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CHARACTERIZATION OF THE GELFAND-SHILOV SPACES OF BEURLING TYPE AND ITS DUAL VIA SHORT-TIME FOURIER TRANSFORM

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Abstract. We characterize Gelfand-Shilov spaces Σ_α^β of test functions of tempered ultradistribution, in terms of their short-time Fourier transform using its symmetric characterization via the Fourier transform. Using Riesz representation theorem, we prove structure theorem for functionals in dual space $(\Sigma_\alpha^\beta)'$.

1. Introduction

In mathematical analysis, distributions (generalized functions) are objects which generalize functions. They extend the concept of derivative to all integrable functions and beyond, and used to formulate generalized solutions of partial differential equations. They play a crucial rule in physics and engineering where many non-continuous problems naturally lead to differential equations whose solutions are distributions, such as the Dirac delta distribution. The theory of generalized functions devised by L. Schwartz was to provide a satisfactory framework for the Fourier transform(see [10]).

Some other types of distributions called ultradistributions have also been studied by Gelfand and Shilov (see [6]) which are well-known in the theory of tempered

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ultradistribution. S. Pilipovic obtained structural theorems for Gelfand-Shilov spaces of Roumieu and Beurling type (see [8], [9]).

In (see [4]), K. Göchenig and G. Zimmermann obtained a characterization of Gelfand-Shilov spaces of Roumieu type via short-time Fourier transform and the Gelfand-Shilov spaces are connected to Modulation spaces.

In this paper, we characterize Gelfand-Shilov spaces of Beurling type of test functions of tempered ultradistribution in terms of their short-time Fourier transform. As a result of this characterization and using Riesz representation theorem, we prove structure theorem for functionals in dual space $(\Sigma_\alpha^\beta)'$.

The symbols C^∞, C_0^∞, L^p, etc., denote the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ indicates the p-norm in the space L^p, where $1 \leq p \leq \infty$. In general, we work on the Euclidean space $\mathbb{R}^n$ unless we indicate other than that as appropriate. The Fourier transform of a function f will be denoted $\mathcal{F}(f)$ or $\widehat{f}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x)\,dx$. With $\mathcal{C}_0$ we denote the Banach space of continuous functions vanishing at infinity with supremum norm.

2. Preliminary definitions and results

J. Chung et al proved symmetric characterizations for Gelfand-Shilov spaces via the Fourier transform in terms of the growth of the function and its Fourier transform which imposes no conditions on the derivative.

Theorem 2.1. *([1]) The space Σ_α^β can be described as a set as well as topologically by*

$$\Sigma_\alpha^\beta = \left\{ \begin{array}{l} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and for all} \\ k = 0, 1, 2, ..., \; p_{k,0}(\varphi) < \infty, \pi_{k,0}(\varphi) < \infty \end{array} \right\},$$

where $p_{k,0}(\varphi) = \left\| e^{k|x|^{1/\alpha}} \varphi \right\|_{\infty}$, $\pi_{k,0}(\varphi) = \left\| e^{k|\xi|^{1/\beta}} \widehat{\varphi} \right\|_{\infty}$.
The space Σ_{α}^{β}*, equipped with the family of semi-norms*

$$\mathcal{N} = \{p_{k,0}, \pi_{k,0} : k \in \mathbb{N}_0\},$$

is a Fréchet space.

Remark 1. For $\alpha > 1$, the function $|\bullet|^{1/\alpha} : [0,\infty) \to [0,\infty)$ has the following properties:

(1) $|\bullet|^{1/\alpha}$ is increasing, continuous and concave,
(2) $|t|^{1/\alpha} \geq a + b\ln(1+t)$ for some $a \in \mathbb{R}$ and some $b > 0$.

Remark 2. Let us observe for future use that if we take $N > \frac{n}{b}$ is an integer, then

$$C_N = \int_{\mathbb{R}^n} e^{-N|x|^{1/\alpha}} dx < \infty, \text{ for all } \alpha > 1,$$

where b is the constant in property 2 of Remark 1. Moreover, property 1 in Remark 1 implies that $|\bullet|^{1/\alpha}$ is subaddative.

Example 2.1. *From Theorem 2.1, it is clear that the Gaussian* $f(x) = e^{-\pi|x|^2}$ *belongs to* Σ_{α}^{β} *for all* $\alpha > 1$ *and* $\beta > 1$.

It is well known that Fourier series are a good tool to represent periodic functions. However, they fail to represent non-periodic functions accurately. To solve this problem, the short-time Fourier transform was introduced by D. Gabor [2]. The short-time Fourier transform works by first cutting off the function by multiplying it by another function called window then apply the Fourier transform. This technique maps a function of time x into a function of time x and frequency ξ .

Definition 2.1. ([3], [4])The short-time Fourier transform (STFT) of a function or distribution f on $\mathbb{R}^n$ with respect to a non-zero window function g is formally defined as

$$\nu_g f(x,\xi) = \int_{\mathbb{R}^n} f(t)\overline{g(t-x)}e^{-2\pi i t.\xi}dt = (\widehat{f\overline{T_x g}})(\xi) =< f, M_\xi T_x g > .$$

where $T_x g(t) = g(t-x)$ is the translation operator and $M_\xi g(t) = e^{2\pi i t.\xi}g(t)$ is the modulation operator.

The composition of T_x and M_ξ is the time-frequency shift

$$(M_\xi T_x g)(t) = e^{2\pi i x.\xi}g(t-x),$$

and its Fourier transform is given by

$$\widehat{M_\xi T_x g} = e^{2\pi i x.\xi}M_{-x}T_\xi \widehat{g}.$$

The main properties of the short-time Fourier transform is given in the following lemma.

Lemma 2.1. *([3], [4])For $f, g \in \Sigma_\alpha^\beta$, the STFT has the following properties.*

(1) (Inversion formula)

$$\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \nu_g f(x,\xi)(M_\xi T_x g)(t)dxd\xi = \|g\|_2^2 f. \tag{2.1}$$

(2) (STFT of the Fourier transforms)

$$\nu_{\widehat{g}}\widehat{f}(x,\xi) = e^{-2\pi i x.\xi}\nu_g f(-\xi, x).$$

(3) (Fourier transform of the STFT)

$$\widehat{\nu_g f}(x,\xi) = e^{2\pi i x.\xi}f(-\xi)\overline{\widehat{g}(x).} \tag{2.2}$$

Remark 3. The space $\nu_g(\Sigma_\alpha^\beta) = \{\nu_g f : f \in \Sigma_\alpha^\beta\}$ has no functions with compact support.

Now we will introduce two auxiliary results that we will use in the proof of the topological characterization of the space Σ_α^β via the short-time Fourier transform.

Lemma 2.2. *([4])Let f and g be two nonnegative measurable functions. If $N > n$, there exists $C > 0$ such that*

$$\left\| e^{k|\bullet|^{1/\alpha}}(f * g) \right\|_\infty \leq C \left\| e^{2(N+k)|\bullet|^{1/\alpha}} f \right\|_\infty \left\| e^{2(N+k)|\bullet|^{1/\alpha}} g \right\|_\infty ,$$

for all $k = 0, 1, 2, ...$ The constant C does not depend on k.

In the following lemma, we include a proof using the topological characterization of Σ_α^β given in Theorem 2.1 which imposes no conditions on the derivative. Our proof is an adaptation of the proof of (Proposition 2.6 stated in [4]).

Lemma 2.3. *Let $g \in \Sigma_\alpha^\beta$ be fixed and assume that $F : \mathbb{R}^{2n} \to \mathbb{C}$ is a measurable function that has a subexponential decay, i.e. such that for each $k = 0, 1, 2, ...,$ there is a constant $C = C_k > 0$ satisfying*

$$|F(x, \xi)| \leq C e^{-k(|x|^{1/\alpha} + |\xi|^{1/\beta})}.$$

Then the integral

$$f(t) = \int\int_{\mathbb{R}^{2n}} F(x, \xi)(M_\xi T_x g)(t) dx d\xi$$

defines a function in Σ_α^β.

Proof. To prove that $f \in \Sigma_\alpha^\beta$, we start with

$$\begin{aligned}
\left|(e^{k|t|^{1/\alpha}} f)(t)\right| &\leq \int\int_{\mathbb{R}^{2n}} (F(x,\xi) e^{k|t|^{1/\alpha}} (M_\xi T_x g)(t)) dx d\xi \\
&\leq \int\int_{\mathbb{R}^{2n}} |F(x,\xi)| \left| M_\xi T_x (e^{k|t+x|^{1/\alpha}} g))(t) \right| dx d\xi \\
&\leq \int\int_{\mathbb{R}^{2n}} |F(x,\xi)| \left| T_x (e^{k|t+x|^{1/\alpha}} g))(t) \right| dx d\xi \\
&\leq \int\int_{\mathbb{R}^{2n}} e^{k|x|^{1/\alpha}} e^{N|\xi|^{1/\beta}} e^{-N|\xi|^{1/\beta}} |F(x,\xi)| \left\| e^{k|\bullet|^{1/\alpha}} g \right\|_\infty dx d\xi \\
&\leq \int\int_{\mathbb{R}^{2n}} e^{(k+N)(|x|^{1/\alpha}+|\xi|^{1/\beta})} e^{-N(|x|^{1/\alpha}+|\xi|^{1/\beta})} |F(x,\xi)| \left\| e^{k|\bullet|^{1/\alpha}} g \right\|_\infty dx d\xi \\
&\leq \left\| e^{k|\bullet|^{1/\alpha}} g \right\|_\infty \left\| e^{(N+k)(|x|^{1/\alpha}+|\xi|^{1/\beta})} F \right\|_\infty \int\int_{\mathbb{R}^{2n}} e^{-N(|x|^{1/\alpha}+|\xi|^{1/\beta})} dx d\xi \\
&\leq C \left\| e^{(N+k)(|x|^{1/\alpha}+|\xi|^{1/\beta})} F \right\|_\infty .
\end{aligned}$$

So,

$$\left\| e^{k|\bullet|^{1/\alpha}} f \right\|_\infty \leq C \left\| e^{(N+k)(|x|^{1/\alpha}+|\xi|^{1/\beta})} F \right\|_\infty . \tag{2.3}$$

This implies that $\left\| e^{k|\bullet|^{1/\alpha}} f \right\|_\infty < \infty$.
To show that $\left\| e^{k|\bullet|^{1/\beta}} \widehat{f} \right\|_\infty < \infty$, we write

$$\widehat{f}(\tau) = \int\int_{\mathbb{R}^{2n}} (F(x,\xi)(M_{-x} T_\xi \widehat{g})(\tau)) e^{2\pi i x.\xi} dx d\xi,$$

using

$$(\widehat{M_\xi T_x g})(\tau) = (M_{-x} T_\xi \widehat{g})(\tau) e^{2\pi i x.\xi}.$$

Using an argument similar to the one leading to the proof of (2.3), we have

$$\left| e^{k|\tau|^{1/\beta}} \widehat{f}(\tau) \right| \leq C \left\| e^{(N+k)(|x|^{1/\alpha}+|\xi|^{1/\beta})} F \right\|_\infty .$$

This completes the proof of Lemma 2.3.

Remark 4. Given $\alpha > 1$.Then for the Gaussian $g(x) = e^{-\pi|x|^2}$ and f with $e^{-k|x|^{1/\alpha}} f \in L^1$ for some $k \in \mathbb{N}_0$, then $\nu_g f$ is well-defined and continuous. In fact,

$$\begin{aligned}
|\nu_g f(x,\xi)| &= \left| \int_{\mathbb{R}^n} f(t)\overline{g(t-x)} e^{-2\pi i t.\xi} dt \right| \\
&\leq \int_{\mathbb{R}^n} \left| f(t)\overline{g(t-x)} e^{-2\pi i t.\xi} \right| dt \\
&= \int_{\mathbb{R}^n} e^{-k|x|^{1/\alpha}} |f(t)| \, e^{k|t|^{1/\alpha}} \left| \overline{g(t-x)} \right| dt \\
&\leq \int_{\mathbb{R}^n} e^{-k|x|^{1/\alpha}} |f(t)| \, e^{k|t-x|^{1/\alpha}} \left| \overline{g(t-x)} \right| e^{k|x|^{1/\alpha}} dt \\
&= \left\| e^{-k|x|^{1/\alpha}} f \right\|_1 \left\| e^{k|x|^{1/\alpha}} g \right\|_\infty e^{k|x|^{1/\alpha}}.
\end{aligned}$$

This shows that $\nu_g f$ is well-defined. Moreover, if we fix $(x_0, \xi_0) \in \mathbb{R}^{2n}$ and let (x_j, ξ_j) be any sequence in $\mathbb{R}^{2n}$ converging to (x_0, ξ_0) as $j \to \infty$, the function $f(t)\overline{g(t-x_j)} e^{-2\pi i t.\xi_j}$ converges to $f(t)\overline{g(t-x_0)} e^{-2\pi i t.\xi_0}$ pointwise as $j \to \infty$ and

$$\begin{aligned}
\left| f(t)\overline{g(t-x_j)} e^{-2\pi i t.\xi_j} \right| &\leq \left| e^{-k|t|^{1/\alpha}} f(t) e^{k|t|^{1/\alpha}} \overline{g(t-x_j)} e^{-2\pi i t.\xi_j} \right| \\
&\leq \left| e^{-k|t|^{1/\alpha}} f(t) e^{k|t-x_j|^{1/\alpha})} \overline{g(t-x_j)} e^{k|x_j|^{1/\alpha}} \right| \\
&\leq C \left| e^{-k|t|^{1/\alpha}} f(t) \right| \left\| e^{k|\bullet|^{1/\alpha}} g \right\|_\infty \\
&\leq C \left| e^{-k|t|^{1/\alpha}} f(t) \right|.
\end{aligned}$$

Since the function $\left| e^{-k|t|^{1/\alpha}} f(t) \right| \in L^1$, we can apply Lebesgue Dominated Convergence Theorem to obtain

$$\nu_g f(x_j, \xi_j) \to \nu_g f(x_0, \xi_0)$$

as $j \to \infty$. This shows the continuity of $\nu_g f$.

□

3. The short-time Fourier transform over Σ_α^β

We use the topological characterization as stated in Theorem 2.1. Our proof imposes no conditions on the derivative.

Theorem 3.1. *Let $\alpha > 1$, $\beta > 1$ and $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then the Gelfand-Shilov space Σ_α^β can be described as a set as well as topologically by*

$$\Sigma_\alpha^\beta = \{f : \mathbb{R}^n \to \mathbb{C}\text{: } e^{-m|x|^{1/\alpha}} f \in L^1 \text{ for some } m \in \mathbb{N}_0 \text{ and } \pi_k(f) < \infty, \forall k \in \mathbb{N}_0, \}$$

$$\text{where } \pi_k(f) = \left\| e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})} \nu_g f \right\|_\infty .$$

Proof. Let us indicate $\mathfrak{B}_\alpha^\beta$ the space defined in (3.1). Observe that the condition $e^{-k|x|^{1/\alpha}} f \in L^1$ for some $k \in \mathbb{N}_0$ implies that $\nu_g f$ is continuous by Remark 4, so the formulation of the condition $\left\| e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})} \nu_g f \right\|_\infty$ makes sense. We define in $\mathfrak{B}_\alpha^\beta$ a structure of Fréchet space by means of the countable family of semi-norms

$$\mathcal{N} = \{\pi_k : k = 0, 1, 2, ...\}.$$

We will show that $\mathfrak{B}_\alpha^\beta = \Sigma_\alpha^\beta$. To do so, we first prove that $\mathfrak{B}_\alpha^\beta \subseteq \Sigma_\alpha^\beta$ continuously. Fix $f \in \mathfrak{B}_\alpha^\beta$, we need to show that $\left\| e^{k|\xi|^{1/\beta}} \widehat{f} \right\|_\infty$ and $\left\| e^{k|x|^{1/\alpha}} f \right\|_\infty$ are finite. Since $f \in \mathfrak{B}_\alpha^\beta$, then $\pi_k(f) < \infty$ for all $k \in \mathbb{N}_0$ which implies that $\nu_g f$ has a subexponential decay. Then by Lemma 2.3 and the inversion formula given in Lemma 2.1, we can write

$$f(t) = \|g\|_2^{-2} \int\int_{\mathbb{R}^{2n}} (\nu_g f(x, \xi)(M_\xi T_x g)(t)) dx d\xi.$$

Using Lemma 2.3, we have that $\left\| e^{k|\xi|^{1/\beta}} \widehat{f} \right\|_\infty$ and $\left\| e^{k|x|^{1/\alpha}} f \right\|_\infty$ are finite for all $k \in \mathbb{N}_0$.
Conversely, let $f \in \Sigma_\alpha^\beta$, then we know that f is continuous and for all $k \in \mathbb{N}_0$ $p_{k,0}(f) < \infty$, $\pi_{k,0}(f) < \infty$.
It is clear that $e^{-k|x|^{1/\alpha}} f \in L^1$ for some $k \in \mathbb{N}_0$ since $f \in \Sigma_\alpha^\beta$.To show that $\pi_k(f) < \infty$

for all $k \in \mathbb{N}_0$, we write

$$\begin{aligned} e^{2k|x|^{1/\alpha})}|\nu_g f(x,\xi)| &= e^{2k|x|^{1/\alpha}} \left| \int_{\mathbb{R}^n} f(t)g(x-t)e^{-2\pi i\xi.t}dt \right| \\ &\leq \| e^{2k|x|^{1/\alpha}}(|f| * |g|) \|_\infty . \end{aligned}$$

Using Lemma 2.2 we get the following estimate

$$\begin{aligned} e^{2k|x|^{1/\alpha}} |\nu_g f(x,\xi)| &\leq \left\| e^{2k|x|^{1/\alpha}}(|f| * |g|) \right\|_\infty \\ &\leq C \left\| e^{2(N+2k)|x|^{1/\alpha}} f \right\|_\infty \left\| e^{2(N+2k)|x|^{1/\alpha}} g \right\|_\infty \\ &\leq C \left\| e^{2(N+2k)|x|^{1/\alpha}} f \right\|_\infty . \end{aligned}$$

Then

$$e^{2k|x|^{1/\alpha}} |\nu_g f(x,\xi)| \leq C \left\| e^{2(N+2k)|x|^{1/\alpha}} f \right\|_\infty . \tag{3.1}$$

Moreover, since we can write $\nu_g f(x,\xi) = e^{-2\pi i\xi.x}\nu_{\widehat{g}}\widehat{f}(\xi,-x)$, we have the following estimate.

$$\begin{aligned} e^{2k|\xi|^{1/\beta}} |\nu_g f(x,\xi)| &\leq e^{2k|\xi|^{1/\beta}} \left| \nu_{\widehat{g}}\widehat{f}(\xi,-x) \right| \\ &\leq \left\| e^{2k|\xi|^{1/\beta}}(\left|\widehat{f}\right| * |\widehat{g}|) \right\|_\infty \end{aligned}$$

Once again, using Lemma 2.2 we obtain

$$\begin{aligned} e^{2k|\xi|^{1/\beta}} |\nu_g f(x,\xi)| &\leq \left\| e^{2k|\xi|^{1/\beta}}(\left|\widehat{f}\right| * |\widehat{g}|) \right\|_\infty \\ &\leq C \left\| e^{2(N+2k)|\xi|^{1/\beta}} \widehat{f} \right\|_\infty \left\| e^{2(N+2k)|\xi|^{1/\beta}} \widehat{g} \right\|_\infty \\ &\leq C \left\| e^{2(N+2k)|\xi|^{1/\beta}} \widehat{f} \right\|_\infty . \end{aligned}$$

Then

$$e^{2k|\xi|^{1/\beta}}|\nu_g f(x,\xi)| \leq C\left\|e^{2(N+2k)|\xi|^{1/\beta}}\widehat{f}\right\|_{\infty}. \tag{3.2}$$

Combining (3.1) and (3.2), we have that

$$e^{2k(|x|^{1/\alpha}+|\xi|^{1/\beta})}|\nu_g f(x,\xi)|^2 \leq C(\left\|e^{2(N+2k)(|x|^{1/\alpha}}f\right\|_{\infty}\left\|e^{2(N+2k)|\xi|^{1/\beta}}\widehat{f}\right\|_{\infty}).$$

This implies that

$$\pi_k(f) \leq C(\left\|e^{2(N+2k)(|x|^{1/\alpha}}f\right\|_{\infty}+\left\|e^{2(N+2k)|\xi|^{1/\beta}}\widehat{f}\right\|_{\infty}). \tag{3.3}$$

So, $f \in \mathfrak{B}^{\beta}_{\alpha}$. Hence $\mathfrak{B}^{\beta}_{\alpha} \subseteq \Sigma^{\beta}_{\alpha}$ and the inclusion is continuous. This completes the proof of Theorem 3.1. □

Remark 5. Let $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then for $f \in \Sigma^{\beta}_{\alpha}(\mathbb{R}^n)$, we have $\nu_g f \in \Sigma^{\beta}_{\alpha}(\mathbb{R}^{2n})$.

4. Characterization of the dual space $(\Sigma^{\beta}_{\alpha})'$

Theorem 4.1. *([7])Given a functional L in the topological dual of the space $\mathcal{C}_0$, there exists a unique regular complex Borel measure μ so that*

$$L(\varphi) = \int_{\mathbb{R}^n}\varphi d\mu.$$

Moreover, the norm of the functional L is equal to the total variation $|\mu|$ of the measure μ. Conversely, any such measure μ defines a continuous linear functional on $\mathcal{C}_0$.

Theorem 4.2. *Let $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then if $L : \Sigma_\alpha^\beta \to \mathbb{C}$, the following statements are equivalent:*
(i) $L \in (\Sigma_\alpha^\beta)'$
(ii) There exist a regular complex Borel measure μ of finite total variation and $k \in \mathbb{N}_0$ so that

$$L = e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g d\mu,$$

in the sense of $(\Sigma_\alpha^\beta)'$.

Proof. $(i) \Rightarrow (ii)$. Given $L \in (\Sigma_\alpha^\beta)'$, there exist k, C so that

$$L(\varphi) \leq C \left\| e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g \varphi \right\|_\infty$$

for all $\varphi \in \Sigma_\alpha^\beta$. Moreover, the map

$$\Sigma_\alpha^\beta(\mathbb{R}^n) \to \mathcal{C}_0(\mathbb{R}^{2n})$$
$$\varphi \to e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g \varphi$$

is well-defined, linear, continuous and injective. Let $\mathcal{R}$ be the range of this map. We define on $\mathcal{R}$ the map

$$l_1\left(e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g \varphi\right) = L(\varphi),$$

for a unique $\varphi \in \Sigma_\alpha^\beta$. The map $l_1 : \mathcal{R} \to \mathbb{C}$ is linear and continuous. By the Hahn-Banach theorem, there exists a functional L_1 in the topological dual $\mathcal{C}_0'(\mathbb{R}^{2n})$ of $\mathcal{C}_0(\mathbb{R}^{2n})$ such that $\|L_1\| = \|l_1\|$ and the restriction of L_1 to $\mathcal{R}$ is l_1. Using Theorem 4.1, there exist a regular complex Borel measure μ of finite total variation so that

$$L_1(f) = \int_{\mathbb{R}^{2n}} f d\mu$$

for all $f \in \mathcal{C}_0(\mathbb{R}^{2n})$. If $f \in \mathcal{R}$, we conclude

$$L(\varphi) = \int_{\mathbb{R}^{2n}} e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g \varphi d\mu$$

for all $\varphi \in \Sigma_\alpha^\beta$. In the sense of $(\Sigma_\alpha^\beta)\prime$,

$$L = \int_{\mathbb{R}^{2n}} e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g d\mu.$$

$(ii) \Rightarrow (i)$. If μ is a regular complex Borel measure satisfying (ii) and $\varphi \in \Sigma_\alpha^\beta$, then

$$L(\varphi) = \int_{\mathbb{R}^{2n}} e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g \varphi d\mu.$$

This implies that

$$\begin{aligned} |L(\varphi)| &\leq \left| \int_{\mathbb{R}^{2n}} e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g \varphi d\mu \right| \\ &\leq |\mu|(\mathbb{R}^{2n}) \left\| e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g \varphi \right\|_\infty \\ &\leq C(\left\| e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g \varphi \right\|_\infty). \end{aligned}$$

It may be noted that μ, employed to obtain the above inequality, is of finite total variation. This completes the proof of Theorem 4.2. □

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