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# RADIO MEAN NUMBER OF SOME SUBDIVISION GRAPHS

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Abstract. A Radio Mean labeling of a connected graph $G$ is a one to one map $f$ from the vertex set $V(G)$ to the set of natural numbers $N$ such that for each distinct vertices $u$ and $v$ of $G$, $d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \text{diam}\ (G)$. The radio mean number of $f$, $rmn(f)$, is the maximum number assigned to any vertex of $G$.The radio mean number of $G$, $rmn(G)$ is the minimum value of $rmn(f)$ taken over all radio mean labeling $f$ of $G$. In this paper we find the radio mean number of subdivision of a star, wheel, bistar.

## 1. Introduction

We will consider finite, simple undirected and connected graphs. $V(G)$ and $E(G)$ respectively denote the vertex set and edge set of $G$. A graph labeling is an assignment of integers to the vertices, or edges, or both, subject to certain conditions. Graph labeling applied in sciences and few of them are communication network, coding theory, database management etc. In particular, radio labeling used for channel assignment problem. The concept of radio labeling was introduced by Chatrand, et al [2] in 2001. Also Liu, et al [5, 6], Ali, et al [1] have found radio number of some graphs. The results on the radio labeling motivated Ponraj et al. [7] to introduced the notion of radio mean labeling of $G$ and they found the radio mean

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number of some graphs like graphs with diameter three, lotus inside a circle, gear graph, Helms and Sunflower graphs. In [8, 9], Ponraj, et al proved the following: $rmn(S(K_{m,n})) = (m+1)(n+1)-1, m > 1, n > 1$. $rmn\,(K_{m,n} \odot P_t) = (m+n)\,(t+1)$, $m \geq 2$, $n \geq 2$, $t \geq 2$. For any integer $t \geq 2$, $rmn\left(C_6^{(t)}\right)$ is $5t+3$ if $t = 2$ or $5t+2$ if $t = 3$ or $5t+1$, otherwise. $rmn(W_n \odot P_m) = (m+1)(n+1)$, $n > 3$ and $m \in \mathbb{Z}^+$. The graph $W_{m,n}$ is obtained from the wheels $W_m$ and $W_n$ by joining the rim vertices of the two wheels with an edge. $rmn(W_{m,n})$ is 10 if $m = 3$, $n = 4$ or $m+n+2$ if $m = 3$, $n \neq 4$ or $m+n+3$ if $m > 3$, $n > 3$. The graph $sp(W_n)$ is obtained from the wheel $W_n$ by subdividing each spoke by a vertex. $rmn(sp(W_n)) = 2n+1$. In this article we find the radio mean number of subdivision of a star, wheel, bistar, $K_2 + mK_1$. We write $d(u, v)$ for the distance between the vertices $u$ and $v$, and use $diam(G)$ to indicate the diameter of $G$. Let $x$ be any real number. Then $\lceil x \rceil$ stands for smallest integer greater than or equal to $x$. Terms and definitions not defined can be found in are follow from Harary [4] and Gallian [3].

## 2. PRELIMINARIES

In this section we present some definitions which are used for the next section.

**Definition 2.1.** A *radio mean labeling* is a one to one mapping $f$ from $V\,(G)$ to $N$ satisfying the condition

$$d\,(u, v) + \left\lceil \frac{f\,(u) + f\,(v)}{2} \right\rceil \geq 1 + \text{diam}\ \ (G) \tag{2.1}$$

for every $u, v \in V\,(G)$. The span of a labeling $f$ is the maximum integer that $f$ maps to a vertex of Graph $G$. The radio mean number of $G$, $rmn\,(G)$ is the lowest span taken over all radio mean labelings of the graph $G$. The condition 2.1 is called radio mean condition.

**Definition 2.2.** Let $G_1$ and $G_2$ be two graphs with vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ respectively. Then their *join* $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1 \text{ and } v \in V_2\}$.

**Definition 2.3.** Let $G_1$, $G_2$ respectively be $(p_1, q_1)$, $(p_2, q_2)$ graphs. The corona of $G_1$ with $G_2$, $G_1 \odot G_2$ is the graph obtained by taking one copy of $G_1$ and $p_1$ copies of $G_2$ and joining the $i^{th}$ vertex of $G_1$ with an edge to every vertex in the $i^{th}$ copy of $G_2$.

**Definition 2.4.** The graph $C_n^{(t)}$ denotes the one point union of $t$ copies of the cycle $C_n : u_1 u_2 \ldots u_n u_1$.

**Definition 2.5.** An edge $x = uv$ of $G$ is said to be subdivided if it is replaced by the edges $uw$ and $wv$ where $w$ is a vertex not in $V(G)$. If every edge of $G$ is subdivided, the resulting graph is called the *subdivision graph* $S(G)$.

**Definition 2.6.** The graph $K_{1,n}$ is called the *star*.

**Definition 2.7.** The graph $C_n + K_1$ is called the *wheel* on $n$ vertices and it is denoted by $W_n$.

**Definition 2.8.** The bistar $B_{m,n}$ is a graph obtained by joining the two central vertices of $K_{1,m}$ and $K_{1,n}$.

**Definition 2.9.** A vertex $v$ of a graph $G$ is called a pendent vertex if its degree is 1. A vertex which is adjacent to a pendent vertex is called a support.

## 3. MAIN RESULTS

**Theorem 3.1.**

$$rmn\left(S\left(K_{1,n}\right)\right) = \begin{cases} 6 & if \quad n = 2 \\ 2n+1 & otherwise \end{cases}$$

*Proof.* Denote the central vertex by $u$, pendent vertices by $u_i$ $(1 \leq i \leq n)$ and the support vertices by $v_i$ $(1 \leq i \leq n)$. We observe that the diameter of $S(K_{1,n}),\ n > 1$ is 4 and that of $S(K_{1,1})$ is 2.

**Case 1.** $n = 2$.

A vertex labeling of $S(K_{1,2})$ with radio mean number 6 is given in figure 1.

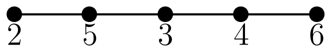


Figure 1

It follows that $rmn(S(K_{1,2})) \leq 6$.

**Claim.** $rmn(S(K_{1,2})) > 5$.

Note that the labels 1 and 2 should be assigned to the vertices with a distance at least 3. Without loss of generality we can assume that the label of one of the pendent vertex is 1 and hence the support of the other pendent vertex is labeled by 2. Therefore 3 can not be labeled to the vertices. This implies $rmn(S(K_{1,2})) > 5$. Hence $rmn(S(K_{1,2})) = 6$.

**Case 2.** $n \neq 2$.

If $n = 1$, then from the figure 2, it is easy to verify that, the given labeling satisfies the radio mean condition and hence $rmn(S(K_{1,1})) = 3$.

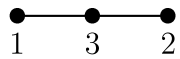


Figure 2

Assume $n \geq 3$. We describe a radio mean labeling $f$ as follows. Assign the labels from $\{1, 2, \ldots, n\}$ to the pendent vertices in any order. Assign the label $n + i + 1$ to the support vertex that is adjacent to the pendent vertex with label $n - i$. Note that in this process, the support of the pendent vertex with label 1 receives the label $2n$. Finally assign the label $2n + 1$ to the central vertex. Now we check the radio mean condition of the labeling $f$.

**Case A.** Check the pair $(u_i, u_j)$, $i \neq j$.

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil \geq 4 + \left\lceil \frac{1+2}{2} \right\rceil \geq 6.$$

**Case B.** Examine the pair $(u_i, v_j)$.

**Subcase 1.** Examine the pair $(u_i, v_j)$ such that $f(u_i) = 1$.

It is easy to check that the pair $(u_i, v_i)$ satisfies the radio mean condition. Suppose $i \neq j$ then

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{1+n+1}{2} \right\rceil \geq 6.$$

**Subcase 2.** Verify the pair $(u_i, v_j)$ where $f(u_i) = 2$.

Obviously, the pair $(u_i, v_i)$ satisfies the radio mean condition. Assume $i \neq j$. In this case,

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{2+n+1}{2} \right\rceil \geq 6.$$

**Subcase 3.** Consider the pair $(u_i, v_j)$ where $f(u_i) \neq 1$ and $f(u_i) \neq 2$.

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{3+n+1}{2} \right\rceil \geq 5.$$

**Case C.** Examine the pair $(v_i, v_j)$.

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{n+1+n+2}{2} \right\rceil \geq 7.$$

**Case D.** Examine the pair $(u, u_i)$.

$$d(u, u_i) + \left\lceil \frac{f(u) + f(u_i)}{2} \right\rceil \geq 2 + \left\lceil \frac{2n+1+1}{2} \right\rceil \geq 6.$$

**Case E.** Verify the pair $(u, v_i)$.

$$d(u, v_i) + \left\lceil \frac{f(u) + f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2n + 1 + n + 1}{2} \right\rceil \geq 7.$$

Hence $rmn(S(K_{1,n})) = 2n + 1$ if $n \neq 2$. □

Next we obtain the radio mean number of Subdivision of a wheel.

**Theorem 3.2.**

$$rmn(S(W_n)) = \begin{cases} 20 & if \quad n = 6 \\ 3n + 1 & otherwise \end{cases}$$

*Proof.* Let $W_n = C_n + K_1$ where $C_n$ is a cycle $u_1 u_2 \dots u_n u_1$ and $V(K_1) = \{u\}$. Let the edge $u_i u_{i+1}$ of the cycle be subdivided by $v_i$ $(1 \leq i \leq n)$ and that of the spoke $u_i u$ by $w_i$ $(1 \leq i \leq n)$. Note that

$$diam(S(W_n)) = \begin{cases} 4 & \text{if} \quad n = 3, 4 \\ 5 & \text{if} \quad n = 5 \\ 6 & \text{if} \quad n \geq 6 \end{cases}$$

From figure 3, it is clear that the $rmn(S(W_n)) = 3n + 1$ where $n = 3, 4, 5, 7, 8$.

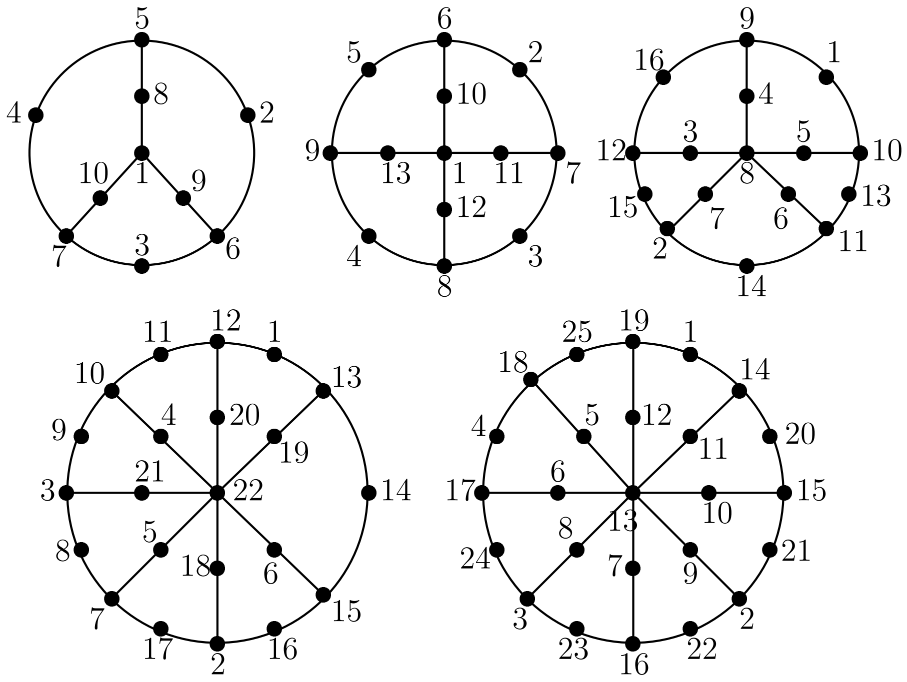

Figure 3

**Case 1.** $n = 6$.

Figure 4 shows that $rmn(S(W_n)) \leq 20$.

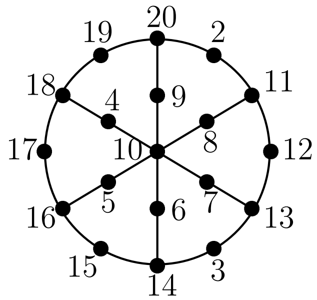

Figure 4

It is clear that the labels 1 and 2 should be labeled at a distance 5 or 6. Also the labels 1 and 3 should be labeled to the vertices which are at a distance 5 or 6 and the distance between the vertices with the labels 2 and 3 should be atleast 4. So we must have 3 vertices $u$, $v$, $w$ such that $d(u, v) = d(u, w) = 5$ or 6, and $d(v, w) \geq 4$. This is not possible. This implies $rmn(S(W_n)) \geq 19$.

**Case 2.** $n \geq 9$.

Assign the labels $1, 2, 3, 4$ to the vertices $v_1, u_4, u_6, v_7$ respectively. Then we move to the vertices $w_i$ $(1 \leq i \leq n)$. Assign the label 5 to $w_n$, 6 to $w_{n-1}$ and in general assign the label $n + 5 - i$ to $w_i$. Clearly the vertex $w_1$ receives the label $n + 4$. Put the label $n + 5$ to the central vertex. Finally, assign the remaining labels from $\{n + 6, n + 7, \ldots, 3n + 1\}$ to the unlabeled rim vertices in any order. Now we check the radio mean condition

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 7$$

holds for every pair of vertices $(u, v)$ such that $u \neq v$. It is easy to verify that any two pairs of the vertices $v_1$, $u_4$, $u_6$, $v_7$ satisfies the radio mean condition.

**Case A.** Check the pair $(u_i, u_j)$.

**Subcase 1.** $i \neq 4, 6$ and $j \neq 4, 6$.

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{n + 6 + n + 7}{2} \right\rceil \geq 18.$$

**Subcase 2.** Verify the pair $(u_4, u_j)$.

$$d(u_4, u_j) + \left\lceil \frac{f(u_4) + f(u_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{2 + n + 6}{2} \right\rceil \geq 11.$$

**Subcase 3.** Verify the pair $(u_6, u_j)$.

$$d(u_6, u_j) + \left\lceil \frac{f(u_6) + f(u_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{3 + n + 6}{2} \right\rceil \geq 11.$$

**Case B.** Check the pair $(v_i, v_j)$.

**Subcase 1.** $i \neq 1, 7$ and $j \neq 1, 7$.

$$d\left(v_i, v_j\right)+\left\lceil\frac{f(v_i)+f(v_j)}{2}\right\rceil \geq 2+\left\lceil\frac{n+6+n+7}{2}\right\rceil \geq 18.$$

**Subcase 2.** Consider the pair $(v_1, v_j)$.

$$d\left(v_1, v_j\right)+\left\lceil\frac{f(v_1)+f(v_j)}{2}\right\rceil \geq 2+\left\lceil\frac{1+n+6}{2}\right\rceil \geq 10.$$

**Subcase 3.** Check the pair $(v_7, v_j)$.

$$d\left(v_7, v_j\right)+\left\lceil\frac{f(v_7)+f(v_j)}{2}\right\rceil \geq 2+\left\lceil\frac{4+n+6}{2}\right\rceil \geq 12.$$

**Case C.** Check the pair $(u_i, v_j)$.

**Subcase 1.** $i \neq 4, 6$ and $j \neq 1, 7$.

$$d\left(u_i, v_j\right)+\left\lceil\frac{f(u_i)+f(v_j)}{2}\right\rceil \geq 1+\left\lceil\frac{n+6+n+7}{2}\right\rceil \geq 17.$$

**Subcase 2.** Examine the pair $(u_4, v_j)$, $j \neq 1, 7$.

$$d\left(u_4, v_j\right)+\left\lceil\frac{f(u_4)+f(v_j)}{2}\right\rceil \geq 1+\left\lceil\frac{2+n+6}{2}\right\rceil \geq 10.$$

**Subcase 3.** Check the pair $(u_6, v_j)$.

$$d\left(u_6, v_j\right)+\left\lceil\frac{f(u_6)+f(v_j)}{2}\right\rceil \geq 1+\left\lceil\frac{3+n+6}{2}\right\rceil \geq 10.$$

**Subcase 4.** Examine the pair $(u_i, v_1)$, $j \neq 4, 6$.

$$d\left(u_i, v_1\right)+\left\lceil\frac{f(u_i)+f(v_1)}{2}\right\rceil \geq 1+\left\lceil\frac{n+6+1}{2}\right\rceil \geq 9.$$

**Subcase 5.** Examine the pair $(u_i, v_7)$, $j \neq 4, 6$.

$$d\left(u_i, v_7\right) + \left\lceil \frac{f(u_i) + f(v_7)}{2} \right\rceil \geq 1 + \left\lceil \frac{n+6+4}{2} \right\rceil \geq 11.$$

**Case D.** Check the pair $(w_i, w_j)$.

$$d\left(w_i, w_j\right) + \left\lceil \frac{f(w_i) + f(w_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{5+6}{2} \right\rceil \geq 8.$$

**Case E.** Consider the pair $(w_i, u_j)$.

**Subcase 1.** $j \neq 4, 6$.

$$d\left(w_i, u_j\right) + \left\lceil \frac{f(w_i) + f(u_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{5+n+6}{2} \right\rceil \geq 11.$$

**Subcase 2.** Examine the pair $(w_i, u_4)$.

Note that $d\left(w_4, u_4\right) = 1$, $f(w_4) = n+1$, $f(u_4) = 2$. Obviously the pair $(w_4, u_4)$ satisfies the radio mean condition. Let us check the pair $(w_i, u_4)$ with $i \neq 4$.

$$d\left(w_i, u_4\right) + \left\lceil \frac{f(w_i) + f(u_4)}{2} \right\rceil \geq 3 + \left\lceil \frac{5+2}{2} \right\rceil \geq 7.$$

**Subcase 3.** Check the pair $(w_i, u_6)$.

Since $d\left(w_6, u_6\right) = 1$, $f(w_6) = n-1$, $f(u_4) = 3$, the pair $(w_6, u_6)$ satisfies the radio mean condition. For $i \neq 6$, we have,

$$d\left(w_i, u_6\right) + \left\lceil \frac{f(w_i) + f(u_6)}{2} \right\rceil \geq 3 + \left\lceil \frac{5+3}{2} \right\rceil \geq 7.$$

**Case F.** Verify the pair $(w_i, v_j)$.

Since $d\left(w_{n-2}, v_1\right) = d\left(w_{n-1}, v_1\right) = d\left(w_n, v_1\right) = 4$, the pairs $(w_{n-2}, v_1)$, $(w_{n-1}, v_1)$ and $(w_n, v_1)$ satisfies the radio mean condition.

**Subcase 1.** Check the pair $(w_i, v_j)$ with $i \neq n$, $n-1$, $n-2$.

$$d\left(w_i, v_j\right)+\left\lceil\frac{f(w_i)+f(v_j)}{2}\right\rceil \geq 2+\left\lceil\frac{8+1}{2}\right\rceil \geq 7.$$

**Subcase 2.** Check the pair $(w_n, v_j)$ with $j \neq 1$.

$$d\left(w_n, v_j\right)+\left\lceil\frac{f(w_n)+f(v_j)}{2}\right\rceil \geq 2+\left\lceil\frac{5+4}{2}\right\rceil \geq 7.$$

**Subcase 3.** Examine the pair $(w_{n-1}, v_j)$ with $j \neq 1$.

$$d\left(w_{n-1}, v_j\right)+\left\lceil\frac{f(w_{n-1})+f(v_j)}{2}\right\rceil \geq 2+\left\lceil\frac{6+4}{2}\right\rceil \geq 7.$$

**Subcase 4.** Verify the pair $(w_{n-2}, v_j)$ with $j \neq 1$.

$$d\left(w_{n-2}, v_j\right)+\left\lceil\frac{f(w_{n-2})+f(v_j)}{2}\right\rceil \geq 2+\left\lceil\frac{7+4}{2}\right\rceil \geq 8.$$

**Case G.** Verify the pair $(u, u_i)$.

$$d\left(u, u_i\right)+\left\lceil\frac{f(u)+f(u_i)}{2}\right\rceil \geq 2+\left\lceil\frac{n+5+2}{2}\right\rceil \geq 10.$$

**Case H.** Verify the pair $(u, v_i)$.

$$d\left(u, v_i\right)+\left\lceil\frac{f(u)+f(v_i)}{2}\right\rceil \geq 3+\left\lceil\frac{n+5+1}{2}\right\rceil \geq 11.$$

**Case I.** Verify the pair $(u, w_i)$.

$$d\left(u, w_i\right)+\left\lceil\frac{f(u)+f(w_i)}{2}\right\rceil \geq 1+\left\lceil\frac{n+5+5}{2}\right\rceil \geq 11.$$

Hence $rmn(S(W_n)) = 3n+1$ if $n \neq 6$. □

**Theorem 3.3.** *The radio mean number of* $S(K_2 + mK_1)$ *is* $3m + 3$.

*Proof.* Let $V(S(K_2 + mK_1)) = \{u_i, v_i, w_i : 1 \leq i \leq m\} \cup \{u, v, w\}$ and $E(S(K_2 + mK_1)) = \{vu, uw\} \cup \{vv_i, v_iu_i, u_iw_i, w_iw : 1 \leq i \leq m\}$. For $m = 1$, the labeling given in figure 5 satisfies the radio mean condition and hence $rmn(S(K_2 + K_1)) = 6$.

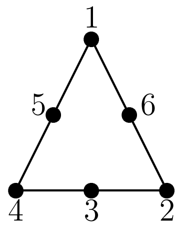


Figure 5

Assume $m \geq 2$. Note that in this case $diam(S(K_2+mK_1)) = 4$. We define a labeling $f$ as follows. Assign the labels $i + 1$ to the vertices $u_i$ $(1 \leq i \leq m)$. Then assign the labels $m+1+i$ to the vertices $v_i$ $(1 \leq i \leq m)$ and then assign the labels $2m+1+i$ to the vertices $w_i$ $(1 \leq i \leq m)$. Finally assign the label $3m+2$, 1, $3m+3$ to the vertices $v$, $u$, $w$ respectively. Now we check the radio mean condition for any two vertices.

**Case 1.** Check the pair $(u_i, u_j)$.

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil \geq 4 + \left\lceil \frac{2+3}{2} \right\rceil \geq 7.$$

**Case 2.** Consider the pair $(v_i, v_j)$.

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{m+1+1+m+1+2}{2} \right\rceil \geq 7.$$

**Case 3.** Examine the pair $(w_i, w_j)$.

$$d(w_i, w_j) + \left\lceil \frac{f(w_i) + f(w_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{2m+1+1+2m+1+2}{2} \right\rceil \geq 9.$$

**Case 4.** Verify the pair $(u, u_i)$.

$$d(u, u_i) + \left\lceil \frac{f(u) + f(u_i)}{2} \right\rceil \geq 3 + \left\lceil \frac{1+2}{2} \right\rceil \geq 5.$$

**Case 5.** Check the pair $(u, v_i)$.

$$d(u, v_i) + \left\lceil \frac{f(u) + f(v_i)}{2} \right\rceil \geq 2 + \left\lceil \frac{1+m+1+1}{2} \right\rceil \geq 5.$$

**Case 6.** Verify the pair $(u, w_i)$.

$$d(u, w_i) + \left\lceil \frac{f(u) + f(w_i)}{2} \right\rceil \geq 2 + \left\lceil \frac{1+2m+1+1}{2} \right\rceil \geq 6.$$

Since $m \geq 2$, $f(v) = 3m + 2$ and $f(w) = 3m + 3$, the pairs $(v, u_i)$, $(v, v_i)$, $(v, w_i)$, $(v, u)$, $(v, w)$, $(w, u)$, $(w, u_i)$, $(w, v_i)$ and $(w, w_i)$ satisfies the radio mean condition. Therefore $rmn(S(K_2 + mK_1)) = 3m + 3$. □

**Theorem 3.4.**

$$rmn\left(S\left(B_{m,n}\right)\right) = \begin{cases} 9 & \textit{if} \quad m = n = 1 \\ 10 & \textit{if} \quad m = 1, n = 2 \\ 12 & \textit{if} \quad m = 1, n = 3 \ \textit{or}\ m = n = 2 \\ 2m + 2n + 3 & \textit{otherwise} \end{cases}$$

*Proof.* **Case 1.** $m = n = 1$.

The labeling in figure 6 is obviously satisfies the radio mean condition.

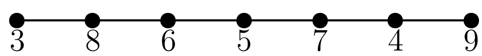

Figure 6

Therefore $rmn(S(B_{1,1})) \leq 9$.

**Claim.** $rmn(S(B_{1,1})) > 8$.

The diameter of $S(B_{1,1})$ is 6. Therefore 1 and 2 should be labeled to the vertices with a distance 5. Without loss of generality we can assume that the label of the one pendent vertex is 1 and hence the other pendent vertex or their support is labeled by 2. It follows that 3, 4 can not be labels of any of the remaining vertices. Hence $rmn(S(B_{1,1})) \leq 9$.

**Case 2.** $m = 1$, $n = 2$.

The labeling given in figure 7 is trivially satisfies the radio mean condition.

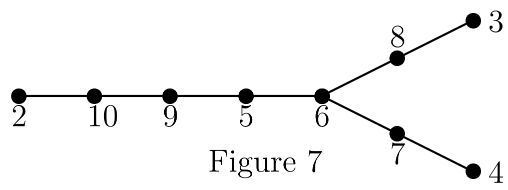


Figure 7

Therefore $rmn(S(B_{1,2})) \leq 10$.

**Claim.** $rmn(S(B_{1,2})) > 9$.

In this case, the diameter is 6. Hence 1 and 2 should be labeled to the vertices with a distance at least 5. If 1 is the label of the pendent vertex of the star or its support with 1 leaf star then 2 must be label of the either pendent vertices or their supports of the other star. In all the cases 4 can not be label of the vertices; if 1 is the pendent vertex of the two leaves star or their support then 2 must be a label of the pendent vertex or its support of the other star. In this case 3 can not be a label of the vertices. Therefore $rmn(S(B_{1,2})) \geq 10$.

**Case 3.** $m = 1$, $n = 3$.

The labeling given in figure 8 satisfies the radio mean condition.

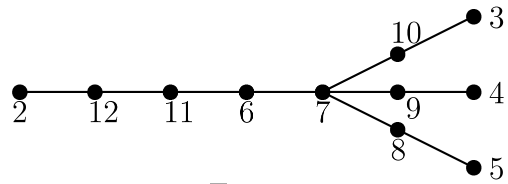


Figure 8

Hence $rmn(S(B_{1,3})) \leq 12$. Since the diameter of $S(B_{1,3}) = 6$, 1 and 2 should be labels of the vertices which are at a distance atleast 5. Also 3 must be a label of a vertex which is atleast at a distance 5 from the vertex with the label 1 and atleast at a distance 4 from the vertex with the label 3. This is impossible. Thus $rmn(S(B_{1,3})) > 11$ and so $rmn(S(B_{1,3})) = 12$.

**Case 4.** $m = n = 2$.

The labeling given in figure 9 satisfies the radio mean condition.

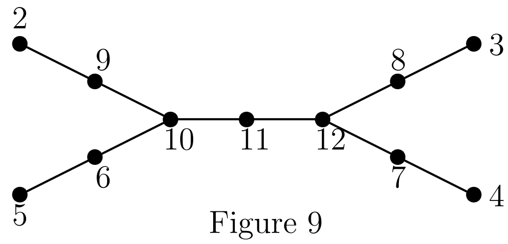


Figure 9

Therefore $rmn(S(B_{1,2})) \leq 12$. For a similar reasoning discussed in case 3, we have $rmn(S(B_{1,2})) > 11$.

**Case 5.** $m + n \geq 5$.

We now define the vertex set and edge set of $S(B_{m,n})$ as follows: Let $V(S(B_{m,n})) = \{u, v, w\} \cup \{u_i, u_i' : 1 \leq i \leq m\} \cup \{v_i, v_i' : 1 \leq i \leq n\}$ and $E(S(B_{m,n})) = \{uw, wv\} \cup \{uu_i, u_iu_i' : 1 \leq i \leq m\} \cup \{vv_i, v_iv_i' : 1 \leq i \leq n\}$. Put the label 1 to $u_1'$ then we move to the vertices $v_i'$. Assign the label $i + 1$ to $v_i'$ $(1 \leq i \leq n)$. Note that the vertex $v_n'$ received the label $n + 1$. Then we move to the vertex $u_2'$ and it is assigned by the label $n + 2$. Assign the label $n + 3$ to the next vertex $u_3'$. Proceed in this manner until we reach the vertex $u_m'$. It is easy to verify that the label of $u_m'$ is $m + n$. We now consider the supports of $u_i'$ $(2 \leq i \leq m)$. Assign the label $m + n + i$

to $u_{i+1}$ $(1 \leq i \leq m-1)$. Then we move to the supports of $v_i^{'}$ $(1 \leq i \leq n)$. Assign the label $2m+2n-i$ to the vertex $v_i$ $(1 \leq i \leq n)$. Note that in this process the vertex $v_n$ received the label $2m+n$. Finally assign the labels $2m+2n$, $2m+2n+1$, $2m+2n+2$, $2m+2n+3$ to the vertices $u_1$, $u$, $w$, $v$ respectively.

Now we show that the above labeling $f$, satisfies the following radio mean condition:

$$d\,(u,v)+\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq diam(S(B_{m,n}))+1=7,$$

for all pairs of vertices $(u,v)$ where $u \neq v$.

One can easily show that the radio mean condition is satisfies for the pairs $(u,w)$, $(u,v)$, $(v,w)$.

**Case a.** Consider the pair $(u,u_i)$.

$$d(u,u_i)+\left\lceil \frac{f(u)+f(u_i)}{2} \right\rceil \geq 1+\left\lceil \frac{2m+2n+1+m+n+1}{2} \right\rceil \geq 10.$$

**Case b.** verify the pair $(u,u_i^{'})$.

$$d(u,u_i^{'})+\left\lceil \frac{f(u)+f(u_i^{'})}{2} \right\rceil \geq 2+\left\lceil \frac{2m+2n+1+1}{2} \right\rceil \geq 8.$$

**Case c.** Check the pair $(u,v_i)$.

$$\begin{aligned} d(u,v_i)+\left\lceil \frac{f(u)+f(v_i)}{2} \right\rceil &\geq 3+\left\lceil \frac{2m+2n+1+2m+n}{2} \right\rceil \\ &\geq \begin{cases} 12 & \text{if} \quad m=1 \ \& \ n \geq 4 \ (or) \ m=2 \ \& \ n \geq 3 \\ 14 & \text{if} \quad m \geq 3 \ \& \ n \geq 3 \end{cases} \end{aligned}$$

**Case d.** Check the pair $(u,v_i^{'})$.

$$d(u,v_i^{'})+\left\lceil \frac{f(u)+f(v_i^{'})}{2} \right\rceil \geq 4+\left\lceil \frac{2m+2n+1+2}{2} \right\rceil \geq 11.$$

Similarly we can prove that the pairs $(w,u_i)$, $(w,u_i^{'})$, $(w,v_i)$, $(w,v_i^{'})$, $(v,u_i)$, $(v,u_i^{'})$, $(v,v_i)$ and $(v,v_i^{'})$ satisfies the radio mean condition.

**Case e.** Examine the pair $(u_i,u_j)$.

$$d(u_i,u_j)+\left\lceil\frac{f(u_i)+f(u_j)}{2}\right\rceil \geq 2+\left\lceil\frac{m+n+1+m+n+2}{2}\right\rceil \geq 9.$$

**Case f.** Examine the pair $(u_i,u_j^{'})$.

For $i\neq j$,

$$d(u_i,u_j^{'})+\left\lceil\frac{f(u_i)+f(u_j^{'})}{2}\right\rceil \geq 3+\left\lceil\frac{1+m+n+1}{2}\right\rceil \geq 7.$$

Suppose $i=j$. Clearly the pair $(u_1,u_1^{'})$ satisfies the radio mean condition. So assume $i\neq 1$. Then

$$d(u_i,u_i^{'})+\left\lceil\frac{f(u_i)+f(u_i^{'})}{2}\right\rceil \geq 1+\left\lceil\frac{m+n+1+n+2}{2}\right\rceil \geq 7.$$

**Case g.** Check the pair $(u_i,v_j)$.

$$\begin{aligned}
d(u_i,v_j)+\left\lceil\frac{f(u_i)+f(v_j)}{2}\right\rceil &\geq 4+\left\lceil\frac{m+n+1+2m+n}{2}\right\rceil \\
&\geq \begin{cases} 10 & \text{if} \quad m=1 \ \& \ n\geq 4 \\ 11 & \text{if} \quad m=2 \ \& \ n\geq 3 \\ 12 & \text{if} \quad m\geq 3 \ \& \ n\geq 3 \end{cases}
\end{aligned}$$

**Case h.** Consider the pair $(u_i,v_j^{'})$.

$$d(u_i,v_j^{'})+\left\lceil\frac{f(u_i)+f(v_j^{'})}{2}\right\rceil \geq 5+\left\lceil\frac{m+n+1+2}{2}\right\rceil \geq 9.$$

**Case i.** Verify the pair $(u_i^{'}, v_i)$.

$$
\begin{aligned}
d(u_i^{'}, v_i) + \left\lceil \frac{f(u_i^{'}) + f(v_i)}{2} \right\rceil &\geq 5 + \left\lceil \frac{1 + 2m + n}{2} \right\rceil \\
&\geq \begin{cases} 9 & \text{if} \quad (m = 1 \ \& \ n \geq 4) \ \mathit{or} \ (m = 2 \ \& \ n \geq 3) \\ 10 & \text{if} \quad m \geq 3 \ \& \ n \geq 3 \end{cases}
\end{aligned}
$$

**Case j.** Verify the pair $(u_i^{'}, v_j^{'})$.

$$
d(u_i^{'}, v_j^{'}) + \left\lceil \frac{f(u_i^{'}) + f(v_j^{'})}{2} \right\rceil \geq 6 + \left\lceil \frac{1 + 2}{2} \right\rceil \geq 8.
$$

**Case k.** Consider the pair $(v_i, v_j)$.

$$
\begin{aligned}
d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil &\geq 2 + \left\lceil \frac{2m + n + 2m + n + 1}{2} \right\rceil \\
&\geq \begin{cases} 9 & \text{if} \quad m = 1 \ \& \ n \geq 4 \\ 11 & \text{if} \quad m = 2 \ \& \ n \geq 3 \\ 12 & \text{if} \quad m \geq 3 \ \& \ n \geq 3 \end{cases}
\end{aligned}
$$

**Case l.** Examine the pair $(v_i, v_j^{'})$.

For $i \neq j$,

$$
\begin{aligned}
d(v_i, v_j^{'}) + \left\lceil \frac{f(v_i) + f(v_j^{'})}{2} \right\rceil &\geq 3 + \left\lceil \frac{2m + n + 1}{2} \right\rceil \\
&\geq \begin{cases} 7 & \text{if} \quad (m = 1 \ \& \ n \geq 4) \ \mathit{or} \ (m = 2 \ \& \ n \geq 3) \\ 8 & \text{if} \quad m \geq 3 \ \& \ n \geq 3 \end{cases}
\end{aligned}
$$

Suppose $i = j$. Then

$$d(v_i, v_i^{'}) + \left\lceil \frac{f(v_i) + f(v_i^{'})}{2} \right\rceil \geq 1 + \left\lceil \frac{2m + 2n - i + i + 1}{2} \right\rceil \geq 7.$$

**Case m.** Consider the pair $(v_i^{'}, v_j^{'})$.

$$d(v_i^{'}, v_j^{'}) + \left\lceil \frac{f(v_i^{'}) + f(v_j^{'})}{2} \right\rceil \geq 4 + \left\lceil \frac{2+3}{2} \right\rceil \geq 7.$$

Hence $rmn(S(B_{m,n})) = 2m + 2n + 3$. □

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