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AN INEQUALITY FOR SOME FUNCTIONS OF CONTINUOUS RANDOM VARIABLES

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ABSTRACT: We establish new inequalities for moment generating function of a continuous random variable; characteristic function of a continuous random variable, power spectral density (or power spectrum) of a continuous-time WSS random process and autocorrelation function of a continuous-time WSS random process, using a form of the Cauchy-Bunyakovsky-Schwarz inequality. Our new inequalities obey the general form $f^2(\alpha x) \leq f\{(\alpha+\beta)x\}f\{(\alpha-\beta)x\} \qquad (\alpha, \beta \in \mathbb{R})$

1. INTRODUCTION

The Cauchy-Schwarz inequality (see for instant; [4]) states that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} {a_i}^2 \sum_{i=1}^{n} {b_i}^2 \qquad (a_i, b_i \in \mathbb{R}),$$

and the well-known Cauchy-Bunyakovsky-Schwarz (CBS) inequality [4] states that

$$\left(\int_a^b u^{\frac{1}{2}}(t)\, v^{\frac{1}{2}}(t)dt\right)^2 \leq \left(\int_a^b u(t)\, dt\right)\left(\int_a^b v(t)\, dt\right), \qquad (1)$$

for all functions u, v : $[a,b] \to (-\infty, \infty)$, such that the integrals exist.

As we know, the Cauchy-Schwarz and Cauchy-Bunyakovsky-Schwarz inequalities play an important role in different branches of modern mathematics such as Hilbert space theory, classical real and complex analysis, numerical analysis, probability and statistics, qualitative theory of differential equations and their applications. To date, a large number

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of generalizations and refinements of these inequalities have been investigated in the literature (see, e.g., [2, 3, 5, 6, 8, 9 and 11]).

A. Laforgia and P. Natalini [1] used the following form of the CBS inequality (1):

$$\left(\int_a^b g(t)\, f^{\frac{m+n}{2}}(t)dt\right)^2 \leq \left(\int_a^b g(t)f^m(t)\, dt\right)\left(\int_a^b g(t)\, f^n(t)\, dt\right). \qquad (2)$$

to establish some new Turán-type inequalities involving the special functions as gamma, Polygamma functions and Riemann zeta function.

Here f and g are two nonnegative functions of a real variable and m and n belong to a set S of real numbers, such that the involved integrals in (2) exist.

Motivated by this remark, we have the idea to replace $u(t)$ and $v(t)$ in (1) by $g(t)\, h^{\alpha+\beta}(t)$ and $g(t)\, h^{\alpha-\beta}(t)$ respectively, to obtain the following new inequality:

$$\left(\int_a^b g(t)\, h^{\alpha}(t)dt\right)^2 \leq \left(\int_a^b g(t)\, h^{\alpha+\beta}(t)\, dt\right)\left(\int_a^b g(t)\, h^{\alpha-\beta}(t)dt\right). \qquad (3)$$

in which $\alpha, \beta \in \mathbb{R}$ and g, h are real integrable functions such that the involved integrals in (3) exist.

The aim of this paper is to apply the continuous inequality (3) for some functions of a continuous random variable in order to get new inequalities of the general form

$$f^2(\alpha x) \leq f\{(\alpha+\beta)x\}f\{(\alpha-\beta)x\} \qquad (\alpha, \beta \in \mathbb{R}). \qquad (4)$$

In particular, for $\alpha = 1$, inequality (4) reduces to

$$f^2(x) \leq f\{(1+\beta)x\}f\{(1-\beta)x\} \qquad (\beta \in \mathbb{R}). \qquad (5)$$

2. THE RESULTS

In this section, we apply the inequality(3) to establish new inequalities of the type(4).

2.1 An inequality for the moment generating function of a continuous random variable:

If X is a continuous random variable, then its density function (probability density function) $f = f_X(x)$ is defined by $f_X(x) = \frac{dF_X(x)}{dx}$, where $F_X(x)$ is known as distribution function (or cumulative distribution function) of X and defined by $F_X(x) = P(X \leq x) \quad -\infty < x < \infty$.

Density functions $f_X(x)$ are characterized by the properties that $f_X(x) \geq 0, \ \forall x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.

The moment generating function of a continuous random variable X with probability density function $f_X(x)$ is defined by

$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$, where t is a real variable (6)

Note that moment generating function will exist only for those values of t for which the integral converges absolutely.

Now if $g(x) = f_X(x)$, $h(x) = e^{tx}$ are substituted in inequality (3) for $[a, b] \to (-\infty, \infty)$, the following inequality is derived

$$\left(\int_{-\infty}^{\infty} e^{\alpha tx} f_X(x)\, dx\right)^2 \le \left(\int_{-\infty}^{\infty} e^{(\alpha+\beta)tx} f_X(x)\, dx\right)\left(\int_{-\infty}^{\infty} e^{(\alpha-\beta)tx} f_X(x)\, dx\right).$$

By (6), above inequality is equivalent to

$M_X{}^2(\alpha t) \le M_X\{(\alpha+\beta)t\} M_X\{(\alpha-\beta)t\}.$ (7)

Clearly this inequality is a special case of (4) and of (5) for $\alpha = 1$ i.e.

$M_X{}^2(t) \le M_X\{(1+\beta)t\} M_X\{(1-\beta)t\}.$ (8)

For example, let us consider a special case of the Gamma distribution [7] in the form

$$f_X(x) = \frac{1}{\Gamma(\lambda)} x^{\lambda-1} e^{-x}, \qquad (0 \le x < \infty; \lambda > 0).$$

whose moment generating function is computed as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\Gamma(\lambda)} x^{\lambda-1} e^{-x}\, dx = \frac{1}{(1-t)^{\lambda}}, (\lambda > 0; t < 1).$$

This means that according to (7), we have

$(1-\alpha t)^{-2\lambda} \le \{1-(\alpha+\beta)t\}^{-\lambda}\{1-(\alpha-\beta)t\}^{-\lambda},$ (9)

where $\alpha t < 1$, $(\alpha+\beta)t < 1$ and $(\alpha-\beta)t < 1$.

If we put α=1 in (9), inequality (9) is transformed to

$(1-t)^{-2\lambda} \le \{1-(1+\beta)t\}^{-\lambda}\{1-(1-\beta)t\}^{-\lambda},$ (10)

where $t < 1$, $(1+\beta)t < 1$ and $(1-\beta)t < 1$.

2.2 An inequality for the characteristic function of a continuous random variable:

The characteristic function of a continuous random variable X with probability density function $f_X(x)$, is defined by

$$\psi_X(\omega) = E(e^{j\omega x}) = \int_{-\infty}^{\infty} e^{j\omega x}\, f_X(x)\, dx, \qquad (11)$$

where ω is a real variable and $j = \sqrt{-1}$.

Note that $\psi_X(\omega)$ is the Fourier transform of $f_X(x)$, so if $\psi_X(\omega)$ is known, $f_X(x)$ can be found from the inverse Fourier transform; that is,

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega x}\, \psi_X(\omega)\, d\omega, \qquad (\omega \in \mathbb{R})$$

Now $|\psi_X(\omega)| = \left|\int_{-\infty}^{\infty} e^{j\omega x}\, f_X(x)\, dx\right| \le \int_{-\infty}^{\infty}\left| e^{j\omega x} f_X(x)\, dx\right| = \int_{-\infty}^{\infty} f_X(x)\, dx = 1 < \infty$. Thus the characteristic function $\psi_X(\omega)$ is always defined even if the moment generating function is not defined.

Hence, if $g(x) = f_X(x), h(x) = e^{j\omega x}$ and $[a, b] \to (-\infty, \infty)$ are considered in inequality (3) then we get

$$\left(\int_{-\infty}^{\infty} e^{j\alpha\omega x} f_X(x)\, dx\right)^2 \le \left(\int_{-\infty}^{\infty} e^{j(\alpha+\beta)\omega x} f_X(x)\, dx\right)\left(\int_{-\infty}^{\infty} e^{j(\alpha-\beta)\omega x} f_X(x)\, dx\right).$$

Using (11), this is in fact equivalent to:

$$\psi_X{}^2(\alpha\omega) \le \psi_X\{(\alpha+\beta)\omega\}\psi_X\{(\alpha-\beta)\omega\}. \qquad (12)$$

Inequality (12) is a special case of the main inequality (4).

If $\alpha = 1$, inequality (12) is transformed to

$$\psi_X{}^2(\omega) \le \psi_X\{(1+\beta)\omega\}\psi_X\{(1-\beta)\omega\}. \qquad (13)$$

For example, let us consider a special case of the Cauchy random variable X with parameter $\mu > 0$ and probability density function given by

$$f_X(x) = \frac{\mu}{\pi(x^2+\mu^2)}, \qquad (-\infty < x < \infty),$$

then the characteristic function of X is given by $\psi_X(\omega) = e^{-\mu|\omega|}$

This means that according to (12), we have

$$\left(e^{-\mu|\alpha\omega|}\right)^2 \le \left(e^{-\mu|(\alpha+\beta)\omega|}\right)\left(e^{-\mu|(\alpha-\beta)\omega|}\right). \qquad (14)$$

If we put α=1 in (14), inequality (14) reduces to

$$\left(e^{-\mu|\omega|}\right)^2 \le \left(e^{-\mu|(1+\beta)\omega|}\right)\left(e^{-\mu|(1-\beta)\omega|}\right). \qquad (15)$$

2.3 An inequality of type (4) for the power spectral density of a continuous-time WSS random process:

The autocorrelation function [7] of a continuous-time WSS (wide–sense stationary) random process $X(t)$ is defined as:

$R_X(\tau) = E[X(t)X(t+\tau)]$, where τ is the time difference.

Now the power spectral density (or power spectrum) $S_X(\omega)$ [10] of a continuous-time WSS (wide–sense stationary) random process $X(t)$ is defined as the Fourier transform of autocorrelation function $R_X(\tau)$; that is,

$$S_X(\omega) = \mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} e^{-j\omega\tau}\, R_X(\tau)\, d\tau, \qquad (16)$$

where ω is a real variable and $j = \sqrt{-1}$, provided that the integral in (16) exists.

Thus, taking the inverse Fourier transform of $S_X(\omega)$, we obtain

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\tau\omega}\, S_X(\omega) d\omega, \qquad (17)$$

where $S_X(\omega)$ is real and $S_X(\omega) \geq 0$.

Equations (16) and (17) are known as the Wiener-Khinchin relations.

Hence, substituting $g(\tau) = R_X(\tau), h(\tau) = e^{-j\omega\tau}$ and $[a,b] \to (-\infty,\infty)$ in inequality (3), the following inequality is derived

$$\left(\int_{-\infty}^{\infty} e^{-j\alpha\omega\tau} R_X(\tau)\, d\tau\right)^2 \leq \left(\int_{-\infty}^{\infty} e^{-j(\alpha+\beta)\omega\tau} R_X(\tau)\, d\tau\right)\left(\int_{-\infty}^{\infty} e^{-j(\alpha-\beta)\omega\tau} R_X(\tau)\, d\tau\right).$$

Using (16), this is in fact equivalent to:

$$S_X{}^2(\alpha\omega) \leq S_X\{(\alpha+\beta)\omega\} S_X\{(\alpha-\beta)\omega\}. \qquad (18)$$

Inequality (18) is a special case of the main inequality (4).

If $\alpha = 1$, inequality (18) is transformed to

$$S_X{}^2(\omega) \leq S_X\{(1+\beta)\omega\} S_X\{(1-\beta)\omega\}. \qquad (19)$$

For example, let us consider a WSS random process $X(t)$ with the autocorrelation function $R_X(\tau) = e^{-\mu|\tau|}$ where μ is a real positive constant. Then power spectral density (or power spectrum) $S_X(\omega)$ of $X(t)$ is

$$S_X(\omega) = \mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} e^{-j\omega\tau} e^{-\mu|\tau|} d\tau = \frac{2\mu}{(\omega^2+\mu^2)}$$

This means that according to (18), we have

$$\left(\frac{2\mu}{(\alpha\omega)^2+\mu^2}\right)^2 \le \left(\frac{2\mu}{((\alpha+\beta)\omega)^2+\mu^2}\right)\left(\frac{2\mu}{((\alpha-\beta)\omega)^2+\mu^2}\right)$$

$$\Rightarrow \left(\frac{1}{(\alpha\omega)^2+\mu^2}\right)^2 \le \left(\frac{1}{((\alpha+\beta)\omega)^2+\mu^2}\right)\left(\frac{1}{((\alpha-\beta)\omega)^2+\mu^2}\right).$$

For $\alpha=1$, this inequality is reduced to

$$\left(\frac{1}{\omega^2+\mu^2}\right)^2 \le \left(\frac{1}{((1+\beta)\omega)^2+\mu^2}\right)\left(\frac{1}{((1-\beta)\omega)^2+\mu^2}\right).$$

2.4 An inequality of type (4) for the autocorrelation function of a continuous-time WSS random process:

As discussed above, the autocorrelation function [10] is the inverse Fourier transform of $S_X(\omega)$ that is,

$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\tau\omega}\, S_X(\omega)d\omega,$

Now if $g(\omega) = S_X(\omega), h(\omega) = e^{j\tau\omega}$ are substituted in inequality (3) for $[a, b] \to (-\infty, \infty)$, the following inequality is derived

$$\left(\int_{-\infty}^{\infty} e^{j\alpha\tau\omega} S_X(\omega)\, d\omega\right)^2 \le \left(\int_{-\infty}^{\infty} e^{j(\alpha+\beta)\tau\omega} S_X(\omega)\, d\omega\right)\left(\int_{-\infty}^{\infty} e^{j(\alpha-\beta)\tau\omega} S_X(\omega)\, d\omega\right).$$

Using (17), this is in fact equivalent to:

$$R_X{}^2(\alpha\tau) \le R_X\{(\alpha+\beta)\tau\}R_X\{(\alpha-\beta)\tau\}. \quad (20)$$

Inequality (20) is a special case of the main inequality (4).

For instance, If $\alpha = 1$, inequality (20) is transformed to

$$R_X{}^2(\tau) \le R\{(1+\beta)\tau\}R_X\{(1-\beta)\tau\}. \quad (21)$$

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REFERENCES

[1]. A. Laforgia and P. Natalini, Turán-type inequalities for some special functions, J.Inequal. Pure Appl. Math., 27 (2006), Issue 1, Art. 32.

[2]. Cristinel Mortici, New Inequalities for some special functions via the Cauchy-Bunyakovsky-Schwarz inequality, Tamkang Journal of Mathematics 42 (2011), 53-57.

[3]. D.K. Callebaut, Generalization of the Cauchy-Schwarz inequality, J. Math. Anal. Appl. 12 (1965), 491-494.

[4]. D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, Classical and New Inequalities in Analysis. Kluwer Academic, Dordrecht (1993).

[5]. H. Alzer, On the Cauchy-Schwarz inequality, J. Math. Anal. Appl. 234 (1999), 6-14.

[6]. L. Zheng, Remark on a refinement of the Cauchy-Schwarz inequality, J. Math. Anal. Appl. 218 (1998), 13-21.

[7]. M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover Publications, Inc., New York, 1965.

[8]. M. Masjed-Jamei, A functional generalization of the Cauchy-Schwarz inequality and some subclasses. Appl. Math. Lett. 22 (2009), 1335-1339.

[9]. M. Masjed-Jamei, S.S. Dragomir, H.M. Srivastava, Some generalizations of the Cauchy-Schwarz and the Cauchy-Bunyakovsky inequalities involving four free parameters and their applications, Math. Computer. Modeling 49(2009), 1960-1968.

[10]. T. Veerarajan, Probability, Statistics and random processes Vol. 3, Tata McGraw- Hill, New Delhi, 2008.

[11]. W.L. Steiger, On a generalization of the Cauchy-Schwarz inequality, Amer. Math. Monthly 76 (1969), 815-816.

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