

PRE-REGULAR P -OPEN SETS AND DECOMPOSITIONS OF COMPLETE CONTINUITY

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ABSTRACT. In this paper, we investigate and characterize pre-regular p -open sets which are defined by Jafari [9] , using the pre-interior and pre-closure operators. By using pre-regular p -open sets, we obtain decompositions of regular open sets and decompositions of complete continuity. It is shown that b^\sharp -closed sets defined in [17] are equivalent to regular open sets. This fact improves many results obtained in [17].

1. INTRODUCTION

The idea of preopen sets on a topological space was introduced by Mashhour et al. [10] in 1982 as a weak form of open sets. In 2006, Jafari [8] introduced a new class of sets, called pre-regular p -open sets, using pre-interior and pre-closure operators. The purpose of this paper is to establish some characterizations of pre-regular p -open sets in topological spaces and to obtain some decompositions of regular open sets by using pre-regular p -open sets. As a consequence, we obtain several decompositions of complete continuity. Furthermore, it is shown that b^\sharp -closed sets defined in [17] are equivalent to regular open sets. This fact improves many results obtained in [17].

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Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) will always denote topological spaces on which no separation axioms are assumed unless explicitly stated.

2. PRELIMINARIES

In general topology, repeated applications of interior (int) and closure (cl) operators give rise to several different classes of sets in the following way.

Definition 2.1. A subset A of a space X is said to be

- i). semi-open [9] if $A \subseteq \text{cl}(\text{int}A)$.
- ii). preopen [10] if $A \subseteq \text{int}(\text{cl}A)$.
- iii). semi-preopen [2] or β -open [1] if $A \subseteq \text{cl}(\text{int}(\text{cl}A))$.
- iv). α -open [12] if $A \subseteq \text{int}(\text{cl}(\text{int}A))$.
- v). regular open [14] if $A = \text{int}(\text{cl}A)$.
- vi). b -open [3] if $A \subseteq \text{cl}(\text{int}A) \cup \text{int}(\text{cl}A)$.
- vii). $b^\#$ -open [17] if $A = \text{cl}(\text{int}A) \cup \text{int}(\text{cl}A)$.

The complements of the above open sets are called their respective closed sets.

Definition 2.2. A subset A of a space X is called

- i) a p -set [15] if $\text{cl}(\text{int}A) \subseteq \text{int}(\text{cl}A)$.
- ii) a q -set [16] δ -set [5] if $\text{int}(\text{cl}A) \subseteq \text{cl}(\text{int}A)$.
- iii) a $D(c, p)$ -set [13] if $\text{int}A = \text{pint}A$.

We recollect some of the relations that, together with their duals, we shall use in the sequel.

Proposition 2.1. *Let A be a subset of a space X . Then*

- i). $\text{pcl}A = A \cup \text{cl}(\text{int}A)$ and $\text{pint}A = A \cap \text{int}(\text{cl}A)$. [2]
- ii). $\text{pcl}(\text{pint}A) = \text{pint}A \cup \text{cl}(\text{int}A)$ and $\text{pint}(\text{pcl}A) = \text{pcl}A \cap \text{int}(\text{cl}A)$. [2]
- iii). $\text{int}(\text{scl}A) = \text{pint}(\text{cl}A) = \text{pint}(\text{scl}A) = \text{scl}(\text{pint}A) = \text{int}(\text{cl}A)$. [2]

$$iv). \text{ } pint(bclA) = bcl(pintA) = pint(pclA).[3]$$

$$v). \text{ } pcl(bintA) = bint(pclA) = pcl(pintA).[3]$$

3. PRE-REGULAR p -OPEN SETS

In this section, we characterize pre-regular p -open sets and study some of their properties.

Definition 3.1. A subset A of a topological space (X, τ) is said to be pre-regular p -open [8] (briefly prp -open) if $A = pint(pclA)$. The complement of a prp -open set is called a prp -closed set. Clearly, A is prp -closed if and only if $A = pcl(pintA)$

We denote the collection of all prp -open (resp. prp -closed) sets of X by $PRO(X)$ (resp. $PRC(X)$).

Theorem 3.1. *Every prp -open set is preopen.*

Proof. This is obvious.

Corollary 3.1. *Every prp -open set is b -open and hence semi-preopen.*

The converse of the above theorem is not true as shown in the following example.

Example 3.1. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $\{a\}$ is open and preopen but it is not prp -open.

Theorem 3.2 shows that prp -open sets and preopen sets are equivalent when A is preclosed.

Theorem 3.2. *Suppose A is preclosed. Then A is prp -open if and only if A is preopen.*

Proof. Let A be prp -open. By Theorem 3.1, A is preopen. Conversely, assume that A is preopen. Then $A = pintA$ and $A = pclA$. Therefore, $pint(pclA) = pintA = A$. Hence A is prp -open.

Theorem 3.3. *Let (X, τ) be a topological space. Then for a subset A of X , the following are equivalent:*

- i). A is prp-open.*
- ii). $A = pclA \cap int(clA)$.*
- iii). $A = pclA \cap scl(pintA)$.*
- iv). $A = pclA \cap int(sclA)$.*
- v). $A = pclA \cap pint(sclA)$.*
- vi). $A = pclA \cap pint(clA)$.*
- vii). $A = (A \cup cl(intA)) \cap int(clA)$.*

Proof. It follows from Proposition 2.1.

Theorem 3.4. *Let (X, τ) be a topological space and $A \subseteq X$. Then A is prp-open if and only if A is b -closed and preopen.*

Proof. Let A be prp-open. Then, by Proposition 2.1(iv) and Theorem 3.1 $A = pint(pclA) = bcl(pintA) = bclA$. Thus A is b -closed. By Theorem 3.1, A is preopen. Conversely assume that A is both b -closed and preopen. Then $A = bclA$ and $A = pintA$. By Proposition 2.1 (iv), $pint(pclA) = pint(bclA) = pintA = A$. Hence A is prp-open.

Theorem 3.5. *Suppose A is a p -set. Then A is prp-open if and only if A is prp-closed.*

Proof. Since A is a p -set, $cl(intA) \subseteq int(clA)$. By using Proposition 2.1 (i) (ii), $pcl(pintA) = pintA \cup cl(intA) = (A \cap int(clA)) \cup cl(intA) = [A \cup cl(intA)] \cap [int(clA) \cup cl(intA)] = [A \cup cl(intA)] \cap (int(clA)) = pclA \cap int(clA) = pint(pclA)$. This completes the proof.

4. CHARACTERIZATIONS OF REGULAR OPEN SETS

In this section, we characterize regular open sets by using prp -open sets.

Theorem 4.1. *Let (X, τ) be a topological space and $A \subseteq X$. Then A is regular open if and only if it is prp -open and open.*

Proof. Let A be regular open. Then $A = \text{int}(clA)$ and hence A is open. By Proposition 2.1 (ii), $\text{pint}(pclA) = pclA \cap \text{int}(clA) = pclA \cap A = A$. Thus A is prp -open. Conversely, assume that A is prp -open and open. By Theorem 3.3 (vii), $A = (A \cup cl(\text{int}A)) \cap \text{int}(clA) = (A \cup clA) \cap \text{int}(clA) = clA \cap \text{int}(clA) = \text{int}(clA)$. Hence A is regular open.

Example 4.1. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b, c\}, X\}$. Then $\{a, c\}$ is prp -open but it is not a q -set and not open. Also $\{b, c\}$ is open but it is not prp -open.*

Remark 1. [See Example 2.2 [8]] Since \emptyset and X are regular open, we have $\emptyset, X \in PRO(X)$. Moreover, the family of prp -open sets $PRO(X)$ is not closed under finite union as well as finite intersection.

Remark 2. Since every open set is a q -set, Theorem 4.1 may be improved as Theorem 4.2. Moreover, in Example 4.1, $\{a, c\}$ is prp -open but it is not a q -set. Therefore, " q -set" and " prp -open set" are independent of each other.

Theorem 4.2. *Let (X, τ) be a topological space and $A \subseteq X$. Then A is regular open if and only if it is prp -open and a q -set.*

Proof. Let A be regular open. Then by Theorem 4.1, A is prp -open and open. Hence A is prp -open and a q -set. Conversely, since A is a q -set, $\text{int}(clA) \subseteq cl(\text{int}A)$. By Theorem 3.3, $A = \text{pint}(pclA) = pclA \cap \text{int}(clA) = (A \cup cl(\text{int}A)) \cap \text{int}(clA) = (A \cap \text{int}(clA)) \cup (cl(\text{int}A) \cap \text{int}(clA)) = (A \cap \text{int}(clA)) \cup (\text{int}(clA)) = \text{int}(clA)$. Hence A is regular open.

Remark 3. If the classes of regular open sets, preopen sets, preclosed sets and q -sets of a topological space (X, τ) are denoted by $RO(X)$, $PO(X)$, $PC(X)$ and $q(X)$, respectively, then Theorems 3.1 and 3.2 show that $PO(X) \cap PC(X) \subseteq PRO(X) \subseteq PO(X)$. Moreover, Theorems 4.1 and 4.2 show that $\tau \cap PRO(X) = RO(X) = q(X) \cap PRO(X)$.

Theorem 4.3. *Let (X, τ) be a topological space. A subset A of X is regular open if and only if it is $b^\#$ -closed.*

Proof. Let A be regular open. Since every regular open set is open, we have $\text{int}(clA) \cap cl(\text{int}A) = A \cap cl(A) = A$. Hence A is $b^\#$ -closed. Conversely, let A be $b^\#$ -closed. Then we have $\text{int}(cl(\text{int}A)) \subset \text{int}(clA) \cap cl(\text{int}A) = A$. Since every $b^\#$ -closed set is semi-open and preopen, we have $A \subset \text{int}(clA) \subset \text{int}(cl(cl(\text{int}A))) = \text{int}(cl(\text{int}A))$. Hence $A = \text{int}(cl(\text{int}A))$. Therefore, we have $\text{int}(clA) = \text{int}(cl(\text{int}(cl(\text{int}A)))) = \text{int}(cl(\text{int}A)) = A$. This shows that A is regular open.

Remark 4. By Theorems 3.1 and 4.3, we have the following diagram:

DIAGRAM I

$$\begin{array}{ccccccc}
 \text{regular open} & \Rightarrow & \text{open} & \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{semi-open} \Rightarrow q\text{-set} \\
 & \Downarrow & & & \Downarrow & & \Downarrow \\
 b^\#\text{-closed} & \Rightarrow & prp\text{-open} & \Rightarrow & \text{preopen} & \Rightarrow & b\text{-open} \Rightarrow \text{semi-preopen}
 \end{array}$$

Corollary 4.1. *Let (X, τ) be a topological space. For a subset A of X , the following properties are equivalent:*

- i). A is regular open.
- ii). A is $b^\#$ -closed.
- iii). A is prp -open and a q -set.

Proof. This is an immediate consequence of Theorems 4.2 and 4.3.

Corollary 4.2. [17] *Let (X, τ) be a topological space. For a subset A of X , the following properties are equivalent:*

- i). A is $b^\#$ -closed.*
- ii). A is prp -open and semi-open.*

Theorem 4.4. *For a subset A of X , the following properties are equivalent:*

- i). A is regular open.*
- ii). A is prp -open and semi-closed.*
- iii). A is prp -open and semi-open.*
- iv). A is prp -open and a $D(c, p)$ -set.*

Proof. $i) \Leftrightarrow ii)$. It is obvious that $i) \Rightarrow ii)$. Conversely, let A be prp -open and semi-closed. Then, since A is semi-closed, $int(clA) \subset A$ and $int(clA) \subset cl(intA)$. Therefore, A is a q -set and by Theorem 4.2 A is regular open.

$i) \Leftrightarrow iii)$. It is obvious that $i) \Rightarrow iii)$. Conversely, let A be prp -open and semi-open. Then since every semi-open set is a q -set, A is a q -set and by Theorem 4.2 A is regular open.

$i) \Leftrightarrow iv)$. Since every open set is a $D(c, p)$ -set, it is obvious that $i) \Rightarrow iv)$. Conversely, suppose A is a $D(c, p)$ -set. Then $intA = pintA$. Therefore, by Proposition 2.1 (iv) and [3, Proposition 2.6 (i)], $A = pint(pclA) = bcl(pintA) = bcl(intA) = int(cl(intA))$. Hence A is open and hence by Theorem 4.1 A is regular open.

5. DECOMPOSITIONS OF COMPLETE CONTINUITY

In this section, the notion of prp -continuous functions is introduced and the decomposition of a completely continuous function is discussed.

Definition 5.1. A function $f : X \rightarrow Y$ is said to be

- i). pre-regular p -continuous (briefly prp -continuous) if $f^{-1}(V)$ is prp -open in X for each open subset V of Y .

- ii). completely continuous [4] if the inverse image of every open subset of Y is a regular open subset of X .
- iii). α -continuous [11] (resp. semi-continuous [9], precontinuous [10], b -continuous [7], β -continuous [1]) if the inverse image of every open subset of Y is α -open (resp. semi-open, preopen, b -open, β -open) in X .
- iv). q -continuous [16] if the inverse image of every open subset of Y is a q -set in X .
- v). contra-semicontinuous [6] if the inverse image of every open subset of Y is semi-closed in X .
- vi). contra $b^\#$ -continuous if the inverse image of every open subset of Y is $b^\#$ -closed in X .

By Theorems 3.1 and 4.3, we obtain the following proposition.

- Proposition 5.1.** *i). Every prp-continuous function is pre-continuous.*
ii). Every contra $b^\#$ -continuous function is prp-continuous.
iii). Contra $b^\#$ -continuity is equivalent to complete continuity.

Remark 5. By Diagram I, we have the following diagram:

DIAGRAM II

$$\begin{array}{ccccccc}
 \text{complete conti.} & \Rightarrow & \text{conti.} & \Rightarrow & \alpha\text{-conti.} & \Rightarrow & \text{semi-conti.} \Rightarrow q\text{-conti.} \\
 \Downarrow & & & & \Downarrow & & \Downarrow \\
 \text{contra } b^\#\text{-conti.} & \Rightarrow & \text{prp-conti.} & \Rightarrow & \text{preconti.} & \Rightarrow & b\text{-conti.} \Rightarrow \beta\text{-conti.}
 \end{array}$$

where "conti." denotes "continuity".

By Theorems 4.1, 4.2 and 4.4, we have the following main theorem.

Theorem 5.1. *For a function $f : X \rightarrow Y$, the following properties are equivalent:*

- i). f is completely continuous.*
- ii). f is continuous and prp-continuous.*
- iii). f is α -continuous and prp-continuous.*

- iv). f is semi-continuous and prp-continuous.*
- v). f is q -continuous and prp-continuous.*
- vi). f is contra-semicontinuous and prp-continuous.*

Remark 6. i). *prp*-continuity and continuity are independent of each other.

ii). *prp*-continuity and α -continuity are independent of each other.

iii). *prp*-continuity and semi-continuity are independent of each other.

iv). *prp*-continuity and q -continuity are independent of each other.

v). *prp*-continuity and contra-semicontinuity are independent notions.

Example 5.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{a, b\}, X\}$. Then

(i). The identity function $f : (X, \tau) \rightarrow (X, \tau)$ is continuous but it is not *prp*-continuous since $f^{-1}(\{a\}) = \{a\}$ is not *prp*-open.

(ii). Consider the function $f : (X, \sigma) \rightarrow (X, \tau)$ defined by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then f is *prp*-continuous but it is not q -continuous, since $f^{-1}(\{a\}) = \{a\}$ is not a q -set in (X, σ) .

Example 5.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, X\}$.

(i). Consider the function $f : (X, \tau) \rightarrow (X, \sigma)$ defined by $f(a) = a$, $f(b) = b$ and $f(c) = b$. Then f is *prp*-continuous but it is not contra-semicontinuous since $f^{-1}(\{a\}) = \{a\}$ is not semi-closed in (X, τ) .

(ii). Consider the function $f : (X, \tau) \rightarrow (X, \sigma)$ defined by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then f is contra semi-continuous but it is not *prp*-continuous, since $f^{-1}(\{a\}) = \{c\}$ is not *prp*-open in (X, τ) .

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