{-}\lim \xi_j = 0\Big\}$*, where* τ *is the topology given in Example 1.1. Then,* $(\mathcal{A}, \tau)$ *is a separable complete locally convex space. Define a sequence* $\{x_n\} \subset \mathcal{A}$ *by*

$$x_1 = \chi_1 \text{ and } x_n = \chi_{n-1},\ n \geq 2,$$

Define a sequence $\{f_n\} \subset \mathcal{A}^*$ *by*

$$f_l(x) = \xi_l,\ l = 1, 2 \text{ and } f_n(x) = \xi_{n-1},\ x = \{\xi_j\} \in \mathcal{A} \quad (n \geq 3).$$

Then, $x = \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i$ *for each* $x \in \mathcal{A}$*. Hence* $\mathcal{F} \equiv \{x_n\}$ *is* τ*-frame for* $\mathcal{A}$*. Let* $\{y_n\} \subset \mathcal{A}$ *be a sequence given by*

$$y_1 = x_1,\ y_2 = x_2,\ y_n = (-1)^{n+1}x_1 + x_n, \quad n \geq 3.$$

It can be easily computed that

$$p\Big(\tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)(x_i - y_i)\Big) \leqslant Mp(x) \text{ for all } x \in \mathcal{A},$$

where M *is an element of the scalar field* $\mathbb{K}$*. Thus, by Theorem 2.2, the sequence* $\{y_n\}$ *is a* τ*-frame for* $\mathcal{A}$*.*

Next result shows that under small perturbation, the perturbed sequence form a τ-frame for $\mathcal{A}$ with coefficient functionals of the given τ-frame.

Theorem 2.3. *Let* $\{x_n\} \subset \mathcal{A}$ *be a* τ*-frame for* $(\mathcal{A}, \tau)$ *and* $\{f_n\} \subset \mathcal{A}'$ *be an associated sequence of functionals. Let* $p_\alpha \in D_\tau$ *be arbitrary. Choose* $\delta > 0$ *such that*

$$p_\alpha\Big(\sum_{k=1}^{N} f_k(x)x_k\Big) < \delta p_\alpha(x) \text{ for all } N \geqslant 1 \text{ and for all } x \in \mathcal{A}.$$

Let $\epsilon \in (0, 1)$ *and let* $\{y_n\} \subset \mathcal{A}$ *be a sequence such that for all* $n < m$*,*

$$p_\alpha\Big(\sum_{k=n}^{m} f_k(x)(x_k - y_k)\Big) < \frac{\epsilon}{\delta} \sup_{n\leqslant l\leqslant m} p_\alpha\Big(\sum_{k=n}^{l} f_k(x)x_k\Big) \text{ for all } n \in \mathbb{N} \text{ and for all } x \in \mathcal{A}.$$

Then, $\{y_n\}$ *is a* τ *frame for* $\mathcal{A}$ *with associated sequence of functionals* $\{f_n\}$*.*

Proof. Define $\Theta : \mathcal{A} \to \mathcal{A}$ by

$$\Theta(x) = \tau\text{-}\lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)(x_k - y_k) \text{ for all } x \in \mathcal{A}.$$

Then, Θ is a well-defined, bounded linear operator on $\mathcal{A}$. For all $x \in \mathcal{A}$, we have

$$\begin{aligned} p_\alpha(\Theta(x)) &= p_\alpha\Big(\tau\text{-}\lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)(x_k - y_k)\Big) \\ &< \tau\text{-}\lim_{n\to\infty} \frac{\epsilon}{\delta} \sup_{1\leqslant l\leqslant n} p_\alpha\Big(\sum_{k=1}^{l} f_k(x)x_k\Big) \\ &< \epsilon p_\alpha(x). \end{aligned}$$

Therefore, by Lemma 1.2, $\Xi = I - \Theta$ is a topological isomorphism from $\mathcal{A}$ onto itself and hence, Ξ is a continuously invertible operator on $\mathcal{A}$. Also, for all $x \in \mathcal{A}$, we have

$$\Xi(x) = x - \Theta(x) = \tau\text{-}\lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)y_k.$$

Hence, $\{y_n\}$ is a τ-frame for $\Xi(\mathcal{A}) = \mathcal{A}$ with associated sequence of functionals $\{f_n\}$. □

The following theorem provides a sufficient condition for perturbation of τ-Schauder frames, where the perturbed sequence is obtained from a block associated with a given τ-Schauder frame.

Theorem 2.4. *Let $\{x_n\}$ be a τ-Schauder frame for $(\mathcal{A}, \tau)$ with an associated sequence of functionals $\{f_n\} \subset \mathcal{A}^*$. Suppose $\{y_n\} \subset \mathcal{A}$ is a sequence given by*

$$y_n = x_n + \tau\text{-} \sum_{i\geqslant n+1} f_i(y_n)x_i, \ n \in \mathbb{N}.$$

For $p \in D_\tau$, assume that there exists $k_p \in \mathbb{N}$ and λ_p with $0 < \lambda_p < 1$ such that

$$p\Big(\sum_{i=k_p}^{m} \alpha_i(y_i - x_i)\Big) \leqslant \lambda_p p\Big(\sum_{i=k_p}^{m} \alpha_i x_i\Big) \tag{2.3}$$

for all finite sequences $\{\alpha_{k_p}, \dots, \alpha_m\}$ *of scalars. Then,* $\{y_n\}$ *is a* τ*-Schauder frame for* $(\mathcal{A}, \tau)$.

Proof. Define $\Theta : \mathcal{A} \to \mathcal{A}$ by

$$\Theta x = \tau\text{-} \sum_{n=1}^{\infty} f_n(x)(y_n - x_n) \text{ for all } x \in \mathcal{A}.$$

Then, Θ is a well defined linear operator on $\mathcal{A}$. Given any $x \in \mathcal{A}$, we can write it $x = u + v$, where

$$u = \sum_{i=1}^{k_p - 1} f_i(x) x_i, \text{ and } v = \tau\text{-} \lim_{m\to\infty} \sum_{i=k_p}^{m} f_i(x) x_i.$$

By definition of Θ, we have

$$\Theta x = \Big(\sum_{i=1}^{k_p - 1} f_i(x)(y_i - x_i) \Big) + \Big(\tau\text{-} \lim_{m\to\infty} \sum_{i=k_p}^{m} f_i(x)(y_i - x_i) \Big) = \Theta u + \Theta v.$$

Since $\{x_n\}$ is a τ-Schauder frame, we have $|f_i(x)| \leqslant p_\alpha(x)$ for all $x \in \mathcal{A}$ $(1 \leqslant i \leqslant k_p - 1)$ for some $p_\alpha \in D_\tau$.

For any $p \in D_\tau$, we have

$$\begin{aligned} p(\Theta u) &= p\Big(\sum_{i=1}^{k_p - 1} f_i(x)(y_i - x_i) \Big) \\ &= \sum_{i=1}^{k_p - 1} |f_i(x)| p(y_i - x_i) \\ &\leqslant \sum_{i=1}^{k_p - 1} p_\alpha(x) p(y_i - x_i) \\ &= p_\alpha(x) c_p, \text{ where } c_p = \sum_{i=1}^{k_p - 1} p(y_i - x_i) < \infty. \end{aligned}$$

By using (2.3), for any $p \in D_\tau$ we compute

$$p(\Theta v) = p\Big(\tau\text{-} \lim_{m\to\infty} \sum_{i=k_p}^{m} f_i(x)(y_i - x_i) \Big)$$

$$\leqslant \lambda_p p\Big(\tau\text{-}\lim_{m\to\infty}\sum_{i=k_p}^{m} f_i(x)x_i\Big)$$

$$= \lambda_p p(v)$$

$$= \lambda_p p(x-u)$$

$$\leqslant \lambda_p p(x) + \lambda_p p(-u)$$

$$= \lambda_p p(x) + \lambda_p p(u)$$

$$= \lambda_p p(x) + \lambda_p p\Big(\sum_{i=1}^{k_p-1} f_i(x)x_i\Big)$$

$$= \lambda_p p(x) + \lambda_p \sum_{i=1}^{k_p-1} |f_i(x)|p(x_i)$$

$$\leqslant \lambda_p p(x) + \lambda_p p_\alpha(x)\sum_{i=1}^{k_p-1} p(x_i)$$

$$= \lambda_p p(x) + \lambda_p p_\alpha(x)d_p, \text{ where } d_p = \sum_{i=1}^{k_p-1} p(x_i) < \infty.$$

Choose p_β =max$\{p_\alpha, p\}$. Then, $p(\Theta x) \leqslant (\lambda_p d_p + \lambda_p + c_p)p_\beta(x)$ for all $x \in \mathcal{A}$. Thus, Θ is a continuous linear map. Therefore, by Lemma 1.2, $\Xi = I - \Theta$ is a topological isomorphism from $(\mathcal{A}, \tau)$ onto itself. Hence $\{y_n\}$ is a τ-Schauder frame for $\Xi(\mathcal{A}) = \mathcal{A}$. □

Now we give a necessary condition for perturbation of τ-frames in terms of an eigenvalue of a matrix associated with the perturbed sequence. This is inspired by [16].

Theorem 2.5. *Let $\{x_n\}$ be a τ-frame for $(\mathcal{A}, \tau)$ and $\{y_k\}_{k=1}^m \subset \mathcal{A}$ ($m \in \mathbb{N}$ is fixed) be linearly independent vectors. Suppose, there exist continuous linear functionals $\{g_k\}_{k=1}^m \subset \mathcal{A}^*$ such that $g_k(x_n) = \alpha_k^{(n)}$ for all $n \in \mathbb{N}$ ($1 \leq k \leq m$). If*

$\{x_n + \sum_{k=1}^{m} \alpha_k^{(n)} y_k\}$ *is a τ-frame for $\mathcal{A}$, then -1 is not an eigenvalue of the matrix*

$$\begin{bmatrix} g_1(y_1) & g_2(y_1) & \dots & g_m(y_1) \\ g_1(y_2) & g_2(y_2) & \dots & g_m(y_2) \\ \dots & \dots & \dots & \dots \\ g_1(y_m) & g_2(y_m) & \dots & g_m(y_m) \end{bmatrix}.$$

Proof. It is sufficient to prove the result for $m = 2$.

Let, if possible, -1 is an eigenvalue of the matrix

$$\begin{bmatrix} g_1(y_1) & g_2(y_1) \\ g_1(y_2) & g_2(y_2) \end{bmatrix}$$

Then

$$\begin{vmatrix} g_1(y_1) + 1 & g_2(y_1) \\ g_1(y_2) & g_2(y_2) + 1 \end{vmatrix} = 0.$$

Therefore, there exists scalars α and β not both zero such that

$$\alpha g_1(y_1) + \beta g_2(y_1) = -\alpha \quad \text{and} \quad \alpha g_1(y_2) + \beta g_2(y_2) = -\beta.$$

Choose $f_0 = -\alpha g_1 - \beta g_2$. Then, f_0 is a non-zero functional in $\mathcal{A}^*$. Indeed, if $f_0 = 0$, then $f_0(y_1) = 0$ and $f_0(y_2) = 0$. This gives $\alpha = 0$ and $\beta = 0$, which is a contradiction. Hence f_0 must be non-zero.

We compute

$$\begin{aligned} & f_0\Big(x_n + \sum_{k=1}^{2} \alpha_k^{(n)} y_k\Big) \\ &= f_0\Big(x_n + \alpha_1^{(n)} y_1 + \alpha_2^{(n)} y_2\Big) \\ &= f_0(x_n) + \alpha_1^{(n)} f_0(y_1) + \alpha_2^{(n)} f_0(y_2) \\ &= (-\alpha g_1 - \beta g_2)(x_n) + \alpha_1^{(n)}(-\alpha g_1 - \beta g_2)(y_1) + \alpha_2^{(n)}(-\alpha g_1 - \beta g_2)(y_2) \\ &= -\alpha g_1(x_n) - \beta g_2(x_n) + \alpha_1^{(n)}(-\alpha g_1(y_1) - \beta g_2(y_1)) + \alpha_2^{(n)}(-\alpha g_1(y_2) - \beta g_2(y_2)) \end{aligned}$$

$$= -\alpha\alpha_1^{(n)} - \beta\alpha_2^{(n)} + \alpha_1^{(n)}\alpha + \alpha_2^{(n)}\beta$$

$$= 0.$$

This gives

$$f_0\Big(x_n + \sum_{k=1}^{2} \alpha_k^{(n)} y_k\Big) = 0 \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

Since, $\{x_n + \sum_{k=1}^{m} \alpha_k^{(n)} y_k\}$ is a τ-frame for $\mathcal{A}$, the sequence $\Big\{x_n + \sum_{k=1}^{2} \alpha_k^{(n)} y_k\Big\}$ is τ-dense in $\mathcal{A}$. By (2.4), the continuous map f_0 is zero on a τ-dense subset of $\mathcal{A}$. Thus, the zero map and f_0 are continuous maps which agree on a τ-dense subset of $\mathcal{A}$. Therefore, by Lemma 1.1, $f_0 \equiv 0$, a contradiction. This completes the proof. □

To conclude the paper, we give a sufficient condition for the perturbation of τ-frames to not be a τ-frame for $\mathcal{A}$.

Proposition 2.1. *Let $\{x_n\}$ be a τ-frame for $(\mathcal{A}, \tau)$. If there exists a continuous linear functional $f_0 \in \mathcal{A}^*$ such that $f_0(x_n) = \lambda$ for all $n \in \mathbb{N}$, where λ is a non-zero scalar in $\mathbb{K}$, then there exists $z_{f_0} \in \mathcal{A}$ such that $\{x_n + z_{f_0}\}$ is not a τ-frame for $\mathcal{A}$.*

Proof. Suppose that $\{x_n + z_{f_0}\}$ is a τ-frame for $\mathcal{A}$. Let $z \in \mathcal{A}$ be such that $f_0(z) = \alpha$, where α is a non-zero scalar in $\mathbb{K}$. Choose $z_{f_0} = \frac{-\lambda}{\alpha} z$. Then, $f_0(x_n + z_{f_0}) = 0$ for all $n \in \mathbb{N}$. Since, $\{x_n + z_{f_0}\}$ is a τ-frame for $\mathcal{A}$, so $\{x_n + z_{f_0}\}$ is τ-dense in $\mathcal{A}$. By Lemma 1.1, $f_0 \equiv 0$, a contradiction. The result is proved. □

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