Jordan Journal of Mathematics and Statistics (JJMS) 9(2), 2016, pp 73 - 92

A GENERALIZATION OF SLANT TOEPLITZ OPERATORS

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Abstract. We ask about the solutions of the equation $\lambda M_z X = X M_{z^2}$, for general complex number λ, which are referred as $\lambda-$slant Toeplitz operators. We completely solve this equation and discuss some algebraic as well as spectral properties of $\lambda-$slant Toeplitz operators. The compactness of the compression of $\lambda-$slant Toeplitz operators is also addressed.

1. Introduction

Let μ denote the normalized Lebesgue measure on the unit circle $\mathbb{T}$ (the boundary of the unit disc $\mathbb{D}$) and L^2 the Hilbert space of all complex-valued measurable functions f defined on $\mathbb{T}$ satisfying

$$\int |f|^2 d\mu \ < \infty.$$

The inner product on L^2 is given by $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\overline{g}d\theta$ for $f, g \in L^2$ and $\{e_n : n \in \mathbb{Z}\}$, where $e_n(z) = z^n$ for each $z \in \mathbb{T}$, denotes the standard orthonormal basis of L^2. The Hardy space H^2 of analytic functions in the open unit disc $\mathbb{D}$ is defined as

$$H^2 = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

2010 *Mathematics Subject Classification.* 47B35.

Key words and phrases. Toeplitz operators, slant Toeplitz operators, compression, compact operator.

Received: April 4, 2015 Accepted: May 5, 2016.

It is customary to identify the functions of H^2 with the space of their boundary functions (see [2,7,16]). The boundary functions correspond to those functions in L^2 whose negative Fourier coefficients vanish. With this identification, H^2 is a closed subspace of L^2. The space of all essentially bounded measurable functions on $\mathbb{T}$ is denoted by L^∞. Toeplitz operators on the space L^2 are nothing but the operators in the commutant of M_z, the bilateral shift and thus can be written as solutions to the operator equation $M_z X = X M_z$. In the year 1911, Toeplitz [20] introduced Toeplitz operators on the Hardy space H^2, which are characterized by the operator equation $U^*XU = X$, where U is the forward unilateral shift operator on the Hardy space H^2. The ideas and methods prevailing in the field of Toeplitz operators are a fascinating illustration of the fruitful interplay between operator theory, complex analysis and a Banach algebra. Barría and Halmos [3] studied a generalization of the equation $U^*XU = X$ and asked about the solutions of the equation $U^*XU = \lambda X$, for general complex number λ, the solutions of which are referred as $\lambda-$Toeplitz operators [6, 13, 14]. This problem was completely solved by S. Sun [19] in the year 1983, where he proved that the only $\lambda-$Toeplitz operator for $|\lambda| > 1$ is the zero operator and the equation $U^*XU = \lambda X$ has non-zero bounded solutions if and only if $|\lambda| \leq 1$. There is an interesting overlapping between $\lambda-$Toeplitz operators and Toeplitz-composition operators (which are expressed as product of Teoplitz operators and composition operators). In fact, $\lambda-$Toeplitz operators are described as Toeplitz-composition operators for $|\lambda| = 1$ and are written as the sum of weighted composition operators and their adjoint for $|\lambda| < 1$ (see [13, 14]). For the theory of composition and weighted composition operators, we refer the reader to [5 and 15]. The essential spectrum and spectrum of $\lambda-$Toeplitz operators for some specific symbols are computed by Ho in [14]. The study is carried forward with the introduction of the notions of analytic $\lambda-$Toeplitz and essentially $\lambda-$Toeplitz operators in [6], where spectral as well as algebraic properties of these operators are explored.

Ho [12] described the notion of slant Toeplitz operators on the space L^2, which are defined as operators whose matrix with respect to the standard orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ is given by

$$\begin{bmatrix} \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{-2} & a_{-3} & \cdots & \cdots & \cdots \\ \cdots & a_0 & a_{-1} & a_{-2} & \cdots & \cdots \\ \cdots & \cdots & a_1 & a_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix},$$

where $\phi = \sum_{n\in\mathbb{Z}} a_n e_n \in L^\infty$ with $a_n = \langle \phi, e_n \rangle$, the n^{th}-Fourier coefficient of ϕ. Equivalently, slant Toeplitz operators on the space L^2 are defined as $A_\phi = WM_\phi$ for $\phi \in L^\infty$, where $We_{2n} = e_n$, $We_{2n+1} = 0$ for each $n \in \mathbb{Z}$ and M_ϕ is the Laurent operator on L^2 induced by the symbol ϕ. Many authors have shown interest in the spectral properties of slant Toeplitz operators and tried to apply them to the theory of wavelets. The smoothness of the wavelets is related to the spectral properties of the slant Toeplitz operators (see [8], [11], [17], [18]). For example, the Besov regularity of solutions of the refinement equation has been associated with the spectral radius of an associated slant Toeplitz operator [21]. Slant Toeplitz operators are also characterized as the solutions of the operator equation $M_z X = XM_{z^2}$ [12]. Motivated by the approach initiated by Halmos and Barría [3] and Sun [19], our interest is prompted in studying the operator equation $\lambda M_z X = XM_{z^2}$ (which is same as $\lambda X = M_{\bar{z}} XM_{z^2}$) for given complex number λ. As the study of the equation $\lambda X = M_{\bar{z}} XM_{z^2}$ is more or less equivalent to the study of the equation $M_{\bar{z}} XM_{z^2} - X = \lambda X$ so using the nomenclature methodology of [2] and [6], the solutions of the equation $\lambda M_z X = XM_{z^2}$ are named as "$\lambda-$slant Toeplitz operators". The existence of non-zero $\lambda-$ Toeplitz operators is shown for $|\lambda| \leq 1$ by Sun [19], whereas we show that the only $\lambda-$slant

Toeplitz operator for $|\lambda| \neq 1$ is the zero operator. It may not be surprising to know that $\lambda-$slant Toeplitz operators and slant Toeplitz operators can be more or less connected through the classical slant Toeplitz operators if $|\lambda| = 1$. However, in this case, $\lambda-$slant Toeplitz operators fall in the category of slant Toeplitz -composition operators, i.e., operators which can be expressed as the product of slant Toeplitz operators and composition operators. The study of slant weighted Toeplitz operators was made in [1] and we extend some results similar to that known for slant weighted Toeplitz operators to $\lambda-$slant Toeplitz operators. This paper also deals with the spectral properties of $\lambda-$slant Toeplitz operators. Properties of compressions of $\lambda-$slant Toeplitz operators to H^2, the Hardy space of analytic functions, are also discussed. The algebra of all bounded operators on the Hilbert space L^2 is denoted by $\mathfrak{B}(L^2)$. We use the symbol $\mathcal{K}$ to represent the set of all compact operators on the Hilbert space L^2. If two operators A and B on L^2 differ by a compact operator then we write it as $A = B\ Mod(\mathcal{K})$. The norm of an operator A in the Calkin algebra $\mathfrak{B}(L^2)/\mathcal{K}$ is denoted by $\|A\|_e$.

2. $\lambda-$slant Toeplitz operators

Slant Toeplitz operators are characterized as the operators satisfying the operator equation $M_z X = X M_{z^2}$. Motivated by the direction initiated by Brown and Halmos [3], we ask about the solutions of the equation $\lambda M_z X = X M_{z^2}$, for general complex number λ. We begin with the following definition.

Definition 2.1. For a fixed complex number λ, an operator X on L^2 is said to be $\lambda-$slant Toeplitz operator if it is a solution of the equation $\lambda M_z X = X M_{z^2}$.

Evidently, $\lambda-$slant Toeplitz operators on L^2 can be viewed as operators satisfying $\langle Ae_{m+2}, e_{n+1}\rangle = \lambda\langle Ae_m, e_n\rangle$ for $m, n \in \mathbb{Z}$, for $\lambda \in \mathbb{C}$. In order to solve the above equation completely, we need the following lemma.

Lemma 2.1. *An operator X on L^2 is a solution of the equation $AX = XM_{z^2}$ $Mod(\mathcal{K})$ if and only if X is compact and is of the form*

$$X = \sum_{n=0}^{\infty} A^n K M_{\overline{z}}^{2n}$$

for some compact operator K, where $\|A\| < 1$.

Proof. Suppose that X satisfies $AX = XM_{z^2}$ $Mod(\mathcal{K})$. Then $AX - XM_{z^2} = K$, for some $K \in \mathcal{K}$. Hence $AXM_{\overline{z}}^2 - X = KM_{\overline{z}}^2$, i.e. $AXM_{\overline{z}}^2 = XMod(\mathcal{K})$. In order to show that X is compact, we show that $\|X\|_e = 0$. On contrary, if we assume that $\|X\|_e \neq 0$ then

$$\begin{aligned}
\|X\|_e &= \|X - AXM_{\overline{z}}^2 + AXM_{\overline{z}}^2\|_e \\
&= \|KM_{\overline{z}}^2 + AXM_{\overline{z}}^2\|_e \\
&\leq \|KM_{\overline{z}}^2\|_e + \|AXM_{\overline{z}}^2\|_e \\
&= \|AXM_{\overline{z}}^2\|_e \\
&\leq \|A\|_e\|X\|_e\|M_{\overline{z}}^2\|_e \\
&< \|X\|_e,
\end{aligned}$$

which is an absurd. Thus, X is a compact operator.

Now define $\tau : \mathfrak{B}(L^2) \to \mathfrak{B}(L^2)$ such that $\tau(X) = AXM_{\overline{z}}^2$. This gives that $\|\tau\| \leq \|A\| < 1$. Therefore $(I - \tau)$ is invertible and $(I-\tau)^{-1} = \sum_{n=0}^{\infty} \tau^n$. This on simple calculations gives that $X = \sum_{n=0}^{\infty} \tau^n(KM_{\overline{z}}^2)$, which means that $X = \sum_{n=0}^{\infty} A^n K M_{\overline{z}}^{2n}$ for some compact operator K.

Conversely, if X is compact and is of the form $X = \sum_{n=0}^{\infty} A^n K M_{\overline{z}}^{2n}$, for a compact operator K, then simple calculations show that X satisfies $AX = XM_{z^2}$ $Mod(\mathcal{K})$. □

Now, we use Lemma 2.1 to conclude the following, which helps us to solve the equation $\lambda M_z X = XM_{z^2}$ for $\lambda \in \mathbb{C}$.

Theorem 2.1. *If X is a solution of $AX = XM_{z^2}$, $\|A\| < 1$, then $X = 0$.*

Proof. If $AX = XM_{z^2}$ then $AXM_{\overline{z}}^2 - X = 0$. Hence X is an eigenvector of τ and invertibility of $(I - \tau)$ provides that $X = 0$. □

If $\lambda \in \mathbb{C}$ is such that $|\lambda| < 1$ then on replacing A by λM_z in Theorem 2.1, we obtain the following.

Corollary 2.1. *If X is a solution of $\lambda M_z X = XM_{z^2}$, $|\lambda| < 1$ then $X = 0$.*

Now, we consider the operator equation $\lambda M_z X = XM_{z^2}$, where $\lambda \in \mathbb{C}$ is such that $|\lambda| > 1$. It is clear that each solution of this equation is an eigenvector of the mapping $\tau : \mathfrak{B}(L^2) \to \mathfrak{B}(L^2)$ defined as $\tau(X) = M_{\overline{z}} X M_{z^2}$, corresponding to the eigenvalue λ. Since $||\tau|| \leq 1$ so λ with $|\lambda| > 1$ is in the resolvent set of τ and the corresponding eigenvector is zero operator. This provides that the only solution of the operator equation $\lambda M_z X = XM_{z^2}$, where $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, is the zero operator.

From above discussion we have observed that for λ, $|\lambda| \neq 1$, the operator equation $\lambda M_z X = XM_{z^2}$ has zero solution only. We now claim the following.

Theorem 2.2. *For $\lambda \in \mathbb{C}$, $|\lambda| = 1$, the solutions of the operator equation $\lambda M_z X = XM_{z^2}$ are always of the form $X = D_{\overline{\lambda}} A$, where A is a slant Toeplitz operator and $D_{\overline{\lambda}}$ is the composition operator on L^2 induced by $z \mapsto \lambda z$, i.e, $D_{\overline{\lambda}} f(z) = f(\lambda z)$ for all $f \in L^2$.*

Proof. Suppose X is an operator of the form $D_{\overline{\lambda}} A$ for some slant Toeplitz operator A. Since $M_z D_{\overline{\lambda}} = \overline{\lambda} D_{\overline{\lambda}} M_z$ and A is a slant Toeplitz operator we get that $\lambda M_z X = XM_{z^2}$.

Suppose that X is an operator satisfying $\lambda M_z X = XM_{z^2}$. Premultiplying by D_λ we get that $M_z D_\lambda X = D_\lambda X M_{z^2}$. Therefore, $X = D_{\overline{\lambda}} A$ for some slant Toeplitz operator A. □

Since slant Toeplitz operators are always of the form $A_\phi (= WM_\phi)$, $\phi \in L^\infty$, hence in view of Theorem 2.2 for each ϕ in L^∞ and λ in $\mathbb{C}$ with $|\lambda| = 1$, we have a $\lambda-$slant

Toeplitz operator $A_{\phi,\lambda} = D_{\overline{\lambda}} A_\phi$. For $\phi = \sum_{n\in\mathbb{Z}} a_n e_n$ in L^∞, the matrix representation of $\lambda-$slant Toeplitz operator $A_{\phi,\lambda}$ with respect to the standard orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ is

$$\begin{bmatrix} \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \lambda^{-1}a_{-1} & \lambda^{-1}a_{-2} & \lambda^{-1}a_{-3} & \lambda^{-1}a_{-4} & \cdots \\ \cdots & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \cdots & \lambda a_3 & \lambda a_2 & \lambda a_1 & \lambda a_0 & \cdots \\ \cdots & \lambda^2 a_5 & \lambda^2 a_4 & \lambda^2 a_3 & \lambda^2 a_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}.$$

Since for $|\lambda| \neq 1$, the only $\lambda-$slant Toeplitz operator is the zero operator so throughout our discussion we take $|\lambda| = 1$. It is clear that for $|\lambda| = 1$, $\phi \in L^\infty$, we have $\|A_{\phi,\lambda}\| \leq \|\phi\|_\infty$.

For $\phi = \sum_{n\in\mathbb{Z}} a_n e_n \in L^\infty$, $\lambda \in \mathbb{C}$, the adjoint of $A_{\phi,\lambda}$ satisfies $A^*_{\phi,\lambda} = (D_{\overline{\lambda}} A_\phi)^* = A^*_\phi D_\lambda$ and for each $i, j \in \mathbb{Z}$, $\langle A^*_{\phi,\lambda} e_j, e_i\rangle = \langle e_j, A_{\phi,\lambda} e_i\rangle = \langle e_j, \sum_{m\in\mathbb{Z}} \lambda^m a_{2m-i} e_m\rangle = \overline{\lambda}^j \overline{a}_{2j-i}$. This helps us to prove the following.

Theorem 2.3. *Adjoint of a non-zero $\lambda-$slant Toeplitz operator is not a $\lambda-$slant Toeplitz operator.*

Proof. Let, if possible, $A^*_{\phi,\lambda}$ be a non-zero $\lambda-$slant Toeplitz operator. Then for each $i, j \in \mathbb{Z}$,

$$\begin{aligned} &\lambda\langle A^*_{\phi,\lambda} e_j, e_i\rangle = \langle A^*_{\phi,\lambda} e_{j+2}, e_{i+1}\rangle \\ &\Rightarrow \lambda\overline{\lambda}^j \overline{a}_{2j-i} = \overline{\lambda}^{(j+2)} \overline{a}_{2j-i+3} \\ &\Rightarrow \overline{a}_{2j-i} = \overline{\lambda}^3 \overline{a}_{2j-i+3} \end{aligned}$$

This on substituting $j = 0$ provides that $\overline{a}_0 = \overline{\lambda}^{3n} \overline{a}_{3n}$, $\overline{a}_1 = \overline{\lambda}^{3n} \overline{a}_{3n+1}$ and $\overline{a}_2 = \overline{\lambda}^{3n} \overline{a}_{3n+2}$ for each $n \in \mathbb{Z}$. Since $a_n \to 0$ as $n \to \infty$, we get that $\overline{a}_0 = \overline{a}_1 = \overline{a}_2 = 0$.

As a consequence $\phi = 0$, which contradicts that $A^*_{\phi,\lambda}$ is non-zero. This completes the proof. □

In order to compute the norm of the $\lambda-$slant Toeplitz operator $A_{\phi,\lambda}$, we prove the following.

Lemma 2.2. *Product of a $\lambda-$slant Toeplitz operator and its adjoint is a Laurent operator.*

Proof. Using [12, Proposition 5], we have $A_{\phi,\lambda}A^*_{\phi,\lambda} = D_{\overline{\lambda}}A_\phi A_\phi{}^* D_\lambda = D_{\overline{\lambda}}M_\psi D_\lambda$, where $\psi = W(|\phi|^2) = \sum_{m\in\mathbb{Z}} \langle \psi, e_m\rangle e_m \in L^\infty$. Now for each $n \in \mathbb{Z}$,

$$\begin{aligned} D_{\overline{\lambda}}M_\psi D_\lambda e_n &= \overline{\lambda}^n D_{\overline{\lambda}} \sum_{m\in\mathbb{Z}} \langle \psi, e_m\rangle e_{m+n} \\ &= (\sum_{m\in\mathbb{Z}} \langle \psi, e_m\rangle \lambda^m e_m)e_n. \end{aligned}$$

Therefore $A_{\phi,\lambda}A^*_{\phi,\lambda} = M_{\xi_\lambda}$, a Laurent operator with symbol ξ_λ in L^∞ given by $\xi_\lambda(z) = \sum_{n\in\mathbb{Z}} \langle \psi, e_n\rangle \lambda^n z^n$ for $z \in \mathbb{T}$. □

This lemma provides the following.

Theorem 2.4. *For $\phi \in L^\infty$ and $\lambda \in \mathbb{C}$, $\|A_{\phi,\lambda}\| = \sqrt{\|\xi_\lambda\|_\infty}$, where $\xi_\lambda(z) = \sum_{n\in\mathbb{Z}} \langle \psi, e_n\rangle \lambda^n z^n$.*

Proof. Proof follows as a consequence of Lemma 2.2 and the fact that $\|A_{\phi,\lambda}\|^2 = \|A_{\phi,\lambda}A^*_{\phi,\lambda}\|$. □

It is apparent to see that the sum of two $\lambda-$slant Toeplitz operators is a $\lambda-$slant Toeplitz operator. However, the following properties of $\lambda-$slant Toeplitz operators, which are known for slant Toeplitz operators (see [12], [1]), can be proved without any extra efforts.

Proposition 2.1. *Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

(a) The mapping $\phi \mapsto A_{\phi,\lambda}$ from L^∞ into $\mathfrak{B}(L^2)$ is linear and one-one.

(b) The set of all $\lambda-$slant Toeplitz operators is weakly closed and hence strongly closed.

(c) For an unimodular complex number μ, $D_{\overline{\mu}\lambda}A_{\phi,\lambda}$ is a $\mu-$slant Toeplitz operator.

(d) A $\lambda-$slant Toeplitz operator $A_{\phi,\lambda}$ for $\phi \in L^\infty$ is compact if and only if $\phi = 0$.

(e) For unimodular ϕ in L^∞, $A_{\phi,\lambda}$ is a coisometry.

In Proposition 2.1 (e), we have shown that for unimodular $\phi \in L^\infty$, $A_{\phi,\lambda}$ is always a coisometry. For the general $\phi \in L^\infty$, we characterize the coisometry of $A_{\phi,\lambda}$ in terms of ϕ in the following form.

Theorem 2.5. *Let $\lambda = e^{\iota\hat{\theta}}$, $\hat{\theta} \in [0, 2\pi[$. Then $A_{\phi,\lambda}$ is coisometry if and only if $|\phi(\frac{\theta}{2})|^2 + |\phi(\frac{\theta+2\pi}{2})|^2 = 2$ for a.e. $\theta \in [0, 2\pi[$.*

Proof. For $f \in L^2$,

$$\begin{aligned}
||A^*_{\phi,\lambda}f||_2^2 &= ||M_{\overline{\phi}}W^*D_\lambda f||_2^2 \\
&= \int_0^{2\pi} |\phi(\theta)|^2 |f(-\hat{\theta}+2\theta)|^2 \frac{d\theta}{2\pi} \\
&= \frac{1}{2}\int_0^{4\pi} |\phi(\frac{\theta}{2})|^2 |f(-\hat{\theta}+\theta)|^2 \frac{d\theta}{2\pi} \\
&= \frac{1}{2}\int_0^{2\pi} \{|\phi(\frac{\theta}{2})|^2 + |\phi(\frac{\theta+2\pi}{2})|^2\} |f(-\hat{\theta}+\theta)|^2 \frac{d\theta}{2\pi} \\
&= ||M_\psi \widehat{f}||_2^2
\end{aligned}$$

where $\psi(\theta) = \{\frac{1}{2}\{|\phi(\frac{\theta}{2})|^2 + |\phi(\frac{\theta+2\pi}{2})|^2\}\}^{1/2}$ and $\widehat{f} = D_\lambda f$. As a consequence, $A_{\phi,\lambda}$ is coisometry if and only if $|\psi| = 1$ a.e. on $\mathbb{T}$, equivalently, $\{|\phi(\frac{\theta}{2})|^2 + |\phi(\frac{\theta+2\pi}{2})|^2\} = 2$ for a.e. $\theta \in [0, 2\pi[$. □

Now we find that the only hyponormal $\lambda-$slant Toeplitz operator on L^2 is the zero operator.

Theorem 2.6. *A $\lambda-$slant Toeplitz operator $A_{\phi,\lambda}$ is hyponormal if and only if $\phi = 0$.*

Proof. Suppose $\lambda-$slant Toeplitz operator $A_{\phi,\lambda}$ is hyponormal. Then for all $f \in L^2$, $||A_{\phi,\lambda}f|| \geq ||A^*_{\phi,\lambda}f||$. On substituting $f = e_0$ in above inequality, we have $\sum_{n\in\mathbb{Z}} |a_{2n}|^2 \geq \sum_{n\in\mathbb{Z}} |\overline{a}_n|^2$, which implies that $a_{2n-1} = 0$ for all $n \in \mathbb{Z}$. Now on substituting $f = e_1$ in the inequality, we find $\sum_{n\in\mathbb{Z}} |a_{2n-1}|^2 \geq \sum_{n\in\mathbb{Z}} |\overline{a}_{2-n}|^2$, which yields that $a_{2-n} = 0$ for all $n \in \mathbb{Z}$. Thus $\phi = 0$.

Converse is obvious. □

As a consequence of Theorem 2.6 and the fact that an isometry is always hyponormal, the set of $\lambda-$slant Toeplitz operators does not contain an isometry.

We now raise the following questions about the product of a $\lambda-$slant Toeplitz operator with operators of other classes.

(1) Is the product of a $\lambda-$slant Toeplitz operator with a slant Toeplitz operator, a $\lambda-$slant Toeplitz operator?

(2) What happens if we replace a slant Toeplitz operator by a Laurent operator in (1)?

We first provide a characterization for the product $A_\psi A_{\phi,\lambda}$ to be a $\lambda-$slant Toeplitz operator, where $\phi, \psi \in L^\infty$, for which we need the following.

Lemma 2.3. *For $\phi \in L^\infty$, $WA_{\phi,\lambda}$ is a $\lambda-$slant Toeplitz operator if and only if $\phi = 0$.*

Proof. If part of the result is obvious. We prove the reverse part. For, suppose that $WA_{\phi,\lambda}$ is a $\lambda-$slant Toeplitz operator. Then

$$\begin{aligned}
&\lambda\langle WA_{\phi,\lambda}e_j, e_i\rangle = \langle WA_{\phi,\lambda}e_{j+2}, e_{i+1}\rangle \\
&\Rightarrow \lambda\langle D_{\overline{\lambda}}A_\phi e_j, e_{2i}\rangle = \langle D_{\overline{\lambda}}A_\phi e_{j+2}, e_{2i+2}\rangle \\
&\Rightarrow \langle A_\phi e_j, e_{2i}\rangle = \lambda\langle A_\phi e_{j+2}, e_{2i+2}\rangle \\
&\Rightarrow a_{4i-j} = \lambda a_{4i-j+2}
\end{aligned}$$

Put $i = 0$, we get $a_{-j} = \lambda a_{2-j}$ for each $j \in \mathbb{Z}$. This implies that $a_0 = \lambda^n a_{2n}$ and $a_1 = \lambda^n a_{2n+1}$. Since $a_n \to 0$ as $n \to \infty$, we get that $a_0 = a_1 = 0$. Therefore $\phi = 0$. □

Theorem 2.7. *Let $\phi, \psi \in L^\infty$. Then $A_\psi A_{\phi,\lambda}$ is a $\lambda-$slant Toeplitz operator if and only if $\psi(\overline{\lambda}z^2)\phi(z) = 0$.*

Proof. Suppose $\phi, \psi \in L^\infty$. Then $A_\psi A_{\phi,\lambda} = WM_\psi D_{\overline{\lambda}} WM_\phi = WD_{\overline{\lambda}}\ M_{\psi(\overline{\lambda}z)} WM_\phi = WD_{\overline{\lambda}} WM_{\psi(\overline{\lambda}z^2)} M_{\phi(z)} = WA_{\psi(\overline{\lambda}z^2)\phi(z),\lambda}$. So the product is $\lambda-$slant Toeplitz operator if and only if $\psi(\overline{\lambda}z^2)\phi(z) = 0$. □

The next result gives a precise answer to question (1) and provides an affirmative answer to it once either a symbol (ϕ or ψ) belongs to the space generated by $\{e_{2n} : n \in \mathbb{Z}\}$. Then we have that $WM_\phi WM_\psi = WM_{\phi\psi}$ (proved in [12]) and this serves a great help to prove the following.

Theorem 2.8. *Let $\phi, \psi \in L^\infty$ be such that either ϕ or ψ is $h(z^2)$ for some $h \in L^\infty$. Then $A_{\phi,\lambda}A_\psi = A_{\phi\psi,\lambda}$.*

Proof. Suppose $\phi = h(z^2)$ or $\psi = h(z^2)$ for some $h \in L^\infty$. Then
$A_{\phi,\lambda}A_\psi = D_{\overline{\lambda}}W(\phi\psi) = D_{\overline{\lambda}}WM_{\phi\psi} = D_{\overline{\lambda}}A_{\phi\psi} = A_{\phi\psi,\lambda}$. □

Now, we consider question (2) and find that it always admits of the positive answer i.e. the product of a $\lambda-$slant Toeplitz operator and a Laurent operator is always a $\lambda-$slant Toeplitz operator.

Theorem 2.9. *$M_\phi A_{\psi,\lambda}$ and $A_{\psi,\lambda}M_\phi$ are always $\lambda-$slant Toeplitz operators for $\phi, \psi \in L^\infty$. Further, $M_\phi A_{\psi,\lambda} = A_{\psi,\lambda}M_\phi$ if and only if $\phi(\overline{\lambda}z^2)\psi(z) = \phi(z)\psi(z)$.*

Proof. Using the definition of $\lambda-$slant Toeplitz operators, we have $\lambda M_z(M_\phi A_{\psi,\lambda}) = (M_\phi A_{\psi,\lambda})M_{z^2}$ and $\lambda M_z(A_{\psi,\lambda}M_\phi) = (A_{\psi,\lambda}M_\phi)M_{z^2}$ for $\phi, \psi \in L^\infty$. Thus both $M_\phi A_{\psi,\lambda}$ and $A_{\psi,\lambda}M_\phi$ are $\lambda-$slant Toeplitz operators. Now, $M_{\phi(z)}A_{\psi(z),\lambda} = M_{\phi(z)}D_{\overline{\lambda}}WM_{\psi(z)} = D_{\overline{\lambda}}M_{\phi(\overline{\lambda}z)}WM_{\psi(z)} = D_{\overline{\lambda}}WM_{\phi(\overline{\lambda}z^2)}M_{\psi(z)} = A_{\phi(\overline{\lambda}z^2)\psi(z),\lambda}$ and $A_{\psi(z),\lambda}M_{\phi(z)} = D_{\overline{\lambda}}WM_{\psi(z)}M_{\phi(z)}$

$= D_{\overline{\lambda}} W M_{\psi(z)\phi(z)} = A_{\phi(z)\psi(z),\lambda}$. Further, $M_{\phi(z)} A_{\psi(z),\lambda} = A_{\psi(z),\lambda} M_{\phi(z)}$ if and only if $A_{\phi(\overline{\lambda}z^2)\psi(z),\lambda} = A_{\phi(z)\psi(z),\lambda}$ if and only if $\phi(\overline{\lambda}z^2)\psi(z) = \phi(z)\psi(z)$. □

We have answered the question (1), but on looking slant Toeplitz operators as 1−slant Toeplitz operators, it becomes genuine to know about a more general case.

When does the product of two λ−slant Toeplitz operators become a λ−slant Toeplitz operator?

We observe the following, which helps to answer our query.

Lemma 2.4. *Let $\phi \in L^\infty$. Then $D_{\overline{\lambda}} W A_{\phi,\lambda}$ is a λ−slant Toeplitz operator if and only if $\phi = 0$.*

Proof. We need to prove one way only. For, suppose $D_{\overline{\lambda}} W A_{\phi,\lambda}$ is a λ−slant Toeplitz operator. Then for integers i, j, we have $\lambda\langle D_{\overline{\lambda}} W A_{\phi,\lambda}\ e_j, e_i\rangle = \langle D_{\overline{\lambda}} W A_{\phi,\lambda} e_{j+2}, e_{i+1}\rangle$. This gives $\langle \sum_k \lambda^k a_{2k-j} e_k, e_{2i}\rangle = \langle \sum \lambda^k\ a_{2k-j-2} e_k, e_{2i+2}\rangle$ or $a_{4i-j} = \lambda^2 a_{4i-j+2}$ for each $i, j \in \mathbb{Z}$. Hence, if $i = 0$ then we get $a_{-j} = \lambda^2 a_{-j+2}$ for each $j \in \mathbb{Z}$. This means that $a_0 = \lambda^{2n} a_{2n}$ and $a_1 = \lambda^{2n} a_{2n+1}$ for all $n \in \mathbb{Z}$. This provide that $a_0 = a_1 = 0$ and hence $\phi = 0$. □

Now, we answer our query in the following form.

Theorem 2.10. *The product of two λ−slant Toeplitz operators is a λ−slant Toeplitz operator if and only if the product is zero.*

Proof. Let $\phi, \psi \in L^\infty$ and $A_{\phi,\lambda}$ and $A_{\psi,\lambda}$ be two λ−slant Toeplitz operators. Now

$$\begin{aligned} A_{\phi,\lambda} A_{\psi,\lambda} &= D_{\overline{\lambda}} W M_\phi D_{\overline{\lambda}} W M_\psi \\ &= D_{\overline{\lambda}} W D_{\overline{\lambda}} M_{\phi(\overline{\lambda}z)} W M_{\psi(z)} \\ &= D_{\overline{\lambda}} W D_{\overline{\lambda}} W M_{\phi(\overline{\lambda}z^2)\psi(z)} \\ &= D_{\overline{\lambda}} W A_{\phi(\overline{\lambda}z^2)\psi(z),\lambda}. \end{aligned}$$

In view of above lemma, $D_{\overline{\lambda}}WA_{\phi(\overline{\lambda}z^2)\psi(z),\lambda}$ is a $\lambda-$slant Toeplitz operator if and only if $\phi(\overline{\lambda}z^2)\psi(z) = 0$. So the product of two $\lambda-$slant Toeplitz operators is $\lambda-$slant Toeplitz operator if and only if the product is zero. □

An immediate information that we receive from this theorem is that the class of $\lambda-$slant Toeplitz operators neither form an algebra nor contain any non-zero idempotent operator.

3. Spectrum of $\lambda-$slant Toeplitz operators

It is shown in Theorem 2.2 that each $\lambda-$slant Toeplitz operator, $|\lambda| = 1$, is induced by a slant Toeplitz operator on multiplying by a unitary composition operator and as a consequence there is a one-one correspondence between the class the $\lambda-$ slant Toeplitz operators and the class of slant Toeplitz operators. In this section, our aim is to investigate information about the spectrum of $\lambda-$slant Toeplitz operators. We also prove that the spectrum of $\lambda-$slant Toeplitz operator contains a closed disc for an invertible symbol in L^∞, which is a well known result in case of slant Toeplitz operators [12, Proposition 10]. For an operator A on a Hilbert space, $\sigma(A)$, $\sigma_p(A)$ and $\Pi(A)$ are used to denote the spectrum, the point spectrum and the approximate spectrum of A respectively. The kernel and range of the operator A are denoted by $N(A)$ and $R(A)$ respectively. To achieve the task, we prove the following lemma.

Lemma 3.1. *If ϕ is invertible in L^∞ then $\sigma_p(A_{\varphi,\lambda}) = \sigma_p(A_{\phi(z^2),\lambda})$, where $\varphi = \sum_{n\in\mathbb{Z}} \langle \phi, e_n\rangle \lambda^n e_n$.*

Proof. Let $\mu \in \sigma_p(A_{\varphi,\lambda})$. Then there exists a non-zero $f \in L^2$ such that $A_{\varphi,\lambda}f = \mu f$ i.e. $A_\varphi f = \mu D_\lambda f$. Write $F = \varphi f$. Invertibility of ϕ in L^∞ and the fact that $\varphi(z) = \phi(\lambda z)$ for each $z \in \mathbb{T}$ provide that φ is invertible in L^∞. Due to this, F is a

non-zero element in L^2. Using the fact that $WM_{\phi(z^2)} = M_\phi W$ (see [12]), we have

$$\begin{aligned} A_{\phi(z^2),\lambda}F &= D_{\overline{\lambda}}WM_{\phi(z^2)}\varphi f \\ &= D_{\overline{\lambda}}M_\phi A_\varphi f \\ &= \mu D_{\overline{\lambda}}M_\phi D_\lambda f. \end{aligned}$$

A simple computation shows that $D_{\overline{\lambda}}M_\phi D_\lambda = M_\varphi$. Therefore $A_{\phi(z^2),\lambda}F = \mu\varphi f = \mu F$ and hence $\mu \in \sigma_p(A_{\phi(z^2),\lambda})$.

Also, if we suppose that $\mu \in \sigma_p(A_{\phi(z^2),\lambda})$ and $0 \neq g \in L^2$ satisfying $A_{\phi(z^2),\lambda}g = \mu g$, equivalently, $A_{\phi(z^2)}g = \mu D_\lambda g$ then by applying the arguments as used above, we find that $G = \varphi^{-1}g$ is a non-zero element in L^2 and $A_{\varphi,\lambda}G = D_{\overline{\lambda}}A_\varphi\varphi^{-1}g = D_{\overline{\lambda}}Wg = D_{\overline{\lambda}}M_{\phi^{-1}}M_\phi Wg = D_{\overline{\lambda}}M_{\phi^{-1}}W(\phi(z^2)g) = \mu D_{\overline{\lambda}}M_{\phi^{-1}}D_\lambda g = \mu G$. Therefore μ is in $\sigma_p(A_{\varphi,\lambda})$. This completes the proof. □

For $\phi \in L^\infty$, we have $M_\varphi = D_{\overline{\lambda}}M_\phi D_\lambda$, where $\varphi = \sum_{n\in\mathbb{Z}}\langle\phi, e_n\rangle\lambda^n e_n$. This provides that

$$\begin{aligned} M_\varphi D_{\overline{\lambda}}W &= D_{\overline{\lambda}}M_\phi W \\ &= D_{\overline{\lambda}}WM_{\phi(z^2)} \\ &= D_{\overline{\lambda}}A_{\phi(z^2)} \\ &= A_{\phi(z^2),\lambda} \end{aligned}$$

A common well known result about any operators A and B on a Hilbert space is that $\sigma(AB)\cup\{0\} = \sigma(BA)\cup\{0\}$. We utilize these facts to conclude the following.

Theorem 3.1. *For $\phi \in L^\infty$, $\sigma(A_{\varphi,\lambda}) = \sigma(A_{\phi(z^2),\lambda})$, where $\varphi = \sum_{n\in\mathbb{Z}}\langle\phi,\ e_n\rangle\lambda^n e_n$, equivalently, $\varphi(z) = \phi(\lambda z)$ for each $z \in \mathbb{T}$.*

Proof. Let $\phi \in L^\infty$ and $\varphi = \sum_{n\in\mathbb{Z}}\langle\phi, e_n\rangle\lambda^n e_n$. Now $A_{\varphi,\lambda} = D_{\overline{\lambda}}A_\varphi = (D_{\overline{\lambda}}W)M_\varphi$ so $\sigma(A_{\varphi,\lambda})\cup\{0\} = \sigma(M_\varphi(D_{\overline{\lambda}}W))\cup\{0\} = \sigma(A_{\phi(z^2),\lambda})\cup\{0\}$. Further, we find that

$\sigma(A_{\phi(z^2),\lambda})$ always contains 0. For, the closure of range of $A^*_{\phi(z^2),\lambda}$
i.e. $\overline{R(A^*_{\phi(z^2),\lambda})} = \overline{R(W^*D_\lambda M_{\overline{\varphi}})} \subseteq \overline{R(W^*)}$, which is the closed linear subspace generated by $\{e_{2n} : n \in \mathbb{Z}\}$. As a consequence, $N(A_{\phi(z^2),\lambda}) \neq 0$ and 0 is an eigenvalue of $A_{\phi(z^2),\lambda}$ so that $0 \in \sigma(A_{\phi(z^2),\lambda})$.

Proof completes once we prove that $\sigma(A_{\varphi,\lambda})$ also contains 0. For this, we consider two cases.

$\underline{Case(1)}$: Let ϕ be invertible in L^∞. By applying Lemma 3.1,
$0 \in \sigma_p(A_{\phi(z^2),\lambda}) = \sigma_p(A_{\varphi,\lambda})$. Hence $0 \in \sigma(A_{\varphi,\lambda})$.

$\underline{Case(2)}$: Let ϕ be non-invertible element of L^∞. The set of all invertible elements in L^∞ is dense in L^∞ hence we get a sequence $< \phi_n >$ of invertible elements in L^∞ satisfying $\|\phi_n - \phi\|_\infty \to 0$ as $n \to \infty$. Now for each $n \in \mathbb{N}$, define $\varphi_n(z) = \phi_n(\lambda z)$ and $\varphi(z) = \phi(\lambda z)$ for $z \in \mathbb{T}$. Then $\|\varphi_n - \varphi\|_\infty = \|\phi_n - \phi\|_\infty \to 0$ as $n \to \infty$. On applying Lemma 3.1 to each ϕ_n, we have $0 \in \sigma_p(A_{\varphi_n,\lambda})$. Hence, for each $n \in \mathbb{N}$, we can find a non-zero element f_n in L^2 such that $A_{\varphi_n,\lambda} f_n = 0$. Without loss of generality, we can assume that $\|f_n\| = 1$. Also, $\|A_{\varphi,\lambda} f_n\| \leq \|A_{\varphi,\lambda} f_n - A_{\varphi_n,\lambda} f_n\| \leq \|\varphi - \varphi_n\| \to 0$ as $n \to \infty$. This yields that $0 \in \Pi(A_{\varphi,\lambda})$ and hence $0 \in \sigma(A_{\varphi,\lambda})$. This completes the proof. □

Our next result for $\lambda-$slant Toeplitz operators, $|\lambda| = 1$, is a motivation of the result [12, Proposition 10] that shows the containment of a closed disc in the spectrum of a slant Toeplitz operator.

Theorem 3.2. *For any invertible ϕ in L^∞, $\sigma(A_{\varphi,\lambda})$ contains a closed disc, where $\varphi = \sum_{n\in\mathbb{Z}} \langle \phi, e_n \rangle \lambda^n e_n$.*

Proof. Let μ be any non-zero complex number. As ϕ is invertible in L^∞ so is ϕ^{-1}. Now suppose that $(A^*_{\overline{\phi}^{-1}(z^2),\lambda} - \mu I)$ is onto. Then for each $f \in L^2$, we have

$$\begin{aligned}(A^*_{\overline{\phi}^{-1}(z^2),\lambda} - \mu I)f &= M_{\phi^{-1}(z^2)} W^* D_\lambda f - \mu(P_e f \oplus P_o f) \\ &= \mu W^* M_{\phi^{-1}}(\mu^{-1} D_\lambda - M_\phi W) f \oplus (-\mu P_o f),\end{aligned}$$

where P_e is the projection on the closed span of $\{e_{2n} : n \in \mathbb{Z}\}$ in L^2 and $P_o = (I - P_e)$ is the projection on the closed span of $\{e_{2n-1} : n \in \mathbb{Z}\}$. Now, pick $0 \neq g_0$ in $P_o(L^2)$. Being $(A^*_{\overline{\phi}^{-1}(z^2),\lambda} - \mu I)$ is onto, on using the computation made above, we find $f \in L^2$ such that

$$g_0 = \mu W^* M_{\phi^{-1}}(\mu^{-1} D_\lambda - M_\phi W) f \oplus (-\mu P_o f).$$

Since $g_0 \in P_o(L^2)$, we have $\mu W^* M_{\phi^{-1}}(\mu^{-1} D_\lambda - M_\phi W) f = 0$. This on using the facts that, $\mu \neq 0$, W is coisometry (i.e. $WW^* = I$) and $M_{\phi^{-1}}$ is invertible, gives that $(\mu^{-1} D_\lambda - M_\phi W) f = 0$. This shows that $0 = (\mu^{-1} I - D_{\overline{\lambda}} M_\phi W) f = (\mu^{-1} I - D_{\overline{\lambda}} W M_{\phi(z^2)}) f = (\mu^{-1} I - A_{\phi(z^2),\lambda}) f$. This implies that $\mu^{-1} \in \sigma_p(A_{\phi(z^2),\lambda})$. Now $(A^*_{\overline{\phi}^{-1}(z^2),\lambda} - \mu I)$ is onto (in fact invertible) for each $\mu \in \rho(A^*_{\overline{\phi}^{-1}(z^2),\lambda})$, the resolvent of $A^*_{\overline{\phi}^{-1}(z^2),\lambda}$, so on applying Lemma 3.1, we get that

$$\{\mu^{-1} : \mu \in \rho(A^*_{\overline{\phi}^{-1}(z^2),\lambda})\} \subseteq \sigma_p(A_{\phi(z^2),\lambda}) = \sigma_p(A_{\varphi,\lambda}) \subseteq \sigma_(A_{\varphi,\lambda}),$$

where $\varphi = \sum_{n \in \mathbb{Z}} \langle \phi, e_n \rangle \lambda^n e_n$. As spectrum of any operator is compact it follows that $\sigma(A_{\varphi,\lambda})$ contains a disc of eigenvalues of $A_{\varphi,\lambda}$. □

Remark 1. Radius of closed disc contained in $\sigma(A_{\varphi,\lambda})$ is $\frac{1}{r(A_{\overline{\varphi}^{-1},\lambda})}$, where $r(A)$ denotes the spectral radius of the operator A. For,

$$\begin{aligned} max\{|\lambda^{-1}| : \lambda \in \rho(A^*_{\overline{\phi}^{-1}(z^2),\lambda})\} &= \frac{1}{min\{|\lambda| : \lambda \in \rho(A^*_{\overline{\phi}^{-1}(z^2),\lambda})\}} \\ &= (r(A^*_{\overline{\phi}^{-1}(z^2),\lambda}))^{-1} \\ &= (r(A_{\overline{\phi}^{-1}(z^2),\lambda}))^{-1} \\ &= (r(A_{\overline{\varphi}^{-1},\lambda}))^{-1}. \end{aligned}$$

Thus the radius of closed disc contained in $\sigma(A_{\varphi,\lambda})$ is $\frac{1}{r(A_{\overline{\varphi}^{-1},\lambda})}$. Since spectral radius of an operator is the radius of smallest disc containing its spectrum so we get that $\frac{1}{r(A_{\overline{\varphi}^{-1},\lambda})} \leq r(A_{\varphi,\lambda})$.

For unimodular $\phi \in L^\infty$, $||A^n_{\varphi,\lambda}||^2 = ||A^n_{\varphi,\lambda}A^{*n}_{\varphi,\lambda}|| = ||I|| = 1$, so that $r(A_{\varphi,\lambda}) = 1$ (using Gelfand formula for spectral radius). Hence, if $|\phi| = 1$, then $\sigma(A_{\varphi,\lambda}) = \overline{\mathbb{D}}$, the closed unit disc.

4. Compressions of $\lambda-$slant Toeplitz operators

We denote the compression of a $\lambda-$slant Toeplitz operator A to H^2 by B. Then by the definition of compression we have $B = PA|_{H^2}$, that is, $BP = PAP$, where P is the orthogonal projection of L^2 onto H^2. The fact that a non-zero $\lambda-$slant Toeplitz operator, $|\lambda| = 1$, is always of the form $A_{\phi,\lambda}$ for $\phi \in L^\infty$, provides that $B = B_{\phi,\lambda} = PD_{\overline{\lambda}}A_\phi|_{H^2}$. Since $PD_{\overline{\lambda}} = D_{\overline{\lambda}}P$, we further have $B = D_{\overline{\lambda}}B_\phi$, where B_ϕ is the compression of slant Toeplitz operator A_ϕ to H^2. Also $\phi \to B_\phi$ is one-one, so $\phi \to B_{\phi,\lambda}$ is also one-one. It is interesting to obtain an equation characterizing the compressions of $\lambda-$slant Toeplitz operators in the following form.

Theorem 4.1. *An operator B on H^2 is the compression of a $\lambda-$slant Toeplitz operator if and only if $\lambda B = U^*BU^2$, where U is the forward unilateral shift.*

Proof. Suppose B is compression of a $\lambda-$slant Toeplitz operator. Then $B = D_{\overline{\lambda}}B_\phi$ for some ϕ in L^∞. Now $U^*BU^2 = U^*D_{\overline{\lambda}}B_\phi U^2 = \lambda D_{\overline{\lambda}}U^*B_\phi\ U^2 = \lambda D_{\overline{\lambda}}B_\phi = \lambda B$.

Conversely, suppose that B is an operator satisfying $\lambda B = U^*BU^2$. Then $\lambda D_\lambda B = D_\lambda U^*BU^2 = \lambda U^*D_\lambda BU^2$. Since $|\lambda| = 1$ we get $D_\lambda B = U^*D_\lambda BU^2$. So $D_\lambda B$ is compression of a slant Toeplitz operator [22]. So $D_\lambda B = B_\phi$ for some ϕ in L^∞. Thus $B = D_{\overline{\lambda}}B_\phi$ for some ϕ in L^∞. □

We now talk about the compactness of compression of a $\lambda-$slant Toeplitz operators. For this, we first prove the following and only the outlines of the proof are given as one can refer [22] for the details of the techniques used.

Lemma 4.1. *Let $|\lambda| = 1$ and $\phi \in L^\infty$. Then we have the following:*

(1) $WB^*_{\phi,\lambda} = D_\lambda T_\psi$, *where* $\psi(z) = W\overline{\phi}(\lambda z)$.

(2) *If* $\overline{\phi}$ *or* ψ *is analytic then* $B_{\phi,\lambda}T_\psi = B_{\phi\psi,\lambda}$.

(3) *If* $\overline{\phi}$ *or* $\overline{\psi}$ *is analytic then* $B_{\phi,\lambda}B^*_{\psi,\lambda}$ *is a Toeplitz operator.*

(4) *If* ϕ *is analytic then* $T_\psi B_{\phi,\lambda}$ *is again compression of a* $\lambda-$*slant Toeplitz operator.*

Proof. Proof of (1) follows as $WB^*_{\phi,\lambda} = WPA^*_\phi D_\lambda|_{H^2} = PM_{W\overline{\phi}}D_\lambda|_{H^2} = D_\lambda P\, M_{W\overline{\phi}(\lambda z)}|_{H^2} = D_\lambda T_\psi$, where $\psi(z) = W\overline{\phi}(\lambda z)$.

Proof of (2) follows using the fact that $B_\phi T_\psi = B_{\phi\psi}$ when either of $\overline{\phi}$ or ψ is analytic [22].

On simple computation we find that if $\overline{\phi}$ or $\overline{\psi}$ is analytic then $B_{\phi,\lambda}B^*_{\psi,\lambda} = D_{\overline{\lambda}}T_{W\overline{\phi}\psi}$ $D_\lambda|_{H^2} = PD_{\overline{\lambda}}M_{W\overline{\phi}\psi}D_\lambda|_{H^2} = PM_\xi|_{H^2} = T_\xi$, where $\xi(z) = W\overline{\phi}\psi(\lambda z)$. This completes the proof of (3).

Now for (4), if ϕ is analytic then $T_\psi B_{\phi,\lambda} = PD_{\overline{\lambda}}M_{\psi(\overline{\lambda}z)}B_\phi|_{H^2} = D_{\overline{\lambda}}\ B_{\psi(\overline{\lambda}z^2)\phi(z)} = B_{\psi(\overline{\lambda}z^2)\phi(z),\lambda}$. Hence the result. □

Now the information gathered here provides that the only compact $B_{\phi,\lambda}$ is the zero operator, which is very common result known for various classes of operators, like, Laurent operators [10], Toeplitz operators [4], slant Toeplitz operators [12].

Theorem 4.2. $B_{\phi,\lambda}$ *is compact if and only if* $\phi = 0$.

Proof. Proof of one part is obvious. For the converse, suppose $B_{\phi,\lambda}$ is compact. Then $WB^*_{\phi,\lambda}$ is compact. So by part (1) of above lemma, $D_\lambda T_\psi$ is compact, where $\psi(z) = W\overline{\phi}(\lambda z)$. But D_λ is unitary hence T_ψ is compact and so $\psi = 0$, that is, $W\overline{\phi}(\lambda z) = 0$. This means that $W\overline{\phi} = 0$. Therefore $\overline{a}_{2n} = \langle\overline{\phi}, e_{2n}\rangle = \langle W\overline{\phi}, e_n\rangle = 0$ for all $n \in \mathbb{Z}$.

Now we use Lemma 4.1(2) that provides the compactness $B_{\phi z,\lambda}$ if $B_{\phi,\lambda}$ is compact. As a consequence $WB^*_{\phi z,\lambda}$ and hence $D_\lambda T_\psi$ is compact, where $\psi(z) = W(\overline{\phi z})(\lambda z)$.

This implies $\psi = W(\overline{\phi z}) = 0$. Therefore $\langle W(\overline{\phi z}), e_n\rangle = 0$, which implies $\langle \overline{\phi z}, e_{2n}\rangle = 0$. Thus $\overline{a}_{-2n-1} = 0$ for all $n \in \mathbb{Z}$. Hence $\phi = 0$. □

Acknowledgement

Insightful suggestions and the thorough review of the referee(s) for the improvement of the paper are gratefully acknowledged.

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