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FURTHER RESULTS ON THE UNIQUENESS OF MEROMORPHIC FUNCTIONS AND THEIR DERIVATIVE COUNTERPART SHARING ONE OR TWO SETS

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Abstract. In this paper we prove a number of results concerning uniqueness of a meromorphic function as well as its derivative sharing one or two sets. In particular, we deal with the specific question raised in [18], [19], [20] and ultimately improve the result of Banerjee-Bhattacharjee [4].

1. Introduction and Definitions

In this paper, we assume that readers familiar with the basic Nevanlinna theory([11]). By $\mathbb{C}$ and $\mathbb{N}$ we mean the set of complex numbers and set of positive integers respectively. Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share the value a CM (counting multiplicities), provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a-IM (ignoring multiplicities), provided

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that $f-a$ and $g-a$ have the same set of zeros, where the multiplicities are not taken into account. In addition we say that f and g share ∞ CM (IM), if $1/f$ and $1/g$ share 0 CM (IM).

We now recall some well-known definitions in the literature of the uniqueness of meromorphic functions sharing sets as it will be pertinent with the follow up discussions.

Definition 1.1. *For a non-constant meromorphic function f and any set $S \subset \mathbb{C}\bigcup\{\infty\}$, we define*

$$E_f(S) = \bigcup_{a\in S}\{(z,p) \in \mathbb{C}\times\mathbb{N} \mid f(z) = a \text{ with multiplicity } p\},$$

$$\overline{E}_f(S) = \bigcup_{a\in S}\{z \in \mathbb{C} \mid f(z) = a, \text{counting without multiplicity}\}.$$

Two meromorphic functions f and g are said to share the set S counting multiplicities(CM), if $E_f(S) = E_g(S)$. They are said to share S ignoring multiplicities(IM), if $\overline{E}_f(S) = \overline{E}_g(S)$.

Definition 1.2. *A set $S \subset \mathbb{C}\bigcup\{\infty\}$ is called a unique range set for meromorphic functions (in short, URSM), if for any two non-constant meromorphic functions f and g the condition $E_f(S) = E_g(S)$ implies $f \equiv g$.*

Similarly we can define unique range set for entire functions(URSE).

Definition 1.3. *A set $S \subset \mathbb{C}\bigcup\{\infty\}$ is called a unique range set for meromorphic functions ignoring multiplicities (in short, URSM-IM), if for any two non-constant meromorphic functions f and g the condition $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$.*

Similarly we can define unique range set for entire functions ignoring multiplicities (URSE-IM).

We further recall the notion of weighted sharing of sets appeared in the literature in 2001 ([12]). As far as relaxations of the nature of sharing of the sets are concerned, this notion has a remarkable influence.

Definition 1.4. *([12]) Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.5. [12] *Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$, the set $\bigcup_{a \in S} E_k(a; f)$. If $E_f(S, k) = E_g(S, k)$, then we say f, g share the set S with weight k.*

Definition 1.6. *([13]) A polynomial P in $\mathbb{C}$, is called a uniqueness polynomial for meromorphic (entire) functions, if for any two non-constant meromorphic (entire) functions f and g, $P(f) \equiv P(g)$ implies $f \equiv g$. We say P is a UPM (UPE) in brief.*

Definition 1.7. *([5], [9]) Let $P(z)$ be a polynomial such that $P^{'}(z)$ has mutually t distinct zeros given by $d_1, d_2, \ldots, d_t$ with multiplicities $q_1, q_2, \ldots, q_t$ respectively then $P(z)$ is said to satisfy critical injection property if $P(d_i) \neq P(d_j)$ for $i \neq j$ where $i, j \in \{1, 2, \cdot\cdot\cdot, t\}$.*

From the definition it is obvious that $P(z)$ is injective on the set of distinct zeros of $P^{'}(z)$ which are known as critical points of $P(z)$. Furthermore any polynomial $P(z)$ satisfying this property is called critically injective polynomial. Thus a critically injective polynomial has at-most one multiple zero.

To this end, we recall two definitions.

Definition 1.8. *([1]) Let z_0 be a zero of $f-a$ of multiplicity p and a zero of $g-a$ of multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p>q\geq 1$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p=q=1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p=q\geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$.*

Definition 1.9. *([1]) Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

In 1976 Gross([10], Question 6) proposed a problem concerning the uniqueness of entire functions that share sets of distinct elements instead of values as follows :

Question A : *Can one find two finite set S_j for $j=1,2$ such that any two non-constant entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2$ must be identical ?*

In ([10]), Gross also asked : "If the answer to Question 6 is affirmative, it would be interesting to know how large both sets would have to be."

Yi ([17]) and independently Fang-Xu ([7]) gave a positive answer to Question A. In fact, Yi ([17]) proved that the smallest cardinalities of S_1 and S_2 are 1 and 3 respectively, where S_1 and S_2 are two finite sets such that any two non-constant entire functions f and g satisfying $E(S_j,f)=E(S_j,g)$ for $j=1,2$ must be identical. And till today this is the best result.

Now it is natural to ask the following question :

Question B :([18],[19],[20]) *Can one find two finite sets* S_j $(j = 1, 2)$ *such that any two non-constant meromorphic functions* f *and* g *satisfying* $E_f(S_j, \infty) = E_g(S_j, \infty)$ *for* $j = 1, 2$ *must be identical ?*

In 1994, Yi ([16]) proved that there exist two finite sets S_1 (with 2 elements) and S_2 (with 9 elements) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$ must be identical.

In ([14]), Li-Yang proved that there exist two finite sets S_1 (with 1 element) and S_2 (with 15 elements) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$ must be identical.

In ([6]), Fang-Guo proved that there exist two finite sets S_1 (with 1 element) and S_2 (with 9 elements) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$ must be identical.

Also in 2002, Yi ([18]) proved that there exist two finite sets S_1 (with 1 element) and S_2 (with 8 elements) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$ must be identical.

In 2008, the first author ([1]) improved the result of Yi ([18]) by relaxing the nature of sharing the range sets by the notion of weighted sharing. He established that there exist two finite sets S_1 (with 1 element) and S_2 (with 8 elements) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_1, 0) = E_g(S_1, 0)$ and $E_f(S_2, 2) = E_g(S_2, 2)$ must be identical.

So the natural query would be whether there exists similar types of unique range sets corresponding to the derivatives of two meromorphic functions. But in this particular direction the number of results are scanty. The following uniqueness results have been obtained when the derivatives of meromorphic functions sharing one or two are studied by the researchers.

Theorem A. *([8, 20]) Let* $S_1 = \{z : z^n + az^{n-1} + b = 0\}$ *and* $S_2 = \{\infty\}$*, where* a*,* b *are nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no repeated root and* n (≥ 7)*,* k *be two positive integers. Let* f *and* g *be two non-constant meromorphic functions such that* $E_{f^{(k)}}(S_1, \infty) = E_{g^{(k)}}(S_1, \infty)$ *and* $E_f(S_2, \infty) = E_g(S_2, \infty)$ *then* $f^{(k)} \equiv g^{(k)}$*.*

In 2010, Banerjee-Bhattacharjee ([3]) improved the above results in the following way :

Theorem B. *([3]) Let* S_i*,* $i = 1, 2$ *and* k *be given as in* Theorem A*. Let* f *and* g *be two non-constant meromorphic functions such that* $E_{f^{(k)}}(S_1, 2) = E_{g^{(k)}}(S_1, 2)$ *and* $E_f(S_2, 1) = E_g(S_2, 1)$ *then* $f^{(k)} \equiv g^{(k)}$*.*

Theorem C. *([3]) Let* S_i*,* $i = 1, 2$ *be given as in* Theorem A*. Let* f *and* g *be two non-constant meromorphic functions such that* $E_{f^{(k)}}(S_1, 3) = E_{g^{(k)}}(S_1, 3)$ *and* $E_f(S_2, 0) = E_g(S_2, 0)$ *then* $f^{(k)} \equiv g^{(k)}$*.*

In 2011, Banerjee-Bhattacharjee ([4]) further improved the above results in the following manner :

Theorem D. *([4]) Let* S_i*,* $i = 1, 2$ *and* k *be given as in* Theorem A*. Let* f *and* g *be two non-constant meromorphic functions such that* $E_{f^{(k)}}(S_1, 2) = E_{g^{(k)}}(S_1, 2)$ *and* $E_f(S_2, 0) = E_g(S_2, 0)$ *then* $f^{(k)} \equiv g^{(k)}$*.*

So far from the above discussions we see that for the two set sharing problems, the best result has been obtained when one set contain 8 elements and the other set contain 1 element. On the other hand, when derivatives of the functions are considered then the cardinality of one set can further be reduced to 7. So it will be natural query whether there can be a single result corresponding to uniqueness of the function sharing two sets which can accommodate the derivative counterpart of the main function as well under relaxed sharing hypothesis with smaller cardinalities than the existing results. This is the motivation of the paper. Our one result present in the paper improves all the preceding theorems stated so far *Theorems A-D* in some sense.

2. Main Results

Suppose for two positive integers m,n we shall denote by $P(z)$ the following polynomial.

$$P(z) = z^n - \frac{2n}{n-m} z^{n-m} + \frac{n}{n-2m} z^{n-2m} + c, \tag{2.1}$$

where c is any complex number satisfying $|c| \neq \frac{2m^2}{(n-m)(n-2m)}$ and $c \neq 0, -\frac{1-\frac{2n}{n-m}+\frac{n}{n-2m}}{2}$.

Following theorems are the main results of the paper. In the first theorem we consider the uniqueness of meromorphic functions and its derivatives counterpart corresponding to single set sharing.

Theorem 2.1. *Let $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$ be three positive integers such that m, n has no common factors. Let $S = \{z : P(z) = 0\}$ where the polynomial $P(z)$ is defined by 2.1. Let f and g be two non-constant meromorphic functions satisfying $E_{f^{(k)}}(S, l) = E_{g^{(k)}}(S, l)$. If one of the following conditions holds:*

(1) $l \geq 2$ *and* $n > \max\{2m + 4 + \frac{4}{k+1}, 4m + 1\}$,

(2) $1 = l$ *and* $n > \max\{2m + 4.5 + \frac{4.5}{k+1}, 4m + 1\}$,

(3) $l = 0$ *and* $n > \max\{2m + 7 + \frac{7}{k+1}, 4m + 1\}$

then $f^{(k)} \equiv g^{(k)}$.

Corollary 2.1. *Let $n(\geq 9)$, $m(= 1)$, $k \geq 1$ be three positive integers. Let $S = \{z : P(z) = 0\}$ where the polynomial $P(z)$ is defined by 2.1. Let f and g be two non-constant meromorphic functions satisfying $E_{f^{(k)}}(S, 2) = E_{g^{(k)}}(S, 2)$. Then $f^{(k)} \equiv g^{(k)}$.*

For $k = 0$ in Theorem 2.1 we get the following :

Corollary 2.2. *Let $n(\geq 1)$, $m(\geq 1)$ be two positive integers having no common factors. Let $S = \{z : P(z) = 0\}$ where the polynomial $P(z)$ is defined by 2.1. Let f and g be two non-constant meromorphic functions satisfying $E_f(S,l) = E_g(S,l)$. If one of the following conditions holds:*

(1) $l \geq 2$ *and* $n > \max\{2m+8, 4m+1\}$,

(2) $1 = l$ *and* $n > \max\{2m+9, 4m+1\}$,

(3) $l = 0$ *and* $n > \max\{2m+14, 4m+1\}$

then $f \equiv g$.

The next theorem focus on the two set sharing problem.

Theorem 2.2. *Let $n(> 4m+1)$, $m(\geq 1)$, $k(\geq 0)$ be three positive integers satisfying* $\gcd\{m,n\} = 1$. *Let $S = \{z : P(z) = 0\}$ where the polynomial $P(z)$ is defined by 2.1. Let f and g be two non-constant meromorphic functions satisfying $E_{f^{(k)}}(S,l) = E_{g^{(k)}}(S,l)$ and $E_{f^{(k)}}(0,q) = E_{g^{(k)}}(0,q)$ where $0 \leq q < \infty$. If*

(1) $l \geq \frac{3}{2} + \frac{2}{n-2m-1} + \frac{1}{(n-2m)q+n-2m-1}$ *and*

(2) $n > 2m + \frac{4}{k+1} + \frac{4}{(k+1)(n-2m-1)} + \frac{2}{(k+1)((n-2m)q+n-2m-1)}$

then $f^{(k)} \equiv g^{(k)}$.

The following example shows that for the two set sharing case, choosing the set S_1 with one element and S_2 with two elements Theorem 2.2 ceases to hold.

Example 2.1. *Let $S_1 = \{a\}$ and $S_2 = \{b,c\}$. Choose $f(z) = p(z) + (b-a)e^z$ and $g(z) = q(z) + (-1)^k(c-a)e^{-z}$, where $p(z)$ and $q(z)$ are polynomial of degree k with the coefficient of z^k in $p(z)$ and $q(z)$ is equal to $\frac{a}{k!}$.*

Clearly $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1,2$ but $f^{(k)} \not\equiv g^{(k)}$.

Corollary 2.3. *Let $n(\geq 8)$, $m(= 1)$ be two positive integers. Let $S = \{z : P(z) = 0\}$ where the polynomial $P(z)$ is defined by 2.1. Let f and g be two non-constant meromorphic functions satisfying*

(1) $E_f(S,3) = E_g(S,3)$ *and* $E_f(0,0) = E_g(0,0)$, *or*

(2) $E_f(S,2) = E_g(S,2)$ *and* $E_f(0,1) = E_g(0,1)$

then $f \equiv g$.

Corollary 2.4. *Let* $n(\geq 6)$, $m(=1)$ *and* $k \geq 1$ *be two positive integers. Let* $S = \{z : P(z) = 0\}$ *where the polynomial* $P(z)$ *is defined by 2.1. Let* f *and* g *be two non-constant meromorphic functions satisfying* $E_{f^{(k)}}(S,3) = E_{g^{(k)}}(S,3)$ *and* $E_{f^{(k)}}(0,0) = E_{g^{(k)}}(0,0)$. *Then* $f^{(k)} \equiv g^{(k)}$.

Remark 2.1. *Corollary 2.4 shows that there exists two sets* S_1 *(with 1 element) and* S_2 *(with* 6 *elements) such that when derivatives of any two non-constant meromorphic functions share them with finite weight yields* $f^{(k)} \equiv g^{(k)}$ *thus improve Theorem D in the direction of Question B.*

The following two examples show that specific form of choosing the set S_1 with five elements and $S_2 = \{0\}$ Corollary 2.4 ceases to hold.

Example 2.2. *Let* $f(z) = \frac{1}{(\sqrt{\alpha\beta\gamma})^{k-1}} e^{\sqrt{\alpha\beta\gamma}\, z}$ *and* $g(z) = \frac{(-1)^k}{(\sqrt{\alpha\beta\gamma})^{k-1}} e^{-\sqrt{\alpha\beta\gamma}\, z}$ $(k \geq 1)$ *and* $S = \{\alpha\sqrt{\beta}, \alpha\sqrt{\gamma}, \beta\sqrt{\gamma}, \gamma\sqrt{\beta}, \sqrt{(\alpha\beta\gamma)}\}$, *where* α, β *and* γ *are three nonzero distinct complex numbers. Clearly* $E_{f^{(k)}}(S) = E_{g^{(k)}}(S)$ *and* $E_{f^{(k)}}(0) = E_{g^{(k)}}(0)$ *but* $f^{(k)} \not\equiv g^{(k)}$.

Example 2.3. *Let* $f(z) = \frac{1}{c^k} e^{cz}$ *and* $g(z) = \omega^4 f(z)$ *and* $S = \{\omega^4, \omega^3, \omega^2, \omega, 1\}$, *where* ω *is the non-real fifth root of unity and* c *is a non-zero complex number. Clearly* $E_{f^{(k)}}(S) = E_{g^{(k)}}(S)$ *and* $E_{f^{(k)}}(0) = E_{g^{(k)}}(0)$ *but* $f^{(k)} \not\equiv g^{(k)}$.

Remark 2.2. *However the following question is still inevitable from the Corollary 2.4 and Example 2.2 that*

Whether there exists two suitable sets S_1 (with 1 element) and S_2 (with 5 elements) such that when derivatives of any two non-constant meromorphic functions share them with finite weight yield $f^{(k)} \equiv g^{(k)}$?

3. Lemmmas

We define
$F=-\frac{(f^{(k)})^{n-2m}((f^{(k)})^{2m}-\frac{2n}{n-m}(f^{(k)})^{m}+\frac{n}{n-2m})}{c}$, $G=-\frac{(g^{(k)})^{n-2m}((g^{(k)})^{2m}-\frac{2n}{n-m}(g^{(k)})^{m}+\frac{n}{n-2m})}{c}$,
where $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are non-negative integers. Henceforth we shall denote by H and Φ the following two functions

$$H=(\frac{F''}{F'}-\frac{2F'}{F-1})-(\frac{G''}{G'}-\frac{2G'}{G-1})$$

and

$$\Phi=\frac{F'}{F-1}-\frac{G'}{G-1}.$$

Let $T(r)=\max\{T(r,f^{(k)}),T(r,g^{(k)})\}$ and $S(r)=o(T(r))$.

Lemma 3.1. *The polynomial*

$$P(z)=z^{n}-\frac{2n}{n-m}z^{n-m}+\frac{n}{n-2m}z^{n-2m}+c$$

is a critically injective polynomial having only simple zeros when $|c|\neq 0,\frac{2m^2}{(n-m)(n-2m)}$.

Proof. Since

$$P'(z)=nz^{n-2m-1}(z^{m}-1)^{2},$$

P is critically injective, because

(1) $P(0)=P(\alpha)$ where $\alpha^m=1$ gives $\alpha=0$ which is a contradiction, and

(2) $P(\beta)=P(\gamma)$ where $\beta^m=1,\gamma^m=1$, gives $\beta^n=\gamma^n$.

Now as $\gcd\{m,n\}=1$, so there exist integers s,t such that $ms+nt=1$.

Thus $\beta=\beta^{ms+nt}=\gamma^{ms+nt}=\gamma$.

For the second part if $P(\alpha)=P'(\alpha)=0$ then either $\alpha=0$ or $\alpha^m=1$.
If $\alpha=0$ then $P(\alpha)=c$, which is not zero by assumption.
If $\alpha^m=1$ then $P(\alpha)=\alpha^n(1-\frac{2n}{n-m}+\frac{n}{n-2m})+c=0$.
As $|\alpha|=1$ so $|c|=\frac{2m^2}{(n-m)(n-2m)}$ which is not possible by assumption. □

Lemma 3.2. *([9]) Suppose that $P(z)$ is a monic polynomial without multiple zero whose derivatives has mutually distinct t zeros given by $d_1, d_2, \ldots, d_t$ with multiplicities $q_1, q_2, \ldots, q_t$ respectively. Also suppose that $P(z)$ is critically injective. Then $P(z)$ will be a uniqueness polynomial if and only if*

$$\sum_{1 \le l < m \le k} q_l q_m > \sum_{l=1}^{t} q_l.$$

In particular the above inequality is always satisfied whenever $t \ge 4$. When $t = 3$ and $\max\{q_1, q_2, q_3\} \ge 2$ or when $t = 2$, $\min\{q_1, q_2\} \ge 2$ and $q_1 + q_2 \ge 5$ then also the above inequality holds.

Lemma 3.3. *F and G are defined as earlier. Then $F \equiv G$ gives $f^{(k)} \equiv g^{(k)}$ when $k \ge 0$ and $n \ge 2m + 4$.*

Proof. $F \equiv G$ implies $P(f^{(k)}) = P(g^{(k)})$.
Since P is critically injective polynomial having no multiple zeros and

$$P'(z) = nz^{n-2m-1}(z^m - 1)^2, \ t = m + 1.$$

So when $n \ge 2m + 4$ we have by the Lemma 3.2 that $f^{(k)} \equiv g^{(k)}$. □

Lemma 3.4. *F and G are defined as earlier, then $FG \not\equiv 1$ for $k \ge 0$ and $n \ge 5$.*

Proof. On contrary, suppose $FG \equiv 1$
Then by Mokhon'ko's Lemma([15]), $T(r, f^{(k)}) = T(r, g^{(k)}) + O(1)$.
Then

$$(f^{(k)})^{n-2m} \prod_{i=1}^{2m}((f^{(k)}) - \gamma_i)(g^{(k)})^{n-2m} \prod_{i=1}^{2m}((g^{(k)}) - \gamma_i) = c^2, \tag{3.1}$$

where γ_i (i=1,2,...,2m) are the roots of the equation $z^{2m} - \frac{2n}{n-m}z^m + \frac{n}{n-2m} = 0$.

Let z_0 be a γ_i point of $f^{(k)}$ of order p. Then z_0 is a pole of g of order q such that $p = n(1 + k)q \ge n$. So

$$\overline{N}(r, \gamma_i; f^{(k)}) \le \frac{1}{n} N(r, \gamma_i; f^{(k)}).$$

Again let z_0 be a zero of $f^{(k)}$ of order t. Then z_0 is a pole of g of order s such that $(n-2m)t = ns(1+k)$.

Thus $t > s(1+k)$ and $2ms(1+k) = (n-2m)(t-s(1+k)) \geq (n-2m)$. Consequently $(n-2m)t = ns(1+k)$ gives $t \geq \frac{n}{2m}$. So

$$\overline{N}(r,0;f^{(k)}) \leq \frac{2m}{n} N(r,0;f^{(k)}).$$

again

$$\begin{aligned} \overline{N}(r,\infty;f^{(k)}) &\leq \overline{N}(r,0;g^{(k)}) + \sum_{i=0}^{2m} \overline{N}(r,\gamma_i;g^{(k)}) \\ &\leq \frac{2m}{n} N(r,0;g^{(k)}) + \frac{1}{n} \sum_{i=0}^{2m} N(r,\gamma_i;g^{(k)}) \\ &\leq \frac{4m}{n} T(r,g^{(k)}) + O(1). \end{aligned}$$

Now by using the Second Fundamental Theorem we get

$$\begin{aligned} &2mT(r,f^{(k)}) \\ \leq\ & \overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,0;f^{(k)}) + \sum_{i=0}^{2m} \overline{N}(r,\gamma_i;f^{(k)}) + S(r,f^{(k)}) \\ \leq\ & \frac{4m}{n} T(r,f^{(k)}) + \frac{2m}{n} T(r,f^{(k)}) + \frac{2m}{n} T(r,f^{(k)}) + S(r,f^{(k)}), \end{aligned} \tag{3.2}$$

which is a contradiction as $n \geq 5$. □

Lemma 3.5. *([2]) If F and G share $(1,l)$ where $0 \leq l < \infty$ then* $\overline{N}(r,1;F) + \overline{N}(r,1;G) - N_E^{1}(r,1,F) + (l-\frac{1}{2})\overline{N}_*(r,1;F,G) \leq \frac{1}{2}(N(r,1;F) + N(r,1;G))$.

Lemma 3.6. *Let F, G, Φ be defined previously and $F \not\equiv G$. If $f^{(k)}$ and $g^{(k)}$ share $(0,q)$ where $0 \leq q < \infty$ and F, G share $(1,l)$, then*

$$\begin{aligned} &\{(n-2m)q + n - 2m - 1\}\, \overline{N}(r,0;f^{(k)} \mid \geq q+1) \\ \leq\ & \overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,\infty;g^{(k)}) + \overline{N}_*(r,1;F,G) + S(r). \end{aligned}$$

Similar expressions hold for g also.

Proof. Case-1 $\Phi = 0$

Then by integration we get

$$F - 1 = A(G - 1)$$

where A is non-zero constant. Since $F \not\equiv G$, we have $A \neq 1$. Thus 0 is an e.v.P. of $f^{(k)}$ and $g^{(k)}$ and hence the lemma follows immediately.

Case-2 $\Phi \neq 0$

Let z_0 be a zero of $f^{(k)}$ of order $t(\geq q+1)$. Then it is a zero of F of order atleast $(q+1)(n-2m)$ and hence z_0 is the zero of Φ of order at least $q(n-2m)+n-2m-1$. Thus

$$\begin{aligned}
& \{(n-2m)q + n - 2m - 1\}\ \overline{N}(r, 0; f^{(k)} \mid \geq q+1) \\
\leq\ & N(r, 0; \Phi) \\
\leq\ & T(r, \Phi) + O(1) \\
\leq\ & N(r, \infty; \Phi) + S(r) \\
\leq\ & \overline{N}(r, \infty; f^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + \overline{N}_*(r, 1; F, G) + S(r).
\end{aligned}$$

□

Lemma 3.7. *Let H be defined previously. If $H \equiv 0$ and $n \geq 4m+2$ with* $\gcd\{m, n\} = 1$*, then $f^{(k)} \equiv g^{(k)}$ for any integer $k \geq 0$.*

Proof. In this case F and G share $(1, \infty)$.

Now by integration we have

$$F = \frac{AG + B}{CG + D}, \tag{3.3}$$

where A, B, C, D are constant satisfying $AD - BC \neq 0$.

Thus by Mokhon'ko's Lemma ([15])

$$T(r,f^{(k)})=T(r,g^{(k)})+S(r). \tag{3.4}$$

As $AD-BC\neq 0$, so $A=C=0$ never occur. Thus we consider the following cases:

Case-1 $AC\neq 0$

In this case

$$F-\frac{A}{C}=\frac{BC-AD}{C(CG+D)}. \tag{3.5}$$

So,

$$\overline{N}(r,\frac{A}{C};F)=\overline{N}(r,\infty;G).$$

Now by using the Second Fundamental Theorem and (3.4), we get

$$\begin{aligned}
& nT(r,f^{(k)})+O(1)=T(r,F)\\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,\frac{A}{C};F)+S(r,F)\\
\leq\ & \overline{N}(r,\infty;f^{(k)})+\overline{N}(r,0;f^{(k)})+2mT(r,f^{(k)})+\overline{N}(r,\infty;g^{(k)})+S(r,f^{(k)})\\
\leq\ & (2m+1+\frac{2}{k+1})T(r,f^{(k)})+S(r,f^{(k)}),
\end{aligned}$$

which is a contradiction as $n\geq 4m+2$.

Case-2 $AC=0$

Subcase-2.1 $A=0$ and $C\neq 0$

In this case $B\neq 0$ and

$$F=\frac{1}{\gamma G+\delta},$$

where $\gamma=\frac{C}{B}$ and $\delta=\frac{D}{B}$.

If F has no 1-point, then by using the Second Fundamental Theorem and (3.4), we get

$$\begin{aligned}
& T(r,F) \\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1;F)+S(r,F) \\
\leq\ & \overline{N}(r,\infty;f^{(k)})+\overline{N}(r,0;f^{(k)})+2mT(r,f^{(k)})+S(r,f^{(k)}) \\
\leq\ & \frac{2m+1+\frac{1}{k+1}}{n}T(r,F)+S(r,F),
\end{aligned}$$

which is a contradiction as $n \geq 4m+2$.

Thus $\gamma+\delta=1$ and $\gamma \neq 0$.

So,

$$F=\frac{1}{\gamma G+1-\gamma},$$

From above we get $\overline{N}(r,0;G+\frac{1-\gamma}{\gamma})=\overline{N}(r,\infty;F)$.

If $\gamma \neq 1$, by using the Second Fundamental Theorem and (3.4), we get

$$\begin{aligned}
& T(r,G) \\
\leq\ & \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,0;G+\frac{1-\gamma}{\gamma})+S(r,G) \\
\leq\ & \overline{N}(r,\infty;g^{(k)})+\overline{N}(r,0;g^{(k)})+2mT(r,g^{(k)})+\overline{N}(r,\infty;f^{(k)})+S(r,g^{(k)}) \\
\leq\ & \frac{2m+1+\frac{2}{k+1}}{n}T(r,F)+S(r,F),
\end{aligned}$$

which is a contradiction as $n \geq 4m+2$.

Thus $\gamma=1$ and $FG \equiv 1$ which is not possible by Lemma 3.4.

Subcase-2.2 $A \neq 0$ and $C=0$

In this case $D \neq 0$ and

$$F=\lambda G+\mu,$$

where $\lambda=\frac{A}{D}$ and $\mu=\frac{B}{D}$.

If F has no 1 point, then similarly as above we get a contradiction.

Thus $\lambda+\mu=1$ with $\lambda \neq 0$.

Clearly $\overline{N}(r,0;G+\frac{1-\lambda}{\lambda})=\overline{N}(r,0;F)$.

Let $\lambda\neq 1$ and $\xi=\frac{(1-\frac{2n}{n-m}+\frac{n}{n-2m})}{c}$. Then $F+\xi=(f^{(k)}-1)^3Q_{n-3}(f^{(k)})$, where $Q_{n-3}(1)\neq 0$ and $Q_{n-3}(z)$ is a $(n-3)$ degree polynomial.

If $\frac{1-\lambda}{\lambda}\neq\xi$, then by using the Second Fundamental Theorem and (3.4), we get

$$
\begin{aligned}
& 2T(r,G) \\
\leq\; & \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,0;G+\frac{1-\lambda}{\lambda})+\overline{N}(r,0;G+\xi)+S(r,G) \\
\leq\; & \overline{N}(r,\infty;g^{(k)})+\overline{N}(r,0;g^{(k)})+2mT(r,g^{(k)})+\overline{N}(r,0;f^{(k)})+2mT(r,f^{(k)}) \\
+\; & \overline{N}(r,1;g^{(k)})+(n-3)T(r,g^{(k)})+S(r,g^{(k)}) \\
\leq\; & \frac{4m+n+\frac{1}{k+1}}{n}T(r,G)+S(r,G),
\end{aligned}
$$

which is a contradiction as $n\geq 4m+2$.

If $\frac{1-\lambda}{\lambda}=\xi$, then $\lambda G=F-\lambda\xi$. As $c\neq-\frac{1-\frac{2n}{n-m}+\frac{n}{n-2m}}{2}$ so $\lambda\neq-1$.

Now applying the Second Fundamental Theorem and (3.4), we get

$$
\begin{aligned}
& 2T(r,F) \\
\leq\; & \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,0;F-\lambda\xi)+\overline{N}(r,0;F+\xi)+S(r,F) \\
\leq\; & \overline{N}(r,\infty;f^{(k)})+\overline{N}(r,0;g^{(k)})+2mT(r,g^{(k)})+\overline{N}(r,0;f^{(k)})+2mT(r,f^{(k)}) \\
+\; & \overline{N}(r,1;f^{(k)})+(n-3)T(r,f^{(k)})+S(r,g^{(k)}) \\
\leq\; & \frac{4m+n+\frac{1}{k+1}}{n}T(r,F)+S(r,F),
\end{aligned}
$$

which is a contradiction as $n>4m+1$.

Thus $\lambda=1$ and $F\equiv G$. Consequently by Lemma 3.3, $f^{(k)}\equiv g^{(k)}$. □

4. Proof of the theorems

Proof of Theorem2.1 . It is clear that $\overline{N}(r,\infty;f^{(k)})\leq\frac{1}{k+1}N(r,\infty;f^{(k)})$.

Case-1 $H\not\equiv 0$

Clearly $F' = -\frac{n}{c}(f^{(k)})^{n-2m-1}((f^{(k)})^m - 1)^2(f^{(k+1)})$,
and $G' = -\frac{n}{c}(g^{(k)})^{n-2m-1}((g^{(k)})^m - 1)^2(g^{(k+1)})$.
Now by simple calculations,

$$\begin{aligned} & N(r,\infty;H) \\ \leq\ & \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2) + \overline{N}(r,\infty;F) \\ +\ & \overline{N}(r,\infty;G) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of zeros of F' which is not zeros of $F(F-1)$. Thus

$$\begin{aligned} & N(r,\infty;H) \\ \leq\ & \overline{N}(r,0;f^{(k)}) + \overline{N}(r,0;g^{(k)}) + \overline{N}(r,0;((g^{(k)})^m - 1)) \\ +\ & \overline{N}(r,0;((f^{(k)})^m - 1)) + \overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,\infty;g^{(k)}) \\ +\ & \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;f^{(k+1)}) + \overline{N}_0(r,0;g^{(k+1)}), \end{aligned} \tag{4.1}$$

where $\overline{N}_0(r,0;f^{(k+1)})$ is the reduced counting function of zeros of $f^{(k+1)}$ which is not zeros of $f^{(k)}((f^{(k)})^m - 1)$ and $(F-1)$.

Clearly

$$\overline{N}(r,1;F|=1) = \overline{N}(r,1;G|=1) \leq N(r,\infty;H). \tag{4.2}$$

Now by using the Second Fundamental Theorem, (4.1), (4.2) and Lemma 3.5 we get

$$
\begin{aligned}
(4.3)\qquad & (n+m)(T(r,f^{(k)})+T(r,g^{(k)}))\\
\leq\ & \overline{N}(r,\infty;f^{(k)})+\overline{N}(r,0;f^{(k)})+\overline{N}(r,\infty;g^{(k)})+\overline{N}(r,0;g^{(k)})\\
+\ & \overline{N}(r,1;F)+\overline{N}(r,1;G)+\overline{N}(r,0;(f^{(k)})^m-1)+\overline{N}(r,0;(g^{(k)})^m-1)\\
-\ & N_0(r,0,f^{(k+1)})-N_0(r,0,g^{(k+1)})+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\ & 2\{\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})\}+2\{\overline{N}(r,0;f^{(k)})+\overline{N}(r,0;g^{(k)})\\
+\ & \overline{N}(r,0;((g^{(k)})^m-1))+\overline{N}(r,0;((f^{(k)})^m-1))\}+\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
-\ & \overline{N}(r,1;F|=1)+\overline{N}_*(r,1;F,G)+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}
$$

$$
\begin{aligned}
(4.4)\qquad & (\frac{n}{2}-m)(T(r,f^{(k)})+T(r,g^{(k)}))\\
\leq\ & 2\{\overline{N}(r,\infty;f^{(k)})+\overline{N}_*(r,\infty;g^{(k)})+\overline{N}(r,0;f^{(k)})+\overline{N}(r,0;g^{(k)})\}\\
+\ & (\frac{3}{2}-l)\overline{N}_*(r,1;F,G)+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}
$$

That is

$$
\begin{aligned}
(4.5)\qquad & (\frac{n}{2}-m-2-\frac{2}{k+1})(T(r,f^{(k)})+T(r,g^{(k)}))\\
\leq\ & (\frac{3}{2}-l)\overline{N}_*(r,1;F,G)+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}
$$

Subcase-1.1 $l\geq 2$

We get a contradiction from (4.5) when $n>2m+4+\frac{4}{k+1}$.

Subcase-1.2 $l=1$

In this case

$$\begin{aligned} & N_*(r,1;F,G) = \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) \\ \leq\ & \frac{1}{2}(N(r,0;f^{(k+1)}|f^{(k)} \neq 0) + N(r,0;g^{(k+1)}|g^{(k)} \neq 0)) \\ \leq\ & \frac{1}{2}(\overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,0;f^{(k)}) + \overline{N}(r,\infty;g^{(k)}) + \overline{N}(r,0;g^{(k)})) \\ +\ & S(r,f^{(k)}) + S(r,g^{(k)}) \\ \leq\ & \frac{1}{2}(1+\frac{1}{k+1})(T(r,f^{(k)}) + T(r,g^{(k)})) + S(r,f^{(k)}) + S(r,g^{(k)}). \end{aligned}$$

Thus (4.5) becomes

$$\begin{aligned} & (\frac{n}{2} - m - 2 - \frac{2}{k+1})(T(r,f^{(k)}) + T(r,g^{(k)})) \\ \leq\ & \frac{1}{4}(1+\frac{1}{k+1})(T(r,f^{(k)}) + T(r,g^{(k)})) + S(r,f^{(k)}) + S(r,g^{(k)}), \end{aligned} \tag{4.6}$$

which is a contradiction when $n > 2m + 4.5 + \frac{4.5}{k+1}$.

Subcase-1.3 $l = 0$

In this case

$$\begin{aligned} & N_*(r,1;F,G) = \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) \\ \leq\ & (N(r,0;f^{(k+1)}|f^{(k)} \neq 0) + N(r,0;g^{(k+1)}|g^{(k)} \neq 0)) \\ \leq\ & (\overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,0;f^{(k)}) + \overline{N}(r,\infty;g^{(k)}) + \overline{N}(r,0;g^{(k)})) \\ +\ & S(r,f^{(k)}) + S(r,g^{(k)}) \\ \leq\ & (\overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,0;f^{(k)}) + \overline{N}(r,\infty;g^{(k)}) + \overline{N}(r,0;g^{(k)})) \\ +\ & S(r,f^{(k)}) + S(r,g^{(k)}) \\ \leq\ & (1+\frac{1}{k+1})(T(r,f^{(k)}) + T(r,g^{(k)})) + S(r,f^{(k)}) + S(r,g^{(k)}). \end{aligned}$$

Thus (4.5) becomes

$$
\begin{aligned}
(4.7) \qquad & (\frac{n}{2} - m - 2 - \frac{2}{k+1})(T(r, f^{(k)}) + T(r, g^{(k)})) \\
\leq \; & \frac{3}{2}(1 + \frac{1}{k+1})(T(r, f^{(k)}) + T(r, g^{(k)})) + S(r, f^{(k)}) + S(r, g^{(k)}),
\end{aligned}
$$

which is a contradiction when $n > 2m + 7 + \frac{7}{k+1}$.

Case-2 $H \equiv 0$

From the Lemma 3.7 we obtained $f^{(k)} \equiv g^{(k)}$ when $n \geq 4m + 2$. □

Proof of Theorem2.2 . **Case-1** $H \not\equiv 0$

Then clearly $F \not\equiv G$.

As f and g share $(0, q)$, we have

$$
\begin{aligned}
(4.8) \qquad & N(r, \infty; H) \\
\leq \; & \overline{N}(r, \infty; f^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + \overline{N}(r, 0; ((g^{(k)})^m - 1)) \\
+ \; & \overline{N}(r, 0; ((f^{(k)})^m - 1)) + \overline{N}_*(r, 0; f^{(k)}, g^{(k)}) \\
+ \; & \overline{N}_*(r, 1; F, G) + \overline{N}_0(r, 0; f^{(k+1)}) + \overline{N}_0(r, 0; g^{(k+1)}),
\end{aligned}
$$

where $\overline{N}_0(r, 0; f^{(k+1)})$ is the reduced counting function of zeros of $f^{(k+1)}$ which is not zeros of $f^{(k)}((f^{(k)})^m - 1)$ and $(F - 1)$.

Now using the Second Fundamental Theorem, (4.2), (4.8) and Lemma 3.5 we get

$$
\begin{aligned}
(4.9)\qquad & (\frac{n}{2}-m)(T(r,f^{(k)})+T(r,g^{(k)})) \\
\leq\ & 2\overline{N}(r,0;f^{(k)})+\overline{N}_*(r,0;f^{(k)},g^{(k)})+2\{\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})\} \\
+\ & (\frac{3}{2}-l)\overline{N}_*(r,1;F,G)+S(r,f^{(k)})+S(r,g^{(k)}) \\
\leq\ & 2\overline{N}(r,0;f^{(k)})+\overline{N}(r,0;f^{(k)}|\geq q+1)+2\{\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})\} \\
+\ & (\frac{3}{2}-l)\overline{N}_*(r,1;F,G)+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}
$$

Thus by the help of Lemma 3.6 we have

$$
\begin{aligned}
(4.10)\ & (\frac{n}{2}-m-\frac{2}{k+1})(T(r,f^{(k)})+T(r,g^{(k)})) \\
\leq\ & 2\overline{N}(r,0;f^{(k)})+\overline{N}(r,0;f^{(k)}|\geq q+1)+(\frac{3}{2}-l)\overline{N}_*(r,1;F,G) \\
+\ & S(r,f^{(k)})+S(r,g^{(k)}) \\
\leq\ & (\frac{2}{(k+1)(n-2m-1)}+\frac{1}{(k+1)((n-2m)q+n-2m-1)})\{T(r,f^{(k)}) \\
+\ & T(r,g^{(k)})\}+(\frac{2}{n-2m-1}+\frac{1}{(n-2m)q+n-2m-1}+\frac{3}{2}-l)\overline{N}_*(r,1;F,G) \\
+\ & S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}
$$

Thus when $l\geq\frac{3}{2}+\frac{2}{n-2m-1}+\frac{1}{(n-2m)q+n-2m-1}$,
and $n>2m+\frac{4}{k+1}+\frac{4}{(k+1)(n-2m-1)}+\frac{2}{(k+1)((n-2m)q+n-2m-1)}$,
we get a contradiction from (4.10).

Case-2 $H\equiv 0$

From the Lemma 3.7 we obtained $f^{(k)}\equiv g^{(k)}$ when $n\geq 4m+2$. □

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References

[1] A. Banerjee, On the uniqueness of meromorphic functions that share two sets, *Georgian Math. J.*, **15**(1), 2008, 1-18.

[2] A. Banerjee, Uniqueness of meromorphic functions sharing two sets with finite weight II, *Tamkang J. Math.*, **41**(4), 379-392, (2010).

[3] A. Banerjee and P. Bhattacharjee, Uniqueness of derivatives of meromorphic functions sharing two or three sets, *Turk. J. Math.*, **34**(2010), 21-34.

[4] A. Banerjee and P. Bhattacharjee, Uniqueness and set sharing of derivatives of meromorphic functions, *Math. Slov.*, **61**(2011), 197-214.

[5] A. Banerjee and I. Lahiri, A Uniqueness Polynomial Generating A Unique Range Set And Vise Versa, *Comput. Method Funct. Theory*, **12**(2)(2012), 527-539.

[6] M. L. Fang, and H. Guo, On meromorphic functions sharing two values, *Analysis*, **17**(1997), 355-366.

[7] M. L. Fang and W S. Xu, A note on a problem of Gross, *Chin Ann Math*, **18A**(1997), 563-568

[8] M. L. Fang and I. Lahiri, Unique range set for certain meromorphic functions, *Indian J. Math.*, **45**(2)(2003), 141-150.

[9] H. Fujimoto, On uniqueness of meromorphic functions sharing finite sets., *Amer. J. Math*, **122**(2000), 1175-1203.

[10] F.Gross, Factorization of meromorphic functions and some open problems, *Proc. Conf. Univ. Kentucky, Leixngton*, Ky(1976); *Lecture Notes in Math.*, **599**(1977), 51-69, Springer(Berlin).

[11] W. K. Hayman, Meromorphic Functions, *The Clarendon Press, Oxford (1964).*

[12] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, *Nagoya Math. J.*, **161**(2001), 193-206.

[13] P.Li and C.C.Yang, Some further results on the unique range set of meromorphic functions, *Kodai Math. J.*, **18**(1995), 437-450..

[14] P.Li and C.C.Yang, On the unique range set for meromorphic functions, *Proc. Amer. Math. Soc.*, **124**(1996), 177-185.

[15] A. Z. Mokhon'ko, On the Nevanlinna characteristics of some meromorphic functions, in "*Theory of Functions, functional analysis and their applications*", Izd-vo Khar'kovsk, Un-ta, **14** (1971), 83-87.

[16] H. X. Yi, Uniqueness of meromorphic functions and a question of Gross, *Science in China*, 1994, **37A**, 802-813

[17] H.X.Yi, On a question of Gross concerning uniqueness of entire functions, *Bull Austral Math Soc*, 1998, **57**, 343-349

[18] H. X. Yi, Meromorphic functions that share two sets, *Acta Math Sinica*, 2002, **45**, 75-82

[19] H.X.Yi and W.R.Lü, Meromorphic functions that share two sets II, *Acta Math. Sci. Ser.B Engl. Ed.*, **24**(2004) (1), 83-90.

[20] H.X.Yi and W.C.Lin, Uniqueness of Meromorphic Functions and a Question of Gross, *Kyungpook Math. J.*, **46**(2006), 437-444.

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