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BOUNDEDNESS OF COMMUTATORS ON HERZ-TYPE HARDY SPACES WITH VARIABLE EXPONENT

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Abstract. In this paper, we obtain the boundedness of some commutators generated by the Calderón-Zygmund singular integral operator, the Littlewood-Paley operator and BMO functions on Herz-type Hardy spaces with variable exponent.

1. Introduction

The theory of function spaces with variable exponent has extensively studied by researchers since the work of Kováčik and Rákosník [4] appeared in 1991. In [1] and [8], the authors proved the boundedness of some integral operators on variable L^p spaces, respectively. In addition, the authors defined the Herz-type Hardy spaces with variable exponent and gave their atomic characterizations in [9].

Motivated by [1], [5] and [7], we will study the boundedness of some commutators generated by the Calderón-Zygmund singular integral operator, the Littlewood-Paley operator and BMO functions on the Herz-type Hardy space with variable exponent.

Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \longrightarrow [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

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This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_{\Omega} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\}.$$

These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$.

For all compact subsets $E \subset \Omega$, the space $L^{p(\cdot)}_{\text{loc}}(\Omega)$ is defined by $L^{p(\cdot)}_{\text{loc}}(\Omega) := \{f : f \in L^{p(\cdot)}(E)\}$. Define $\mathcal{P}(\Omega)$ to be set of $p(\cdot) : \Omega \longrightarrow [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1. ([4]) *Let $p(\cdot) \in \mathcal{P}(\Omega)$. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, then fg is integrable on Ω and*

$$\int_{\Omega} |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.2. ([2]) *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1},$$

and

$$\frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 *are constants with* $0 < \delta_1, \delta_2 < 1$ *and* χ_S, χ_B *are the characteristic functions of* S, B *respectively.*

Throughout this paper δ_2 is the same as in Lemma 1.2.

Lemma 1.3. ([2]) *Suppose* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Then there exists a constant* $C > 0$ *such that for all balls* B *in* $\mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

Recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_* = \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx < \infty,$$

where $f_Q = |Q|^{-1}\int_Q f(y)dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of Q.

Lemma 1.4. ([3]) *Let* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, k *be a positive integer and* B *be a ball in* $\mathbb{R}^n$. *Then we have that for all* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and all* $j, i \in \mathbb{Z}$ *with* $j > i$,

$$\frac{1}{C}\|b\|_*^k \le \sup_{B:ball} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b - b_B)^k\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*^k,$$

$$\|(b - b_{B_i})^k\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(j - i)^k\|b\|_*^k\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $B_i = \{x \in \mathbb{R}^n : |x| \le 2^i\}$ *and* $B_j = \{x \in \mathbb{R}^n : |x| \le 2^j\}$.

Next we recall the definition of the Herz-type spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$.

Definition 1.1. *([2]) Let* $\alpha \in \mathbb{R}, 0 < p \leq \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.$$

The non-homogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.$$

In [9], the authors gave the definition of Herz-type Hardy space with variable exponent and the atomic decomposition characterizations. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N(f)(x)$ be the grand maximal function of $f(x)$ defined by

$$G_N(f)(x) = \sup_{\phi \in \mathcal{A}_N} |\phi^*_\nabla(f)(x)|,$$

where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \leq N} |x^\alpha D^\beta \phi(x)| \leq 1\}$ and $N > n+1$, ϕ^*_∇ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 1.2. *([9]) Let* $\alpha \in \mathbb{R}, 0 < p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $N > n+1$.

(i) The homogeneous Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \right\}$$

and $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

(ii) The non-homogeneous Herz-type Hardy space with variable exponent $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \right\}$$

and $\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

For $x \in \mathbb{R}$ we denote by $[x]$ the largest integer less than or equal to x. Similar to [9], we have

Definition 1.3. Let $n\delta_2 \le \alpha < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n), s \in \mathbb{N}$ and $b \in L^{q'(\cdot)}_{\text{loc}}(\mathbb{R}^n)$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot), s; b)$-atom, if it satisfies

(1) $\operatorname{supp} a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |B(0,r)|^{-\alpha/n}$.

(3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = \int_{\mathbb{R}^n} a(x)b(x)x^\beta dx = 0, |\beta| \le s$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot), s; b)$-atom of restricted type, if it satisfies the conditions (2), (3) above and

(1)$'$ $\operatorname{supp} a \subset B(0,r), r \ge 1$.

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 1.3, then the corresponding central $(\alpha, q(\cdot), s; b)$-atom is called a dyadic central $(\alpha, q(\cdot), s; b)$-atom.

Lemma 1.5. *Let $n\delta_2 \le \alpha < \infty$, $0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$(or $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$) if and only if*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \left(\text{or} \sum_{k=0}^{\infty} \lambda_k a_k \right), \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k *is a central* $(\alpha, q(\cdot), s; b)$*-atom(or central* $(\alpha, q(\cdot), s; b)$*-atom of restricted type) with support contained in* B_k *and* $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$*(or* $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$*). Moreover,*

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} \left(\text{or} \quad \|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p} \right),$$

where the infimum is taken over all above decompositions of f*.*

2. Boundedness of the commutator of the Calderón-Zygmund singular integral operator

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and let T be the Calderón-Zygmund singular integral operator of Coifman and Meyer, that is,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy, \quad x \notin \operatorname{supp} f,$$

with the kernel $K(x,y) : \mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$ satisfying

$$|K(x,y)| \leq \frac{C}{|x-y|^n},$$

$$|K(x,y) - K(x,z)| \leq C \frac{|y-z|^{\varepsilon}}{|x-y|^{n+\varepsilon}}, \quad 2|y-z| < |x-y|,$$

$$|K(x,y) - K(w,y)| \leq C \frac{|x-w|^{\varepsilon}}{|x-y|^{n+\varepsilon}}, \quad 2|x-w| < |x-y|,$$

for some $\varepsilon \in (0,1]$. The commutator $[b,T]$ generated by b and T is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

In [1], the authors proved that the commutator $[b,T]$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ for $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. In this section we will generalize the result to the case of Herz-type Hardy spaces with variable exponent.

Theorem 2.1. *Suppose* $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < p < \infty$, $n\delta_2 \leq \alpha < n\delta_2 + \varepsilon$ *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$*. Then the commutator* $[b,T]$ *is bounded from* $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *(or* $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$*) to* $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *(or* $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$*).*

Proof. We only prove homogeneous case. The non-homogeneous case can be proved in the same way. Let $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. By Lemma 1.5, we get $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p}$ (the infimum is taken over above decompositions of f), and a_j is a dyadic central $(\alpha, q(\cdot), 0; b)$-atom with the support B_j. We have

$$
\begin{aligned}
\|[b,T]f\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|\chi_k [b,T] f\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|\chi_k [b,T] a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|\chi_k [b,T] a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
&=: E_1 + E_2.
\end{aligned} \tag{2.1}
$$

Now we estimate E_2, by the $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness of the commutator $[b,T]$ we have

$$
\begin{aligned}
E_2 &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
&\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha} \right)^p \\
&\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p.
\end{aligned} \tag{2.2}
$$

Let us now estimate E_1. For each $j \le k-2$, we first compute $\|\chi_k [b,T] a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}$. Let b_{B_j} be the mean value of b on the ball B_j. Note that if $x \in A_k$, $y \in B_j$ and $j \le k-2$, then $2|y| < |x|$. Using the vanishing moments of a_j and the generalized

Hölder inequality we have

$$
\begin{aligned}
&|[b,T]a_j(x)| \\
&\le |b(x)-b_{B_j}|\int_{B_j}|k(x,y)-k(x,0)||a_j(y)|dy+\int_{B_j}|k(x,y)-k(x,0)||b_{B_j}-b(y)||a_j(y)|dy \\
&\le C2^{j\varepsilon-k(n+\varepsilon)}\left\{|b(x)-b_{B_j}|\int_{B_j}|a_j(y)|dy+\int_{B_j}|b_{B_j}-b(y)||a_j(y)|dy\right\} \\
&\le C2^{j\varepsilon-k(n+\varepsilon)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\{|b(x)-b_{B_j}|\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}.
\end{aligned}
$$

So by Lemma 1.1-1.4 we have

$$
\begin{aligned}
&\|\chi_k[b,T]a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C2^{j\varepsilon-k(n+\varepsilon)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\times\left\{\|(b-b_{B_j})\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right\} \\
&\le C2^{j\varepsilon-k(n+\varepsilon)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\times\left\{(k-j)\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|b\|_*\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right\} \\
&\le C2^{j\varepsilon-k(n+\varepsilon)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}(k-j)\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\le C(k-j)2^{(j-k)\varepsilon}\|b\|_*\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \\
&\le C(k-j)2^{-j\alpha+(j-k)(\varepsilon+n\delta_2)}\|b\|_*.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
E_1 &\le C\|b\|_*^p\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|(k-j)2^{-j\alpha+(j-k)(\varepsilon+n\delta_2)}\right)^p \\
&= C\|b\|_*^p\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|(k-j)2^{(j-k)(\varepsilon+n\delta_2-\alpha)}\right)^p.
\end{aligned}
$$

When $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $\varepsilon + n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$
\begin{aligned}
E_1 &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p/2} \right) \\
&\qquad \times \left(\sum_{j=-\infty}^{k-2} (k-j)^{p'} 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p'/2} \right)^{p/p'} \\
&\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p/2} \right) \\
&= C\|b\|_*^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p/2} \right) \\
&\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p.
\end{aligned} \tag{2.3}
$$

When $0 < p \le 1$, we have

$$
\begin{aligned}
E_1 &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p (k-j)^p 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p} \right) \\
&= C\|b\|_*^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=j+2}^{\infty} (k-j)^p 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p} \right) \\
&\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p.
\end{aligned} \tag{2.4}
$$

Therefore, by (2.1)-(2.4) we complete the proof of Theorem 2.1.

3. Boundedness of the commutator of the Littlewood-Paley operator

Let $\varepsilon > 0$ and fix a function ψ satisfying the following properties:

(1) $\int_{\mathbb{R}^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+\varepsilon)}$,

(3) $|\psi(x+y) - \psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

The Littlewood-Paley operator is defined by

$$S_\psi f(x) = \left(\int_{\Gamma(x)} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Let b be a locally integrable function. The commutator of the Littlewood-Paley operator is defined by

$$[b, S_\psi]f(x) = \left(\int_{\Gamma(x)} |F_{b,t}(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where

$$F_{b,t}(f)(x,y) = \int_{\mathbb{R}^n} \psi_t(y-z) f(z)(b(x) - b(z))dz.$$

In [1], the authors proved that the operator S_ψ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ for $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Similar to the method of [1], we can easily obtain that the commutator $[b, S_\psi]$ is also bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ for $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. In this section we will generalize the result to the case of Herz-type Hardy spaces with variable exponent.

Theorem 3.1. *Suppose $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < p < \infty$, $n\delta_2 \leq \alpha < n\delta_2 + \varepsilon$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then the commutator $[b, S_\psi]$ is bounded from $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (or $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$) to $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$).*

Proof. We only prove homogeneous case. The non-homogeneous case can be proved in the same way. Let $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. By Lemma 1.5, we get $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p}$ (the infimum is taken over above decompositions of f), and a_j is a dyadic central $(\alpha, q(\cdot), 0; b)$-atom with the support B_j. We have

$$
\begin{aligned}
\|[b,S_\psi]f\|^p_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|\chi_k[b,S_\psi]f\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|\chi_k[b,S_\psi]a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\quad + C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|\chi_k[b,S_\psi]a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&=: F_1+F_2.
\end{aligned}
\tag{3.1}
$$

Now we estimate F_2, by the $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness of the commutator $[b,S_\psi]$ we have

$$
\begin{aligned}
F_2 &\le C\|b\|_*^p\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\le C\|b\|_*^p\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{\infty}|\lambda_j|2^{(k-j)\alpha}\right)^p \\
&\le C\|b\|_*^p\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}
\tag{3.2}
$$

Let us now estimate F_1. Similar to the proof of Theorem 2.1, for each $j\le k-2$, we first compute $\|\chi_k[b,S_\psi]a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}$. Let $b_{B_j} = |B_j|^{-1}\int_{B_j} b(x)dx$. Note that if $x\in A_k$, $y\in B_j$ and $j\le k-2$, then $2|y|<|x|$. Using the vanishing moments of a_j

and the generalized Hölder inequality we have

$$\begin{aligned}
&|[b,S_\psi]a_j(x)|\\
&\le \left[\int_{\Gamma(x)}\left(\int_{B_j}|\psi_t(y-z)-\psi_t(y-0)||a_j(z)||b(x)-b(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\
&\le C\left[\int_{\Gamma(x)}\left(\int_{B_j}t^{-n}\frac{(|z|/t)^\varepsilon}{(1+|y|/t)^{n+1+\varepsilon}}|a_j(z)||b(x)-b(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\
&\le C2^{j\varepsilon}\left[\int_{\Gamma(x)}\left(\int_{B_j}\frac{t}{(t+|y|)^{n+1+\varepsilon}}|a_j(z)||b(x)-b(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\
&\le C2^{j\varepsilon}\left[\int_{\Gamma(x)}\frac{t^{1-n}}{(t+|y|)^{2(n+1+\varepsilon)}}dydt\right]^{1/2}\int_{B_j}|a_j(z)||b(x)-b(z)|dz\\
&\le C2^{j\varepsilon}\left[\int_0^\infty\frac{t}{(t+|x|)^{2(n+1+\varepsilon)}}dt\right]^{1/2}\int_{B_j}|a_j(z)||b(x)-b(z)|dz\\
&\le C2^{j\varepsilon-k(n+\varepsilon)}\left\{|b(x)-b_{B_j}|\int_{B_j}|a_j(z)|dz+\int_{B_j}|b_{B_j}-b(z)||a_j(z)|dz\right\}\\
&\le C2^{j\varepsilon-k(n+\varepsilon)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\{|b(x)-b_{B_j}|\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}.
\end{aligned}$$

So by Lemma 1.1-1.4 we have

$$\begin{aligned}
&\|\chi_k[b,S_\psi]a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{j\varepsilon-k(n+\varepsilon)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\left\{\|(b-b_{B_j})\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b)\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}\\
&\le C2^{j\varepsilon-k(n+\varepsilon)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\left\{(k-j)\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|b\|_*\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}\\
&\le C2^{j\varepsilon-k(n+\varepsilon)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}(k-j)\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\le C(k-j)2^{(j-k)\varepsilon}\|b\|_*\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\
&\le C(k-j)2^{-j\alpha+(j-k)(\varepsilon+n\delta_2)}\|b\|_*.
\end{aligned}$$

Thus we obtain

$$\begin{aligned} F_1 &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|(k-j)2^{-j\alpha+(j-k)(\varepsilon+n\delta_2)} \right)^p \\ &= C\|b\|_*^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|(k-j)2^{(j-k)(\varepsilon+n\delta_2-\alpha)} \right)^p. \end{aligned}$$

When $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $\varepsilon + n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned} F_1 &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p/2} \right) \\ &\qquad \times \left(\sum_{j=-\infty}^{k-2} (k-j)^{p'} 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p'/2} \right)^{p/p'} \\ &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p/2} \right) \\ &= C\|b\|_*^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p/2} \right) \\ &\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p. \end{aligned} \tag{3.3}$$

When $0 < p \le 1$, we have

$$\begin{aligned} F_1 &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p (k-j)^p 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p} \right) \\ &= C\|b\|_*^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=j+2}^{\infty} (k-j)^p 2^{(j-k)(\varepsilon+n\delta_2-\alpha)p} \right) \\ &\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p. \end{aligned} \tag{3.4}$$

Therefore, by (3.1)-(3.4) we complete the proof of Theorem 3.1.

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