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SOLVING LINEAR FUZZY FREDHOLM INTEGRAL EQUATIONS SYSTEM BY TRIANGULAR FUNCTIONS

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Abstract. In this paper, we present a numerical method to solve linear fuzzy Fredholm integral equations system of the second kind. This method converts the given fuzzy system into a linear system of algebraic equations by using triangular orthogonal functions. The proposed method is tested by two examples and also results are compared with the exact solution, by using computer simulations.

1. Introduction

It is known that the fuzzy differential and integral equations are one of the important parts of the fuzzy analysis theory that play major role in numerical analysis. In 1990, Wu and Ma [11] investigated the fuzzy Fredholm integral equations for the first time. Then numerous numerical methods are investigated which have been focusing on the solution of fuzzy integral equations. For example, Tricomi, in his book [9], introduced the classical method of successive approximations for nonlinear integral equations. Variational iteration method [5] and Adomian decomposition method [2] were effective and convenient for solving integral equations . Recently, a new set of triangular orthogonal functions have been applied for solving integral equation by

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Babolian et al. [1]. Mirzaee et al. [6] have used the triangular functions for solving fuzzy Fredholm integral equation of second kind (FFIE-2).

The aim of this paper is to apply the triangular functions for the linear fuzzy Fredholm integral equations system of the second kind (FFIES-2). We show that the proposed method well performs for linear FFIES-2.

This paper is organized as follows. Preliminaries of triangular orthogonal functions and their properties is briefly presented in Section 2. In Section 3, we give an overview of elementary concepts of the fuzzy calculus. In Section 4, numerical method for solving system of fuzzy Fredholm integral equations of the second kind is presented. Convergence analysis for the method is given in Section 5. Finally, we illustrate in Section 6 some numerical examples to show the efficiency and accuracy of the proposed method.

2. Preliminaries of triangular functions

Definition 2.1. Two m-sets of triangular functions (TFs) are defined over the interval [0,T] as

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \leq t < (i+1)h, \\ 0, & o.w, \end{cases}$$

and

$$T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \leq t < (i+1)h, \\ 0, & o.w, \end{cases}$$

where $i = 0, 1, \cdots, m-1, h = \frac{T}{m}$, with a positive integer value for m.

Also, consider $T1_i$ as the ith left-handed triangular function and $T2_i$ as the ith right-handed triangular function. In this paper, it is assumed that $T = 1$. Consider the first m terms of the left-handed triangular functions and the first m terms of the right-handed triangular functions and write them concisely as m-vectors:

$$T1(t) = [T1_0(t), T1_1(t), \cdots, T1_{m-1}(t)]^T, \tag{2.1}$$

$$T2(t) = [T2_0(t), T2_1(t), \cdots, T2_{m-1}(t)]^T, \tag{2.2}$$

where $T1(t)$ and $T2(t)$ are called left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector, respectively. We have

$$\int_0^1 T1(t)T1^T(t)\mathrm{d}t = \int_0^1 T2(t)T2^T(t)\mathrm{d}t \simeq \frac{h}{3}I, \tag{2.3}$$

and

$$\int_0^1 T1(t)T2^T(t)\mathrm{d}t = \int_0^1 T2(t)T1^T(t)\mathrm{d}t \simeq \frac{h}{6}I, \tag{2.4}$$

where I is an $m \times m$ identity matrix. We denote the TF vector $T(t)$ as

$$T(t) = [T1(t) \quad T2(t)]^T. \tag{2.5}$$

By using Eqs.(2.3) and (2.4), we can write

$$\int_0^1 T(t)T^T(t)\mathrm{d}t \simeq \begin{pmatrix} \frac{h}{3}I_m & \frac{h}{6}I_m \\ \frac{h}{6}I_m & \frac{h}{3}I_m \end{pmatrix} = D \tag{2.6}$$

where D is $2m \times 2m$ matrix [6]. Let $f(t)$ belong to $L^2[0,1)$, the expansion of $f(t)$ with respect to TFs, can be defined as follows

$$f(t) \simeq \sum_{i=0}^{m-1}[f_i T1_i(t) + f_{i+1}T2_i(t)] = F1^T T1(t) + F2^T T2(t) = \mathcal{F}^T T(t) \tag{2.7}$$

where the sequence of constant coefficients $\{f_i\}_{i=0}^m$ are given as $f_i = f(ih)$ for $i = 0, 1, \ldots, m$. Moreover, for each function $f(s,t) \in L^2([0,1) \times [0,1))$, we can rewrite the TFs expansion as

$$\begin{aligned} f(t,s) \simeq & T1^T(t).F11.T1(s) + T1^T(t).F12.T2(s) \\ & + T2^T(t).F21.T1(s) + T2^T(t).F22.T2(s), \end{aligned}$$

or

$$f(t,s) \simeq T^T(t).F.T(s) \tag{2.8}$$

where $F11, F12, F21$ and $F22$ are $m \times m$ matrices and can be obtained easily as follows

$$\begin{aligned} (F11)_{ij} &= f(ih, jh), \\ (F12)_{ij} &= f(ih, (j+1)h), \\ (F21)_{ij} &= f((i+1)h, jh), \\ (F22)_{ij} &= f((i+1)h, (j+1)h), \end{aligned} \tag{2.9}$$

for $i, j = 0, 1, \ldots, m-1$, and $T(t)$, $T(s)$ are $2m_1$ and $2m_2$ dimensional TFs and F is a $2m_1 \times 2m_2$ TFs coefficient matrix [1]. For simplicity, we put $m_1 = m_2 = m$, so matrix F can be written as

$$F = \begin{pmatrix} (F11)_{m\times m} & (F12)_{m\times m} \\ (F21)_{m\times m} & (F22)_{m\times m} \end{pmatrix} \tag{2.10}$$

where $F11$, $F12$, $F21$ and $F22$ are previously defined in Eq. (2.9). (For more details see, [1, 6])

3. The basic concepts of fuzzy equations

In this section, the basic notations in fuzzy calculus and integral equations, which we will use, are briefly introduced. We started by defining the fuzzy number.

Definition 3.1. A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \to [0, 1]$ such that:

(a) u is upper semi-continuous,

(b) $u(x) = 0$ outside some interval [a, d],

(c) There are real numbers b, c such as $a \le b \le c \le d$ and

   (i) $u(x)$ is monotonically increasing on [a, b],

(ii) $u(x)$ is monotonically decreasing on [c, d],

(iii) $u(x) = 1, b \leq x \leq c$.

The set of all fuzzy numbers is denoted by E^1 and it is a convex cone [3, 8].

An alternative definition or parametric form of a fuzzy number which yields to the same E^1 is given by Kaleva [4] as follows

Definition 3.2. A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, where $0 \leq r \leq 1$, such that

(a) $\underline{u}(r)$ is abounded monotonic increasing left continuous function,

(b) $\overline{u}(r)$ is abounded monotonic decreasing left continuous function,

(c) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and real number k, we define

(a) Equality: $u = v$ if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$,

(b) Addition: $u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,

(c) Scalar multiplicationand: $k \otimes u = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0, \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0. \end{cases}$

Definition 3.3. For arbitrary numbers $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$,

$$D(u, v) = max\{\sup_{0 \leq r \leq 1} |\underline{u}(r) - \underline{v}(r)|, \sup_{0 \leq r \leq 1} |\overline{u}(r) - \overline{v}(r)|\} \tag{3.1}$$

is the distance between u and v. It is proved that (E^1, D) is a complete metric space with its properties [7], and

(a) $\forall u, v, w \in E^1; D(u \oplus w, v \oplus w) = D(u, v)$,

(b) $\forall u, v \in E^1, \forall k \in \mathbb{R}; D(k \otimes u, k \otimes v) = |k| D(u, v)$,

(c) $\forall u, v, w, e \in E^1; D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e)$.

Definition 3.4. Let $f, g : [a, b] \to E^1$ be fuzzy real number valued functions. The uniform distance between f and g is defined by [10]

$$D_U(f, g) = \sup\{D(f(x), g(x)) | x \in [a, b]\}.$$

Definition 3.5. Suppose $f : [a, b] \to E^1$ is a fuzzy function. For each partition $P = \{x_0, x_1, \ldots, x_n\}$ and orbitrary ξ_i, $x_{i-1} \leq \xi_i \leq x_i$, $i = 1, 2, \ldots, n$, let $R_P = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$, and $\lambda = \max_{1\leq i\leq n} |x_i - x_{i-1}|$. The integral of $f(x)$ over [a,b] is defined as

$$\int_a^b f(x)\mathrm{d}x = \lim_{\lambda \to 0} R_P$$

provided that the limit exists in the metric D.

If the fuzzy function $f(x)$ is continuous in the metric D, then the definite integral in last equation exists [3], and also we have

$$\underline{\int_a^b f(x, r)\mathrm{d}x)} = \int_a^b \underline{f}(x, r)\mathrm{d}x$$

and

$$\overline{(\int_a^b f(x, r)\mathrm{d}x)} = \int_a^b \overline{f}(x, r)\mathrm{d}x$$

where $(\underline{f}(x, r), \overline{f}(x, r))$ is the parametric form of $f(x)$. More details about the properties of the fuzzy integral are given in [3, 4].

4. Solving linear fuzzy Fredholm integral equations system

In this section, we present a TFs method to solve linear FFIES-2. First consider the following FFIE-2:

$$u(x) = g(x) \oplus \lambda \otimes \int_0^1 k(x, t) \otimes u(t)\mathrm{d}t$$

where $k(x,t)$ is an orbitary kernel function over the square $0 \leq x,t \leq 1$ and $u(x)$ is a fuzzy real valued function. We introduce the FFIES-2 in the form

$$\begin{cases} u_1(x) = g_1(x) \oplus \sum_{j=1}^{n} \lambda_{1j} \otimes \int_0^1 k_{1j}(x,t) \otimes u_j(t)\mathrm{d}t, \\ u_2(x) = g_2(x) \oplus \sum_{j=1}^{n} \lambda_{2j} \otimes \int_0^1 k_{2j}(x,t) \otimes u_j(t)\mathrm{d}t, \\ \quad \vdots \\ u_n(x) = g_n(x) \oplus \sum_{j=1}^{n} \lambda_{nj} \otimes \int_0^1 k_{nj}(x,t) \otimes u_j(t)\mathrm{d}t, \end{cases} \tag{4.1}$$

where $k_{ij}(x,t)$ is an orbitary kernel function over the square $0 \leq x,t \leq 1$ and $\lambda_{ij} \neq 0$ for $i,j = 1,2,\ldots,n$ are real constants. In system (4.1), the fuzzy function $g_i(x)$ and kernel $k_{ij}(x,t)$ are given and assumed to be differentiable as the discussion required with respect to all their arguments on the interval $0 < x,t < 1$. Also, $u_i(x)$ is a fuzzy real valued function and $U(x) = [u_1(x), u_2(x), \ldots, u_n(x)]^T$ is the solution to be determined. For simplicity, we consider the ith equation of system (4.1) as

$$u_i(x) = g_i(x) \oplus \sum_{j=1}^{n} \lambda_{ij} \otimes \int_0^1 k_{ij}(x,t) \otimes u_j(t)\mathrm{d}t. \tag{4.2}$$

Let $(\underline{g}_i(x,r), \overline{g}_i(x,r))$ and $(\underline{u}_i(x,r), \overline{u}_i(x,r))$, $0 \leq r \leq 1$ and $x \in [0,1)$ be parametric forms of $g_i(x)$ and $u_i(x)$, respectively. Therefore, by using definition (3), we get

$$k_{ij}(x,t) \otimes u_j(x) = \begin{cases} (k_{ij}(x,t)\underline{u}_j(x,r), k_{ij}(x,t)\overline{u}_j(x,r)), & k_{ij}(x,t) \geq 0, \\ (k_{ij}(x,t)\overline{u}_j(x,r), k_{ij}(x,t)\underline{u}_j(x,r)), & k_{ij}(x,t) < 0. \end{cases}$$

In this paper, we assumed that $k_{ij}(x,t) \geq 0$. Now, for solving (4.1) we write the parametric form of the given fuzzy integral equations system as follows

$$\underline{u}_i(x,r) = \underline{g}_i(x,r) + \sum_{j=1}^{n} \lambda_{ij} \int_0^1 k_{ij}(x,t)\underline{u}_j(t,r)\mathrm{d}t \tag{4.3}$$

and

$$\overline{u}_i(x,r) = \overline{g}_i(x,r) + \sum_{j=1}^{n} \lambda_{ij} \int_0^1 k_{ij}(x,t)\overline{u}_j(t,r)\mathrm{d}t \tag{4.4}$$

for $i, j = 1, 2, \ldots, n$. Let us expand $\underline{u}_i(x, r)$, $\underline{g}_i(x, r)$ and $k_{ij}(x, t)$ by using Eq. (2.8) as follow

$$\begin{aligned} \underline{u}_i(x,r) &\simeq T^T(x).U_i.T(r), \\ \underline{g}_i(x,r) &\simeq T^T(x).G_i.T(r), \\ \underline{k}_{ij}(x,t) &\simeq T^T(x).K_{ij}.T(t), \end{aligned} \tag{4.5}$$

where U_i, G_i and K_{ij} a in Eq. (2.10). Substituting Eqs. (4.5) in Eq. (4.3), we have

$$T^T(x)U_iT(r) \simeq T^T(x)G_iT(r) + \sum_{j=1}^{n} \lambda_{ij} T^T(x) K_{ij} \left(\int_0^1 T(t)T^T(t)\mathrm{d}t \right) U_j T(r). \tag{4.6}$$

Substituting Eqs. (2.6) in Eq. (4.6), we get

$$T^T(x)U_iT(r) \simeq T^T(x)G_iT(r) + T^T(x)(\sum_{j=1}^{n} \lambda_{ij} K_{ij} D U_j)T(r).$$

Thus, we have

$$U_i = G_i + \sum_{j=1}^{n} \lambda_{ij} K_{ij} D U_j.$$

Then, we get the following system

$$\sum_{j=1}^{n} (\Delta_{ij} - \lambda_{ij} K_{ij} D)\, U_j = G_i \tag{4.7}$$

where

$$\Delta_{ij} = \begin{cases} I & i = j \\ 0 & i \neq j \end{cases}$$

for $i, j = 1, 2, \ldots, n$ and I is a $2m \times 2m$ identity matrix. By solving this matrix system, we can find U_i where is a $2m \times 2m$ matrix. So $\underline{u}_i(x, r) \simeq T^T(x)U_iT(r)$. The same procedure is used for $\overline{u}_i(x, r)$ in Eq. (4.4). For solving system (4.1), we need to solve the two systems of (4.7).

5. Convergence Analysis

In this section, we obtain error estimate for the numerical method proposed in previous section. We first present the following lemma.

Lemma 5.1. [12] *If f and $g : [a,b] \subseteq \mathbb{R} \to E^1$ are fuzzy continuous function, then the function $F : [a,b] \to R_+$ by $F(x) = D(f(x), g(x))$ is continuous on $[a,b]$ and*

$$D\left(\int_a^b f(x)\mathrm{d}x, \int_a^b g(x)\mathrm{d}x\right) \leq \int_a^b D(f(x), g(x))\mathrm{d}x.$$

Theorem 5.1. *If $k_{ij}(x,t)$, $i,j = 1,2,\ldots,n$ and $0 \leq x,t \leq 1$ are bounded and continuous, then approximate solution of System (4.1), converges to the exact solution.*

Proof. Suppose that $\tilde{u}_i(x)$ is the approximate solution of exact solution $u_i(x)$. Therefore, $\tilde{u}_i(x) \simeq \mathcal{U}_i^T T(x)$ (see Eq. (2.7)). Then,

$$\begin{aligned} D(u_i(x), \tilde{u}_i(x)) &= D\left(\sum_{j=1}^{n} \lambda_{ij} \int_0^1 k_{ij}(x,t)u_j(t)\mathrm{d}t, \sum_{j=1}^{n} \lambda_{ij} \int_0^1 k_{ij}(x,t)\mathcal{U}_j^T T(t)\mathrm{d}t\right) \\ &\leq M \sum_{j=1}^{n} \int_0^1 D\left(u_j(t), \mathcal{U}_j^T T(t)\right)\mathrm{d}t, \quad M = \max_{0 \leq x,t \leq 1} |\lambda_{ij} k_{ij}(x,t)|. \end{aligned}$$

Also, $\lim_{m\to\infty} \mathcal{U}_i^T T(x) = u_i(x)$, [6]. So $D(u_i(x), \mathcal{U}_i^T T(x)) \to 0$ as $m \to \infty$. Since M is bounded, $\lim_{m\to\infty} D(u_i(x), \tilde{u}_i(x)) \to 0$. Hence, the proof is completed.

6. Numerical examples

In this section, we present two examples of linear FFIES-2 and the results will be compared with the exact solutions. All results are computed by using a program written in the Matlab.

Example 6.1. *Consider the system of fuzzy linear Fredholm integral equations with*

$$(\underline{g}_1(x,r), \overline{g}_1(x,r)) = x^2(r^2 + 2r + 2, 7 - 2r) + \frac{x}{3}(r^2 + r + 1, 4 - r)$$

$$(\underline{g}_2(x,r), \overline{g}_2(x,r)) = x(r^2 + 3r + 3, 10 - 3r), \quad 0 \leq x,t \leq 1, for \quad 0 \leq r \leq 1,$$

and kernel functions:

$$k_{11}(x,t) = x, \quad k_{12}(x,t) = 2x^2,$$

$$k_{21}(x,t) = 4xt, \quad k_{22}(x,t) = 2x,$$

and $\lambda_{ij} = -1$ *for* $i, j = 1, 2$*. The exact solution in this case is given by*

$$(\underline{u}_1(x,r), \overline{u}_1(x,r)) = x^2(r^2 + r + 1, 4 - r),$$

$$(\underline{u}_2(x,r), \overline{u}_2(x,r)) = x(r + 1, 3 - r).$$

We solve this system by the proposed method with $m = 32$, $x = 0.5$ and $r \in [0, 0.9]$ we noticed that the absolute error is zero. The absolute error of $(\underline{u}_1(x,r), \overline{u}_1(x,r))$ and $(\underline{u}_2(x,r), \overline{u}_2(x,r))$ for this Example is listed in Table 1. Moreover, the errors are also shown in Table 2, with $m = 16, 32, 64, 128, 256, 512, 1024$ and 2048. We see that the absolute error converges to zero as $m \to \infty$. Also Fig. 1, shows comparison between the exact solution and the approximate solution using the presented method. In addition, the absolute error functions obtained by the present method shown in Fig. 2.

Example 6.2. *Consider the system of fuzzy linear Fredholm integral equations with*

$$\underline{g}_1(x,r) = r(e^x + (1+x)(2-e) - \frac{1}{4}(1-x^2)(2r+5r^2)),$$

$$\overline{g}_1(x,r) = (2-r)(e^x + (1+x)(2-e)) - \frac{1}{4}(1-x^2)(9-2r^5),$$

$$\underline{g}_2(x,r) = (2r^2+5r^3)(x - \frac{11}{12}(1-x^3)) + 3rx - 2rxe,$$

$$\overline{g}_2(x,r) = 15x - x(2r^5 + 3r + 4e - 2re) + (1-x^3)(\frac{11}{6}r^5 - \frac{33}{4}),$$

where $0 \leq x, t \leq 1$*, for* $0 \leq r \leq 1$*, and kernel functions:*

$$k_{11}(x,t) = t^2(1+x), \quad k_{12}(x,t) = t^2(1-x^2),$$

$$k_{21}(x,t) = x(1+t^2), \quad k_{22}(x,t) = (t-2)^2(1-x^3),$$

and $\lambda_{ij} = 1$ *for* $i, j = 1, 2$. *The exact solution in this case is given by*

$$(\underline{u}_1(x,r), \overline{u}_1(x,r)) = e^x(r, 2-r),$$

$$(\underline{u}_2(x,r), \overline{u}_2(x,r)) = x(2r^2 + 5r^3, 9 - 2r^5).$$

The results are presented for Example 2 in Table 3 and Figs. 3 and 4. Moreover, the errors are also presented in Table 4, with $m = 16, 32, 64, 128, 256, 512$, using $E_1 = |\underline{u}_1(x,r) - \tilde{\underline{u}}_1(x,r)|$ and $E_2 = |\overline{u}_1(x,r) - \tilde{\overline{u}}_1(x,r)|$. Fig. 4 shows the absolute error functions obtained by the present method. We see that the absolute error converges to zero as $m \to \infty$.

TABLE 1. Numerical results for Example 1, with $x = 0.5, m = 32$.

r	Absolute error $\underline{u}_1(x,r)$	Absolute error $\overline{u}_1(x,r))$	Absolute error $\underline{u}_2(x,r)$	Absolute error $\overline{u}_2(x,r)$
0.0	3.4882e-05	1.3953e-04	6.9764e-05	2.7906e-04
0.1	3.3792e-07	1.3604e-04	7.7449e-05	2.7208e-04
0.2	1.5332e-05	1.3255e-04	8.6524e-05	2.6510e-04
0.3	1.0099e-05	1.2906e-04	9.6989e-05	2.5813e-04
0.4	1.5359e-05	1.2558e-04	1.0884e-04	2.5115e-04
0.5	6.1044e-05	1.2209e-04	1.2209e-04	2.4417e-04
0.6	2.9312e-05	1.1860e-04	1.3675e-04	2.3720e-04
0.7	1.7806e-05	1.1511e-04	1.5280e-04	2.3022e-04
0.8	2.6527e-05	1.1162e-04	1.7024e-04	2.2325e-04
0.9	5.5473e-05	1.0813e-04	1.8907e-04	2.1627e-04

TABLE 2. Numerical results for Example 1, with $x = 0.7, r = 0.1$

m	Absolute error $\underline{u}_1(x,r)$	Absolute error $\overline{u}_1(x,r))$	Absolute error $\underline{u}_2(x,r)$	Absolute error $\overline{u}_2(x,r)$
16	1.0238e-03	1.9816e-03	4.3420e-04	1.5243e-03
32	3.0426e-04	7.9992e-04	1.0843e-04	1.8091e-04
64	6.3944e-05	1.2378e-04	2.7102e-05	9.5218e-05
128	1.9013e-05	4.9988e-05	6.7750e-06	2.3804e-05
256	3.9963e-06	7.7362e-06	1.6937e-06	5.9509e-06
512	1.1883e-06	3.1242e-06	4.2343e-07	1.4877e-06
1024	2.4977e-07	4.8351e-07	1.0586e-07	3.7193e-07
2048	7.4267e-08	1.9526e-07	2.6464e-08	9.2983e-08

TABLE 3. Numerical results for Example 2, with $x = 0.3, m = 64$.

r	Absolute error $\underline{u}_1(x,r)$	Absolute error $\overline{u}_1(x,r))$	Absolute error $\underline{u}_2(x,r)$	Absolute error $\overline{u}_2(x,r)$
0.0	0.0000e-00	9.6557e-04	0.0000e-00	1.1249e-03
0.1	3.7316e-05	9.2891e-04	2.1803e-05	1.0863e-03
0.2	7.6432e-05	8.9223e-04	2.4615e-05	1.0486e-03
0.3	1.1813e-04	8.5545e-04	5.2431e-05	1.0142e-03
0.4	1.6319e-04	8.1838e-04	4.0274e-05	9.9034e-04
0.5	2.1236e-04	7.8064e-04	2.3819e-04	9.2738e-04
0.6	2.6646e-04	7.4156e-04	1.0944e-04	9.6112e-04
0.7	3.2622e-04	7.0024e-04	2.3040e-04	9.1817e-04
0.8	3.9244e-04	6.5533e-04	2.9456e-04	9.0901e-04
0.9	4.6589e-04	6.0509e-04	2.8055e-04	9.8385e-04

TABLE 4. Numerical results for Example 2, with $r = 0.1, x = 0.7$

m	16	32	64	128	256	512
E_1	7.238e-04	1.881e-04	4.501e-05	1.174e-05	2.813e-06	7.340e-07
E_2	2.098e-04	5.304e-05	1.382e-05	3.377e-06	8.640e-07	2.119e-07

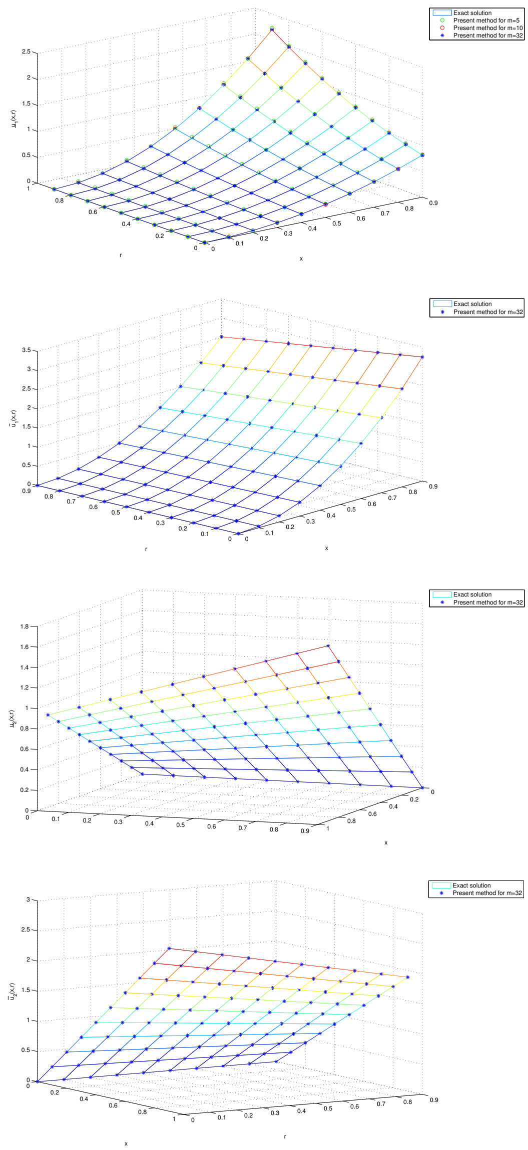

FIGURE 1. Comparison between the exact solution and the approximate solution of Example 1.

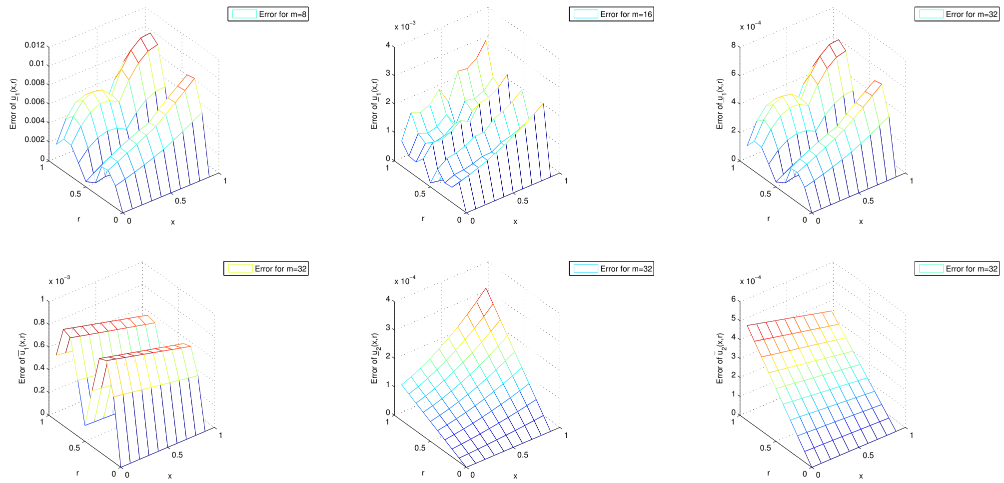

FIGURE 2. Absolute error functions obtained by the present method of Example 1.

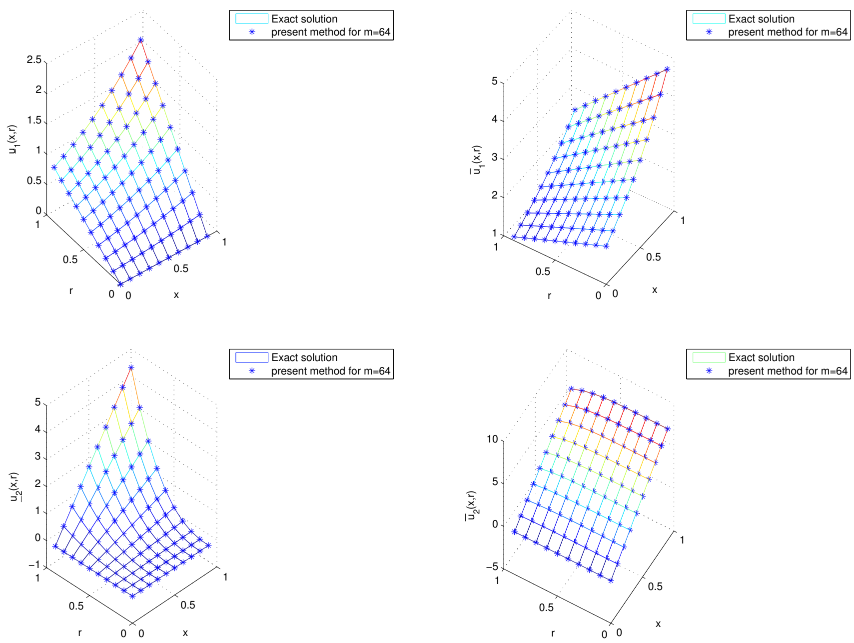

FIGURE 3. Comparison between the exact solution and the approximate solution of Example 2.

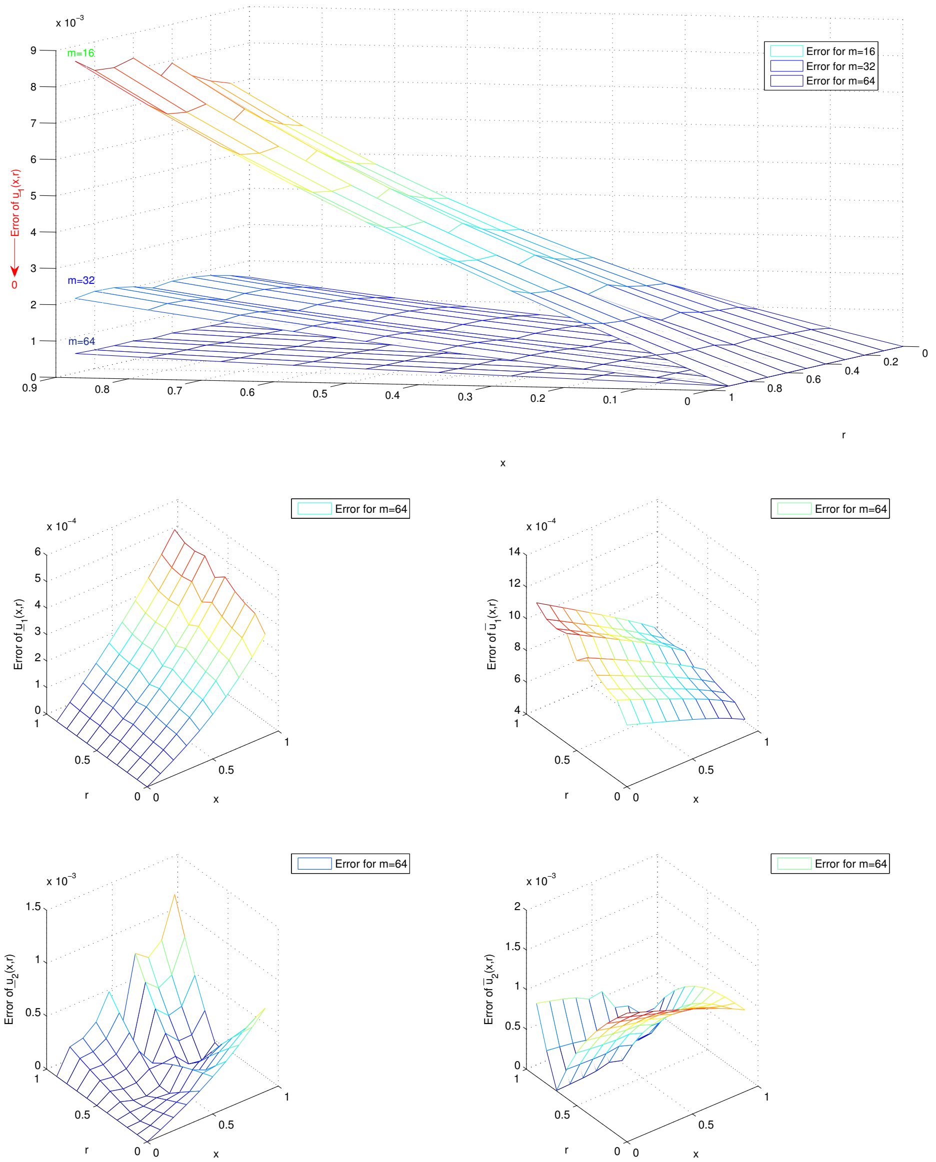

FIGURE 4. Absolute error functions obtained by the present method of Example 2.

7. Conclusion

In this paper, we introduce TFs method for approximating the solution of linear FFIES-2. The structural properties of TFs are utilized to reduce the FFIES-2 to a system of algebraic equations, without using any integration. In the above presented numerical example we see that the proposed method is very accurate and efficient for linear FFIES-2.

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