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WEAK COMPATIBILTY AND FIXED POINT THEOREM IN FUZZY METRIC SPACES USING IMPLICIT RELATION

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Abstract. In this paper, we prove a common fixed point theorem for weakly compatible mappings in fuzzy metric space by removing the assumption of continuity and replacing the completeness of the space with a set of alternative conditions.

1. Introduction

It is a well known fact that the concept of fuzzy set given by Zadeh [9] is a foundation stone in the field of non linear analysis for the development of Fuzzy metric spaces in fixed point theorems and its applications. The main break through in the said field was given by Kramosil and Mechalik [4] who followed Grabiec [1] to obtain succesfully the fuzzy version of Banach's fixed point theorem. Working on the same line, Mishra et. al [5] used the concept of compatibilty in fuzzy metric spaces and proved some common fixed point theorems for the same. Popa [6] came out with a concept of implicit relation and used it to prove some fixed point theorems for compatible mapping. The introduction of the notion of weak compatibile maps by Jungck and Rhoades [3] and thereby proving that compatible maps are weakly compatible but converse need not be true opens new boundaries in the domain of fixed

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point theory and its applications in the allied fields. Singh and Jain [8] studied semi-compatibility and compatibility of mappings to prove some fixed point theorems in fuzzy metric space. The aim of this paper is to prove a common fixed point theorem for five mappings under weak compatibility by striking off the condition of continuity and replacing the completeness of the space with a set of alternative conditions.

Definition 1.1. [7] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $\forall\, a, b, c, d \in [0,1]$.

Definition 1.2. [4]The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set ,$*$ is a continuous $t-$ norm, and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions ,$\forall\, x, y, z \in X$ and $s, t > 0$:

$$M(x, y, 0) = 0; \tag{1.1}$$

$$M(x, y, t) = 1, \forall\, t > 0 \iff x = y; \tag{1.2}$$

$$M(x, y, t) = M(y, x, t); \tag{1.3}$$

$$M(x, y, t) * M(y, z, s) \geq M(x, z, t + s); \tag{1.4}$$

$$M(x, y, .) : [0, \infty) \to [0, 1]\, is\, left\, continuous. \tag{1.5}$$

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1, \forall\, t > 0$. The folowing example shows that every metric space induces a fuzzy metric space.

Example 1.1. *[2] Let* (X, d) *be a metric space. Define* $ab = min\{a, b\}$ *and* $\forall\, a, b \in X$,

$$M(x, y, t) = t/(t + d(x, y)), \forall\, t > 0;$$

$$M(x, y, 0) = 0;$$

then $M(x, y, *)$ *is a fuzzy metric space. It is called the fuzzy metric induced by metric* d.

Lemma 1.1. *[1]* $\forall\, x, y \in X, M(x, y, .)$ *is a non-decreasing function.*

Remark 1. Since $*$ is continuous, it follows from (1.4) that the limit of a sequence in fuzzy metric space is unique. Let $(X, M, *)$ be a fuzzy metric space with the following condition:

$$\lim_{t\to\infty} M(x, y, t) = 1, \forall\, x, y \in X. \tag{1.6}$$

Lemma 1.2. *[5] If* $\forall\, x, y \in X, t > 0$ *and for a number* $k \in (0, 1)$, $M(x, y, kt) \geq M(x, y, t)$ *then* $x = y$.

Lemma 1.3. *[5] Let* $\{y_n\}$ *be a sequence in a fuzzy metric space* $(X, M, *)$ *with the condition (1.6). If there exists a number* $k \in (0, 1)$ *such that* $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$, $\forall t > 0$ *and* $n = 1, 2, ...$ *then* $\{y_n\}$ *is a cauchy sequence in* X.

Definition 1.3. [3] A pair of self-mappings S and T in a Fuzzy metric space are said to be weakly compatible if they commute at coincidence points; i.e. if $Tu = Su$ for some $u \in X$, then $TSu = STu$. It is easy to see that if S and T are compatible , then they are weakly compatible and the converse is not true in general.

Definition 1.4. (Implicit Relation) Let ψ be the set of all real continuous functions $\phi : (R^+)^4 \to R$, non-decreasing in first argument and satisfying the following conditions. For $u, v \geq 0$,

$$\phi(u, v, v, u) \geq 0, or, \phi(u, v, u, v) \geq 0, \tag{1.7}$$

implies that $u \geq v$

$$\phi(u, u, 1, 1) \geq 0 \tag{1.8}$$

implies that $u \geq 1$.

Remark 2. $\phi(u, 1, u, 1) \geq 0 \, or \, \phi(u, 1, 1, u) \geq 0$ implies that $u \geq 1$ follows from (1.7).

Example 1.2. *Define* $\phi(t_1, t_2, t_3, t_4) = 15t_1 - 13t_2 + 5t_3 - 7t_4$. *Then* $\phi \in \psi$.

2. MAIN RESULTS

Theorem 2.1. *Let* $(X, M, *)$ *be a fuzzy metric space with* $t * t \geq t, \forall t \in [0, 1]$ *and the condition* (1.6). *Let* A, B, S, T *and* P *be mappings of* X *into itself such that*

$$P(X) \subseteq AB(X), P(X) \subseteq ST(X); \tag{2.1}$$

$$for\, some\, \phi \in \psi, there\, exist\, k \in (0, 1)\, such\, that\, \forall\, x, y \in X and\, t > 0 \tag{2.2}$$

$$\phi\Big[M(Px, Py, kt), M(ABx, Px, t), M(ABx, STy, t), M(STy, Py, kt)\Big] \geq 0\,;$$

$$If\, one\, of\, P(X), AB(X)\, or\, ST(X)\, is\, a\, complete\, subspace\, of\, X\, then; \tag{2.3}$$

$$(i)\, P\, and\, AB\, have\, a\, coincidence\, point;$$

$$(ii)\, P\, and\, ST\, have\, a\, coincidence\, point;$$

(2.4) $$PB = BP,\ PT = TP,\ AB = BA,\ ST = TS;$$

(2.5) *Further, if the pair (P, AB) and (P, ST) is weak compatible,*

then A, B, S, T and P have a unique common fixed point in X.

Proof. *Let $x_0 \in X$ be any arbitrary point as $P(X) \subseteq AB(X)$ for any $x_0 \in X$ there exists a point $x_1 \in X$ such that $Px_0 = ABx_1$. Since $P(X) \subseteq ST(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Px_1 = STx_2$. Inductively we can define a sequence $\{y_n\} \in X$ as follows:*

$y_{2n} = Px_{2n} = ABx_{2n+1}, y_{2n+1} = Px_{2n+1} = STx_{2n+2}$, for $n = 0, 1, 2, \ldots$. Now using (2.2) with $x = x_{2n+1}, y = x_{2n+2}$,we get

$$\phi\Big[M(Px_{2n+1}, Px_{2n+2}, kt), M(ABx_{2n+1}, Px_{2n+1}, t), M(ABx_{2n+1}, STx_{2n+2}, t),$$
$$M(STx_{2n+2}, Px_{2n+2}, kt)\Big] \geq 0, \text{ that is}$$
$$\phi\Big[M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, kt)\Big] \geq 0,$$

Using (1.7), we get

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t).$$

Similarly by putting $x = x_{2n+2}$ and $y = x_{2n+3}$ in (2.2), we have

$$\phi\Big[M(Px_{2n+2}, Px_{2n+3}, kt), M(ABx_{2n+2}, Px_{2n+2}, t), M(ABx_{2n+2}, STx_{2n+3}, t), M(STx_{2n+3},$$
$$Px_{2n+3}, kt)\Big] \geq 0,$$
$$\phi\Big[M(y_{2n+2}, y_{2n+3}, kt), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+3}, kt),\Big] \geq 0,$$

Using (1.7), we get

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t).$$

Thus for any n and t, we have

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t).$$

Hence by Lemma(1.3),$\{y_n\}$ is a cauchy sequence in X, hence the subsequence

$\{y_{2n}\} = \{ABx_{2n+1}\} \subset AB(X)$ is a cauchy subsequence in $AB(X)$. Assume that $AB(X)$ is complete. Therefore, $\{y_n\}$ converges to a point $z = ABu$ for some $u \in X$. Hence, the sequence $\{y_{2n}\}$ converges also to z and the subsequences $\{Px_{2n}\}, \{Px_{2n+1}\}, \{STx_{2n+2}\}$ converges also to z.

Suppose $AB(X)$ is a complete subspace of X there exists a point $u = (AB)^{-1}z$ i.e $ABu = z$, by (2.2), we have

$$\phi\Big[M(Pu, Px_{2n+2}, kt), M(ABu, Pu, t), M(ABu, STx_{2n+2}, t), M(STx_{2n+2}, Px_{2n+2}, kt)\Big] \geq 0,$$

Letting $n \to \infty$ and using continuity of ϕ, we obtain

$$\phi\Big[M(Pu, z, kt), M(z, Pu, t), M(z, z, t), M(z, z, kt)\Big] \geq 0,$$

$$\phi\Big[M(Pu, z, kt), M(z, Pu, t), 1, 1\Big] \geq 0,$$

As ϕ is non-decreasing in first argument, we have

$$\phi\Big[M(Pu, z, t), M(z, Pu, t), 1, 1\Big] \geq 0,$$

Using (1.8), we get $M(Pu, z, t) \geq 1, \forall t > 0$, which gives $M(Pu, z, t) = 1$, i.e. $Pu = z$, therefore $Pu = ABu = z$, i.e. u is a coincidence point of P and AB. This proves (i).

Since $P(X) \subseteq ST(X), Pu = z$ implies that $z \in ST(X)$. Let $v \in (ST)^{-1}z$. Then $STv = z$. By (2.2), we have

$$\phi\Big[M(Px_{2n+1}, Pv, kt), M(ABx_{2n+1}, Px_{2n+1}, t), M(ABx_{2n+1}, STv, t), M(STv, Pv, kt)\Big] \geq 0,$$

Taking the limit $n \to \infty$, we get on solving

$\phi\Big[M(z, Pv, kt), 1, 1, M(z, Pv, kt)\Big] \geq 0$. Using (1.8), we get $M(Pv, z, kt) \geq 1, \forall\, t > 0$, which gives $M(Pv, z, t) = 1$ i.e. $Pv = z$, therefore $Pv = STv = z$, i.e. v is a coincidence point of P and ST. Hence $STv = ABu = Pv = Pu = z$ .Therefore (i) and (ii) is established.

The proof is similar if we suppose that one of $ST(X)$ or $P(X)$ is complete instead of

$AB(X)$.

Since the pair $\{P,AB\}$ *is weakly compatible , therefore* P *and* AB *commute at their coincidence point i.e.* $(AB)Pu = P(ABu)$ *or* $ABz = Pz$. *By (2.2) , we have*

$$\phi\Big[M(Pz, Px_{2n+2}, kt), M(ABz, Pz, t), M(ABz, STx_{2n+2}, t), M(STx_{2n+2}, Px_{2n+2}, kt)\Big] \geq 0,$$

Taking the limit $n \to \infty$, *we get*

$$\phi\Big[M(Pz, z, kt), M(z, Pz, t), 1, 1\Big] \geq 0.$$

As ϕ *is non-decreasing in first argument, we have*

$$\phi\Big[M(Pz, z, t), M(z, Pz, t), 1, 1\Big] \geq 0.$$

Using (1.8),we get

$M(Pz, z, t) \geq 1, \forall t > 0$, *which gives* $M(Pz, z, t) = 1$ *i.e.* $Pz = z$, *therefore* $Pz = ABz = z$. *Again, since the pair* $\{P,ST\}$ *is weakly compatible , therefore* P *and* ST *commute at their coincidence point i.e.* $(ST)Pv = P(STv)$ *or* $Pz = STz$. *By (2.2) , we have*

$$\phi\Big[M(Px_{2n+1}, Pz, kt), M(ABx_{2n+1}, Px_{2n+1}, t), M(ABx_{2n+1}, STz, t), M(STz, Pz, kt)\Big] \geq 0,$$

Taking the limit $n \to \infty$, *we get*

$$\phi\Big[M(z, Pz, kt), M(z, z, t), M(z, z, t), M(z, Pz, kt)\Big] \geq 0,$$

This gives $\phi\Big[M(z, Pz, kt), 1, 1, M(z, Pz, kt) \geq 0$,

Therefore by using (1.8), we have

$M(Pz, z, kt) \geq 1, \forall t > 0$, *which gives* $M(Pz, z, t) = 1$ *i.e.* $Pz = z$, *therefore* $Pz = STz = z$, *and so* $Pz = ABz = STz = z$. *By (2.2),* $x = Bz$ *and* $y = z$, *we have*

$$\phi\Big[M(P(Bz), Pz, kt), M(AB(Bz), P(Bz), t), M(AB(Bz), STz, t), M(STz, Pz, kt)\Big] \geq 0,$$

$\phi\Big[M(Bz,z,kt),M(Bz,Bz,t),M(Bz,z,t),M(z,z,kt)\Big] \geq 0,$
$\phi\Big[M(Bz,z,kt),1,M(Bz,z,t),1\Big] \geq 0,$
As ϕ is non-decreasing in first argument, we have
$\phi\Big[M(Bz,z,t),1,M(Bz,z,t),1\Big] \geq 0,$
using (1.8), we have

$M(Bz,z,t) \geq 1, \forall\, t > 0$, *which gives* $M(Bz,z,t) = 1$ *i.e.* $Bz = z$, *therefore* $Az = Bz = Pz = z$. *By (2.2),* $x = z$ *and* $y = Tz$, *we have*
$\phi\Big[M(Pz,P(Tz),kt),M(ABz,Pz,t),M(ABz,ST(Tz),t),M(ST(Tz),P(Tz),kt)\Big]$
$\geq 0,$
$\phi\Big[M(z,Tz,kt),M(z,z,t),M(z,Tz,t),M(Tz,Tz,kt)\Big] \geq 0,$
$\phi\Big[M(z,Tz,kt),1,M(z,Tz,t),1\Big] \geq 0,$
As ϕ is non-decreasing in first argument, we have
$\phi\Big[M(z,Tz,t),1,M(z,Tz,t),1\Big] \geq 0,$
using (1.8), we have

$M(Tz,z,t) \geq 1, \forall\, t > 0$, *which gives* $M(Tz,z,t) = 1$ *i.e.* $Tz = z$, *therefore* $Tz = Sz = Pz = z$. *Hence* $Az = Bz = Sz = Tz = Pz = z$, *i.e.* z *is a common fixed point of* $A, B, S, T,$ *and* P.

For uniqueness, let $(z \neq w)$ *be another common fixed point of* $A, B, S, T,$ *and* P. *By (2.2), we have*
$\phi\Big[M(Pz,Pw,kt),M(ABz,Pz,t),M(ABz,STw,t),M(STw,Pw,kt)\Big] \geq 0,$
$\phi\Big[M(z,w,kt),M(z,z,t),M(z,w,t),M(w,w,kt)\Big] \geq 0,$
$\phi\Big[M(z,w,kt),1,M(z,w,t),1\Big] \geq 0,$
As ϕ is non-decreasing in first argument, we have

$\phi\Big[M(z,w,t), 1, M(z,w,t), 1\Big] \geq 0,$
using (1.8), we have

$M(z,w,t) \geq 1, \forall\, t > 0,$ *which gives* $M(z,w,t) = 1$ *i.e.* $z = w$. *Thus* $A, B, S, T,$ *and* P *has a unique common fixed point.*

If we take the space to be complete and $P = I$ (the identity mapping on X) in Theorem (2.1), the conditions (2.1),(2.3),(2.4),(2.5) are satisfied trivially and we have the following corollary for four self maps:

Corollary 2.1. *Let* $(X, M, *)$ *be a complete fuzzy metric space with* $t{*}t \geq t, \forall\, t \in [0,1]$ *and the condition* (1.6). *Let* A, B, S *and* T *be mappings of* X *into itself such that* $for\, some\, \phi \in \psi, there\, exist\, k \in (0,1)\, satisfying$

(2.6) $$\phi\Big[M(x,y,kt), M(ABx,x,t), M(ABx,STy,t), M(STy,y,kt)\Big] \geq 0,$$

$\forall x, y \in X and\, t > 0.$

Then A, B, S, and T have a unique common fixed point in X.

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