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PASTING LEMMA FOR αgrw-CONTINUOUS FUNCTIONS

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Abstract. In this paper, some properties of αgrw-continuous functions are discussed and the notion of αgrw-closed graph is introduced.

1. Introduction

The Pasting lemma for continuous functions has applications in algebraic topology. The continuous functions defined on closed sets of a locally finite covering of a topological space can be pasted to form a continuous function on the whole space. Several mathematicians have established pasting lemmas for some stronger and weaker forms of continuous functions. In this paper pasting lemma for αgrw-continuous functions is proved and also αgrw-closed graph functions are introduced in topological spaces.

Throughout this paper, the space (X, τ) (or simply X) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, $cl(A)$, $int(A)$ and $X - A$ (or A^c) denote the closure of A, the interior of A and the complement of A in X respectively.

2. Preliminaries

Definition 2.1. A subset A of a topological space (X, τ) is called

(1) regular open [12] if $A = int(cl(A))$ and regular closed if $A = cl(int(A))$.

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(2) pre-open [7] if $A \subseteq int(cl(A))$ and pre-closed if $cl(int(A)) \subseteq A$

(3) β-open [1] if $A \subseteq cl(int(cl(A)))$ and β-closed if $int(cl(int(A))) \subseteq A$.

(4) α-open [8] if $A \subseteq int(cl(int(A)))$ and α-closed [6] if $cl(int(cl(A))) \subseteq A$

Definition 2.2. [3] A subset A of a space (X, τ) is called regular semi-open if there is a regular open set U such that $U \subseteq A \subseteq cl(U)$. The family of all regular semi-open sets of X is denoted by $RSO(X)$.

Definition 2.3. [9] A subset A of a topological space (X, τ) is said to be *αgrw-closed* if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semi-open.

A subset A of a topological space (X, τ) is said to be *αgrw-open* [11] if A^c is αgrw-closed.

The set of all αgrw-closed sets and αgrw-open sets are denoted by $\alpha grwC(X)$ and $\alpha grwO(X)$ respectively.

Definition 2.4. [6] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be α-closed if $f(U)$ is an α-closed set of (Y, σ) for every closed set U of (X, τ).

Definition 2.5. [10] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be αgrw-continuous if $f^{-1}(V)$ is an αgrw-closed set of (X, τ) for every closed set V of (Y, σ).

Definition 2.6. [10] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *αgrw-irresolute* if $f^{-1}(V)$ is an αgrw-closed set of (X, τ) for every αgrw-closed set V of (Y, σ).

Definition 2.7. [8] A topological space (X, τ) is an α-space if every α-closed subset of (X, τ) is closed in (X, τ).

Definition 2.8. [5] A function $f : (X, \tau) \to (Y, \sigma)$ has an α-closed graph if for each $(x, y) \notin G(f)$, there exists an α-open set U and an open set V containing x and y respectively such that $(U \times cl(V)) \cap G(f) = \emptyset$.

Lemma 2.1. [2] *Let $A \subset Y \subset X$, where X is a topological space and Y is open subspace of X. If $A \in RSO(X)$, then $A \in RSO(Y)$.*

3. αgrw-CONTINUOUS FUNCTIONS

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called regular semi-open* (resp. regular semi-closed*) if $f(V)$ is regular semi-open(resp. regular semi-closed) in (Y, σ) for every regular semi-open(resp. regular semi-closed) set V in (X, τ).

Definition 3.2. A function $f : (X, \tau) \to (Y, \sigma)$ is called regular semi-irresolute if $f^{-1}(V)$ is regular semi-open in (X, τ) for every regular semi-open V in (Y, σ).

Proposition 3.1. *If A is αgrw-closed in a α-space (X, τ) and if $f : (X, \tau) \to (Y, \sigma)$ is regular semi-irresolute and α-closed, then $f(A)$ is αgrw-closed in (Y, σ).*

Proof. Let U be any regular semi-open in (Y, σ) such that $f(A) \subseteq U$. Then $A \subseteq f^{-1}(U)$ and by assumption, $\alpha cl(A) \subseteq f^{-1}(U)$. This implies $f(\alpha cl(A)) \subseteq U$ and $f(\alpha cl(A))$ is α-closed. Now, $\alpha cl(f(A)) \subseteq \alpha cl(f(\alpha cl(A))) = f(\alpha cl(A)) \subseteq U$. Therefore $\alpha cl(f(A)) \subseteq U$ and hence $f(A)$ is αgrw-closed in (Y, σ).

Remark 1. The following examples show that no assumption of the above proposition can be removed.

Example 3.1. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$, $Y = \{p, q, r\}$ and $\sigma = \{\emptyset, \{p\}, \{r\}, \{p, r\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = p, f(b) = f(c) = r$ and $f(d) = q$. Then the function f is regular semi-irresolute and α-closed but $A = \{a\}$ is not an αgrw-closed in a α-space (X, τ) and so $f(A)$ is not an αgrw-closed set in (Y, σ).*

Example 3.2. *In Example* 3.1, *let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = f(c) = r, f(b) = p$ and $f(d) = q$. Then $A = \{a, c\}$ is αgrw-closed, f is α-closed and X is*

α-space but f is not regular semi-irresolute and so $f(A)$ is not an αgrw-closed set in (Y,σ).

Example 3.3. *In Example 3.1, let $f:(X,\tau)\to(Y,\sigma)$ be defined by $f(a)=f(d)=p$ and $f(b)=f(c)=r$. Then $A=\{a,d\}$ is αgrw-closed, f is regular semi-irresolute and X is α-space but f is not α-closed and so $f(A)$ is not an αgrw-closed set in (Y,σ).*

Example 3.4. *Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,\{a\},\{b\},\{a,b\},X\}$, $Y=\{p,q,r\}$ and $\sigma=\{\emptyset,\{p\},\{r\},\{p,r\},Y\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by $f(a)=p, f(b)=f(d)=r$ and $f(c)=q$. Then the function f is regular semi-irresolute, f is α-closed and $A=\{d\}$ is αgrw-closed but X is not α-space and so $f(A)$ is not an αgrw-closed set in (Y,σ).*

Theorem 3.1. *Let f be an αgrw-continuous and regular semi-closed* function from a space (X,τ) to an α-space (Y,σ). Then f is an αgrw-irresolute function.*

Proof. Let A be an αgrw-open subset in (Y,σ) and let F be any regular semi-closed set in (X,τ) such that $F\subseteq f^{-1}(A)$. Then $f(F)\subseteq A$. Since f is regular semi-closed*, $f(F)$ is regular semi-closed. Therefore $f(F)\subseteq \alpha int(A)$ by Theorem 3.1 [11] and so $F\subseteq f^{-1}(\alpha int(A))$. Since f is αgrw-continuous and Y is an α-space, $f^{-1}(\alpha int(A))$ is αgrw-open in (X,τ). Thus $F\subseteq \alpha int(f^{-1}(\alpha int(A))\subseteq \alpha int(f^{-1}(A))$ and so $f^{-1}(A)$ is αgrw-open in (X,τ) by Theorem 3.1 [11]. The proof is similar for αgrw-closed set.

Remark 2. The following examples show that no assumption of the above theorem can be removed.

Example 3.5. *Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,\{a\},\{b,c\},\{a,b,c\},X\}$, $Y=\{p,q,r\}$ and $\sigma=\{\emptyset,\{q\},\{p,q\},\{q,r\},Y\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by $f(a)=f(b)=r, f(c)=q$ and $f(d)=p$. Then the function f is αgrw-continuous and Y is α-space but f is not regular semi-closed* and so f is not αgrw-irresolute.*

Example 3.6. *Let* $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ *and* $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be an identity map. Then the function* f *is* αgrw*-continuous and regular semi-closed* but* Y *is not an* α*-space and so* f *is not* αgrw*-irresolute.*

Example 3.7. *Let* $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$, $Y = \{p, q, r\}$ *and* $\sigma = \{\emptyset, \{p\}, \{r\}, \{p, r\}, Y\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = f(d) = p, f(b) = r$ *and* $f(c) = q$. *Then the function* f *is regular semi-closed* and* Y *is* α*-space but* f *is not* αgrw*-continuous and so* f *is not* αgrw*-irresolute.*

Corollary 3.1. *If* $f : (X, \tau) \to (Y, \sigma)$ *is* αgrw*-continuous and regular semi-closed* and if* A *is* αgrw*-closed (or* αgrw*-open) subset of an* α*-space* (Y, σ)*, then* $f^{-1}(A)$ *is* αgrw*-closed (or* αgrw*-open) in* (X, τ).

Corollary 3.2. *Let* $(X, \tau), (Z, \eta)$ *be a topological spaces and* (Y, σ) *be an* α*-space. If* $f : (X, \tau) \to (Y, \sigma)$ *is* αgrw*-continuous and regular semi-closed* and* $g : (Y, \sigma) \to (Z, \eta)$ *is* αgrw*-continuous then* $g \circ f : (X, \tau) \to (Z, \eta)$ *is* αgrw*-continuous.*

Proof. Let F be any closed set in (Z, η). Since g is αgrw-continuous, $g^{-1}(F)$ is αgrw-closed. By assumption and by Theorem 3.1, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is αgrw-closed in (X, τ) and so $g \circ f$ is αgrw-continuous.

Proposition 3.2. *Let* $f : (X, \tau) \to (Y, \sigma)$ *is* αgrw*-continuous then for each point* x *in* X *and each open set* V *in* Y *with* $f(x) \in V$*, there is an* αgrw*-open set* U *in* X *such that* $x \in U$ *and* $f(U) \subseteq V$.

Proof. Let V be an open set in (Y, σ) and let $f(x) \in V$. Then $x \in f^{-1}(V) \in \alpha grwO(X)$, since f is αgrw-continuous. Let $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subseteq V$.

Remark 3. The converse of the above proposition need not be true as seen from the following example.

Example 3.8. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, $Y = \{p, q, r, s\}$ *and* $\sigma = \{\emptyset, \{p\}, \{q\}, \{p, q\}, Y\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = p, f(b) = q$ *and* $f(c) = r$. *Then for each point* x *in* X *and each open set* V *in* Y *with* $f(x) \in V$, *there is an* αgrw*-open set* U *in* X *such that* $x \in U$ *and* $f(U) \subseteq V$ *but* f *is not* αgrw*-continuous.*

The following theorem is the Pasting lemma for αgrw-continuous functions.

Theorem 3.2. *Let* $X = A \cup B$, *where* A *and* B *are* αgrw*-closed and regular open in* X. *Let* $f : (A, \tau_A) \to (Y, \sigma)$ *and* $g : (B, \tau_B) \to (Y, \sigma)$ *be* αgrw*-continuous such that* $f(x) = g(x)$ *for every* $x \in A \cap B$. *Then the combination* $f \nabla g : (X, \tau) \to (Y, \sigma)$ *defined by* $(f \nabla g)(x) = f(x)$ *if* $x \in A$ *and* $(f \nabla g)(x) = g(x)$ *if* $x \in B$ *is* αgrw*-continuous.*

Proof. Let U be any closed set in Y. Then $(f\nabla g)^{-1}(U) = [(f\nabla g)^{-1}(U) \cap A] \cup [(f\nabla g)^{-1}(U) \cap B] = f^{-1}(U) \cup g^{-1}(U) = C \cup D$, where $C = f^{-1}(U)$ and $D = g^{-1}(U)$. Since f is αgrw-continuous, we have C is αgrw-closed in (A, τ_A) and also since A is αgrw-closed and regular open in X, C is αgrw-closed in X by Proposition 7 [4]. Similarly, D is αgrw-closed in X and by Theorem 3.19[9], $(f\nabla g)^{-1}(U) = C \cup D$ is αgrw-closed in X. Hence $f\nabla g$ is αgrw-continuous.

Definition 3.3. A function $f : (X, \tau) \to (Y, \sigma)$ has an αgrw-closed graph if for each $(x, y) \notin G(f)$, there exists an αgrw-open set U and an open set V containing x and y respectively such that $(U \times cl(V)) \cap G(f) = \emptyset$.

Example 3.9. *Let* $X = \{a, b, c\}$ *with topology* $\tau = \{\emptyset, \{a\}, X\}$ *and* $Y = \{p, q, r\}$ *with topology* $\sigma = P(Y)$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = p, f(b) = q$ *and* $f(c) = r$. *Then* f *has an* αgrw*-closed graph.*

Proposition 3.3. *A function with* α*-closed graph has an* αgrw*-closed graph.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ has an α-closed graph. Then there exists an α-open set U and an open set V containing x and y respectively if for each $(x, y) \in G(f)$ such that $(U \times cl(V)) \cap G(f) = \emptyset$. Since every α-open set is an αgrw-open set[9]. Therefore U is an α-open set. Hence f has an αgrw-closed graph.

Remark 4. The converses of the above proposition need not be true in general. In Example 3.9, the function f has an αgrw-closed graph but not has an α-closed graph.

Lemma 3.1. *The function $f : (X, \tau) \to (Y, \sigma)$ has an αgrw-closed graph if and only if for each $(x, y) \in X \times Y$ such that $f(x) \neq y$, there exist an αgrw-open set U and an open set V containing x and y respectively, such that $f(U) \cap cl(V) = \emptyset$.*

Proof. **Necessity.** Let for each $(x, y) \in X \times Y$ such that $f(x) \neq y$. Then there exist an αgrw-open set U and an open set V containing x and y, respectively, such that $(U \times cl(V)) \cap G(f) = \emptyset$, since f has an αgrw-closed graph. Hence for each $x \in U$ and $y \in cl(V)$ with $y \neq f(x)$, we have $f(U) \cap cl(V) = \emptyset$.

Sufficiency. Let $(x, y) \notin G(f)$. Then $y \neq f(x)$ and so there exist an αgrw-open set U and an open set V containing x and y, respectively, such that $f(U) \cap cl(V) = \emptyset$. This implies, for each $x \in U$ and $y \in cl(V)$, $f(x) \neq y$. Therefore $(U \times cl(V)) \cap G(f) = \emptyset$. Hence f has an αgrw-closed graph.

Theorem 3.3. *If f is an αgrw-continuous function from a space X into a Hausdorff space Y, then f has an αgrw-closed graph.*

Proof. Let $(x, y) \notin G(f)$. Then $y \neq f(x)$. Since Y is Hausdorff space, there exist two disjoint open sets V and W such that $f(x) \in W$ and $y \in V$. Since f is αgrw-continuous, there exists an αgrw-open set U such that $x \in U$ and $f(U) \subseteq W$ by Proposition 3.2. Thus $f(U) \subseteq Y - cl(V)$. Therefore $f(U) \cap cl(V) = \emptyset$ and so f has an αgrw-closed graph.

Theorem 3.4. *If f is a surjective function with an αgrw-closed graph from a space X onto a space Y, then Y is Hausdorff.*

Proof. Let y_1 and y_2 be two distinct points in Y. Then there exists a point $x_1 \in X$ such that $f(x_1) = y_1 \neq y_2$. Thus $(x_1, y_2) \notin G(f)$. Since f has an αgrw-closed graph, there exist an αgrw-open set U and an open set V containing x_1 and y_2, respectively, such that $f(U) \cap cl(V) = \emptyset$ and so $f(x_1) \notin cl(V)$. Hence Y is Hausdorff.

Proposition 3.4. *The space X is Hausdorff if and only if the identity mapping $f : X \to X$ has an αgrw-closed graph.*

Proof. Obvious from Theorem 3.3 and 3.4.

References

[1] M.E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *β-open sets and β-continuous mappings,* Bull. Fac. Sci. Assiut Univ., **12**(1983), 77-90.

[2] S. S. Benchalli and R.S. Wali, *On RW-closed sets in topological spaces*, Bull. Malaysian. Math. Sci. Soc., **(2) 30(2)** (2007), 99-110.

[3] D. E. Cameron, *Properties of S-closed spaces*, Proc. Amer Math. Soc. **72**(1978), 581-586.

[4] Ennis Rosas, N. Selvanayaki and Gnanambal Ilango, *A note on αgrw-closed sets*, Euro. J. Pure and Appl. Math.**9(1)**(2016), 27-33.

[5] I. A. Hasanein, *Topological applications on some supraopen sets*, Ph. D. Thesis, Assiut University (1982).

[6] A. S. Mashhour, I. A. Hasanein and S.N. El-Deeb, *α-continuous and α-open mappings*, Acta. Math. Hungar. **41**(1983), 213-218.

[7] A. S. Mashhour, M. E. Abd. El-Monsef and S. N. El-Deeb, *On pre continuous mappings and weak pre-continuous mappings,* Proc. Math, Phys. Soc. Eqipt., **53** (1982), 47-53.

[8] O. Njastad, *On some classes of nearly open sets*, Pacific J. Math. **15**(1965), 961-970.

[9] N. Selvanayaki and Gnanambal Ilango, *On α-generalized regular weakly closed sets in topological spaces*, Scientia Magna, **9(1)**(2013), 52-58.

[10] N. Selvanayaki and Gnanambal Ilango, *On α-generalized regular weakly continuous functions in topological spaces*, Bull. Kerala Math. Asso., **11(1)** (2014), 103-112.

[11] N. Selvanayaki and Gnanambal Ilango, *Quasi αgrw-open maps in topological spaces*, Jordan J. Math. Stat., **8(2)** (2015), 169 -177.

[12] M. Stone, *Application of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. **41**(1937), 374-481.

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