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ON QUASI SEMIPRE-θ-CLOSED SETS IN TOPOLOGY

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Abstract. N. Levine [8] introduced and investigated the notion of generalized closed set. It is the object of this paper to offer a new notion of generalized closed set called quasi semipre-θ-closed set. This class is obtained by generalizing semipre-θ-closed sets. We also study and characterize the class of $\beta\theta$-$T_{\frac{1}{2}}$ spaces.

1. Introduction and Preliminaries

Is common viewpoint of many topologists that generalized open sets are important ingredients in General Topology and they are now the research topics of many topologists worldwide of which lots of important and interesting results emerged. Indeed a significant theme in General Topology and Real Analysis concerns the variously forms of continuity, separation axioms etc by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of β-open sets or semipreopen sets introduced by Abd El Monsef et al. [1] and Andrijević [2], respectively. In 2003, Noiri [7] used this notion and the β-closure [1] of a set to introduce the concepts of $\beta\theta$-open and $\beta\theta$-closed sets which provide a formulation of the $\beta\theta$-closure of a set in a topological space. Caldas, jafari and Ekici [3, 4, 5, 6] continued the work of Noiri and defined other concepts utilizing $\beta\theta$-closed sets.

In this paper we employ a new technique to obtain the class of sets, called quasi

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semipre-θ-closed sets (briefly qspθ-closed sets). This class is obtained by generalizing $\beta\theta$-closed sets using the notion of $\beta\theta$-open sets and $\beta\theta$-closure. As an application of qspθ-closed sets, we introduced and characterize a new class of spaces namely $\beta\theta$-$T_{\frac{1}{2}}$ spaces. We also introduce and study two classes of functions, qspθ-continuity and qspθ-irresoluteness by using qspθ-closed sets and study some of their fundamental properties. We introduce $Q\beta\theta$-connectedness by involving qspθ-open sets and show that if a topological space (X, τ) is $\beta\theta$-$T_{\frac{1}{2}}$, then the notions of $Q\beta\theta$-connectedness, and $\beta\theta$-connectedness are equivalent with each other.

Now we begin to recall some known notions which will be used in the sequal.
Abd El Monsef et al. [1] and Andrijević [2] introduced the notion of β-open set, which Andrijević called semipreopen, completely independent of each other. In this paper, we adopt the word β-open for the sake of clarity. A subset S of a topological space (X, τ) is called β-open if $S \subseteq Cl(Int(Cl(S)))$, where $Cl(S)$ and $Int(S)$ denote the closure and the interior of S, respectively. The complement of a β-open set is called β-closed. The intersection of all β-closed sets containing S is called the semipre-closure of S and is denoted by $spCl(S)$. The family of all β-open (resp. β-closed, open) subsets of X is denoted by $\beta O(X, \tau)$ or $\beta O(X)$ (resp. $\beta C(X, \tau), O(X, \tau)$). We set $\beta O(X, x) = \{U : x \in U \in \beta O(X, \tau)\}$.
A subset S is called β-regular [6], if it is both β-open and β-closed. The family of all β-regular sets of (X, τ) is denoted by $\beta R(X, \tau)$.
The semipre θ-closure of S [7], denoted by $spCl_\theta(S)$, is defined to be the set of all $x \in X$ such that $spCl(O) \cap S \neq \emptyset$ for every $O \in \beta O(X, \tau)$ with $x \in O$. A subset S is called $\beta\theta$-closed [7] if $S = spCl_\theta(S)$. The complement of a $\beta\theta$-closed set is called $\beta\theta$-open. The family of all $\beta\theta$-open subsets of X is denoted by $\beta\theta O(X, \tau)\}$. We set $\beta\theta O(X, x) = \{U : x \in U \in \beta\theta O(X, \tau)\}$.

Lemma 1.1. [7] *Let S be a subset of a topological space (X, τ).*
(1) If $S \in \beta O(X, \tau)$, then $spCl(S)$ is β-regular and $spCl(S) = spCl_\theta(S)$.
(2) S is β-regular if and only if S is $\beta\theta$-closed and $\beta\theta$-open.
(3) S is β-regular if and only if $S = spInt(spCl(S))$.

Lemma 1.2. [7] *For any subset S of a topological space (X,τ), $spCl_\theta(S)$ is $\beta\theta$-closed for every $S \subset X$.*

Definition 1.1. [6] Let X be a topological space and $S \subset X$. The β-θ-kernel of S, denoted by β-θ-$ker(S)$, is defined to be the set $\beta\text{-}\theta\text{-}ker(S) = \cap\{U \in \beta\theta O(X,\tau) : S \subset U\}$.

Definition 1.2. [5] A subset B of a topological space (X,τ) is called a Λ_θ^β-set if B=β-θ-$ker(B)$.

By Λ_θ^β, we denote the family of all Λ_θ^β-sets of (X,τ).

Proposition 1.1. *Let A, B and $\{B_\lambda \colon \lambda \in \Omega\}$ be subsets of a topological space (X,τ). Then the following properties are valid:*
(a) $B \subseteq \beta\text{-}\theta\text{-}ker(B)$.
(b) If $A \subseteq B$, then $\beta\text{-}\theta\text{-}ker(A) \subseteq \beta\text{-}\theta\text{-}ker(B)$.
(c) $\beta\text{-}\theta\text{-}ker[\beta\text{-}\theta\text{-}ker(B)] = \beta\text{-}\theta\text{-}ker(B)$. (i.e., $\beta\text{-}\theta\text{-}ker(B) \in \Lambda_\theta^\beta$).
(d) $\beta\text{-}\theta\text{-}ker[\bigcup_{\lambda\in\Omega} B_\lambda] = \bigcup_{\lambda\in\Omega} \beta\text{-}\theta\text{-}ker(B_\lambda)$.
(e) If $A \in \beta\theta O(X,\tau)$, then $A = \beta\text{-}\theta\text{-}ker(A)$. (i.e., $\beta\theta O(X,\tau) \subset \Lambda_\theta^\beta$).
(f) $\beta\text{-}\theta\text{-}ker[\bigcap_{\lambda\in\Omega} B_\lambda] \subseteq \bigcap_{\lambda\in\Omega} \beta\text{-}\theta\text{-}ker(B_\lambda)$.
(g) The subsets $\emptyset$ and X are Λ_θ^β-sets.
(h) Every union of Λ_θ^β-sets is a Λ_θ^β-set.
(i) Every intersection of Λ_θ^β-sets is a Λ_θ^β-set.

Proof. (a) Clear by the definition .
(b) Suppose that $x \notin \beta\text{-}\theta\text{-}ker(B)$. Then there exists a subset $O \in \beta\theta O(X,\tau)$ such that $O \supseteq B$ with $x \notin O$. Since $B \supseteq A$, then $x \notin \beta\text{-}\theta\text{-}ker(A)$ and thus $\beta\text{-}\theta\text{-}ker(A) \subseteq \beta\text{-}\theta\text{-}ker(B)$.
(c) Follows from (a) and the definition.
(d) Suppose that there exists a point x such that $x \notin \beta\text{-}\theta\text{-}ker[\bigcup_{\lambda\in\Omega} B_\lambda]$ Then, there exists a subset $O \in \beta\theta O(X,\tau)$ such that $\bigcup_{\lambda\in\Omega} B_\lambda \subseteq O$ and $x \notin O$. Thus, for each

$\lambda \in \Omega$ we have $x \notin \beta$-θ-$ker(B_\lambda)$. This implies that $x \notin \bigcup_{\lambda\in\Omega} \beta$-$\theta$-$ker(B_\lambda)$.
Conversely, suppose that there exists a point $x \in X$ such that $x \notin \bigcup_{\lambda\in\Omega} \beta$-$\theta$-$ker(B_\lambda)$. Then by definition, there exist subsets $O_\lambda \in \beta\theta O(X,\tau)$ (for each $\lambda \in \Omega$) such that $x \notin O_\lambda$, $B_\lambda \subseteq O_\lambda$. Let $O = \bigcup_{\lambda\in\Omega} O_\lambda$. Then we have that $x \notin \bigcup_{\lambda\in\Omega} O_\lambda$, $\bigcup_{\lambda\in\Omega} B_\lambda \subseteq O$ and $O \in \beta\theta O(X,\tau)$. This implies that $x \notin \beta$-θ-$ker[\bigcup_{\lambda\in\Omega} B_\lambda]$. Thus, the proof of (d) is completed.
(e) By definition and since $A \in \beta\theta O(X,\tau)$, we have β-θ-$ker(A) \subseteq A$. By (a) we have that β-θ-$ker(A) = A$.
(f) Suppose that there exists a point x such that $x \notin \bigcap_{\lambda\in\Omega} \beta$-$\theta$-$ker(B_\lambda)$. Then, for $\lambda \in \Omega$, such that $x \notin \beta$-θ-$ker(B_\lambda)$. Hence there exists $O \in \beta\theta O(X,\tau)$ such that $O \supseteq B_\lambda$ and $x \notin O$. Thus $x \notin \beta$-θ-$ker[\bigcap_{\lambda\in\Omega} B_\lambda]$.
(g) is obvious.
(h) Let $\{B_\lambda \colon \lambda \in \Omega\}$ be a family of Λ_θ^β-sets in a topological space (X,τ). Then by definition and (d), $\bigcup_{\lambda\in\Omega} B_\lambda = \bigcup_{\lambda\in\Omega} \beta$-$\theta$-$ker(B_\lambda) = \beta$-$\theta$-$ker[\bigcup_{\lambda\in\Omega} B_\lambda]$.
(i) Let $\{B_\lambda : \lambda \in \Omega\}$ be a family of Λ_θ^β-sets in (X,τ). Then by (f) and definition , β-θ-$ker[\bigcap_{\lambda\in\Omega} B_\lambda] \subseteq \bigcap_{\lambda\in\Omega} \beta$-$\theta$-$ker(B_\lambda) = \bigcap_{\lambda\in\Omega} B_\lambda$. Hence, by (a), $\bigcap_{\lambda\in\Omega} B_\lambda = \beta$-$\theta$-$ker[\bigcap_{\lambda\in\Omega} B_\lambda]$.

Remark 1.1. In general β-θ-$ker(B_1 \bigcap B_2) \neq \beta$-$\theta$-$ker(B_1) \bigcap \beta$-$\theta$-$ker(B_2)$, as the following example shows.

Example 1.1. *Let (X,τ) be a space with $X = \{a,b,c\}$ and $\tau = \{\emptyset, \{a\}, \{a,b\}, X\}$. Let $B_1 = \{a\}$ and $B_2 = \{b\}$. Then we have β-θ-$ker(B_1 \bigcap B_2)$ $= \emptyset$ but β-θ-$ker(B_1) \bigcap \beta$-θ-$ker(B_2) = X$.*

2. Quasi semipre θ-closed sets

Definition 2.1. A subset A of a topological space (X,τ) is called quasi semipre θ-closed (briefly qspθ-closed) if $spCl_\theta(A) \subset U$ whenever $A \subset U$ and U is $\beta\theta$-open in (X,τ).

The following Theorem characterize the qspθ-closed sets.

Theorem 2.1. A set A of a topological space (X, τ) is qspθ-closed if and only if $spCl_\theta(A) \subseteq \beta\text{-}\theta\text{-}ker(A)$.

Proof. Necessity. Let $x \in X$ such that $x \notin \beta\text{-}\theta\text{-}ker(A)$. So there exists a $\beta\theta$-open subset O such that $A \subset O$ with $x \notin O$. This means that $x \notin spCl_\theta(A)$ since A is qspθ-closed.
Sufficiency. Obvious.

Theorem 2.2. A subset A of a topological space (X, τ) is $\beta\theta$-closed if and only if A is qspθ-closed and $A = \beta\text{-}\theta\text{-}ker(A) \cap spCl_\theta(A)$.

Proof. Necessity. It is obvious, since $A \subset \beta\text{-}\theta\text{-}ker(A) \cap spCl_\theta(A) \subset spCl_\theta(A) = A$. This means that A is qspθ-closed and $A = \beta\text{-}\theta\text{-}ker(A) \cap spCl_\theta(A)$.
Sufficiency. Since A is qspθ-closed, then by Theorem 2.1 $spCl_\theta(A) \subset \beta\text{-}\theta\text{-}ker(A)$. Now $A = \beta\text{-}\theta\text{-}ker(A) \cap spCl_\theta(A) = spCl_\theta(A)$. Hence A is a $\beta\theta$-closed set.

The complement of a quasi semipre θ-closed set is called quasi semipre θ-open (briefly qspθ-open)

Theorem 2.3. A subset A of a topological space (X, τ) is qspθ-open if and only if $F \subset spInt_\theta(A)$ whenever F is $\beta\theta$-closed in X and $F \subset A$.

Proof. Necessity. Let A be qspθ-open and $F \subset A$, where F is $\beta\theta$-closed. It is obvious that A^c (complement of A) is contained in F^c. This implies that
$spCl_\theta(A^c) \subset F^c$. Hence $spCl_\theta(A^c) = (spInt_\theta(A))^c \subset F^c$, i.e., $F \subset spInt_\theta(A)$.
Sufficiency. If F is $\beta\theta$-closed set with $F \subset spInt_\theta(A)$ whenever $F \subset A$, then it follows that $A^c \subset F^c$ and $(spInt_\theta(A))^c \subset F^c$. Therefore A^c is qspθ-closed and therefore A is qspθ-open. Hence the proof.

Proposition 2.1. *Let A be a qspθ-closed subset of a space (X, τ). Then,*
(1) $spCl_\theta(A) \setminus A$ does not contain any nonempty $\beta\theta$-closed set.
(2) If $A \subset B \subset spCl_\theta(A)$, then B is also a qspθ-closed set.
(3) $spCl_\theta(A) \setminus A$ is qspθ-open.

Proof. (1) Suppose that A is qspθ-closed and let F be a nonempty $\beta\theta$-closed set with $F \subset spCl_\theta(A) \setminus A$. Since F^c is $\beta\theta$-open and $A \subset F^c$, it follows that $spCl_\theta(A) \subset F^c$, i.e. $F \subset (spCl_\theta(A))^c$ a contradiction.
(2) Let O be a $\beta\theta$-open set of the topological space (X,τ) such that $B \subset O$. Then $A \subset O$. Since A is qspθ-closed, we obtain $spCl_\theta(A) \subset O$. Then $spCl_\theta(B) \subset spCl_\theta(spCl_\theta(A)) = spCl_\theta(A) \subset O$. Therefore B is also a qspθ-closed set of (X,τ).
(3) If A is qspθ-closed and F is a $\beta\theta$-closed set such that $F \subset spCl_\theta(A) \setminus A$, then by (1), F is empty and therefore $F \subset spInt_\theta(spCl_\theta(A) \setminus A)$. By Theorem 2.3, $spCl_\theta(A) \setminus A$ is qspθ-open.

Theorem 2.4. For a topological space (X,τ), the following properties hold:
For each $x \in X$, the singleton $\{x\}$ is qspθ-closed in (X,τ).

Proof. Suppose that $\{x\} \subset U \in \beta\theta O(X)$. This implies that there exists
$V \in \beta O(X,\tau)$ such that $x \in V \subset spCl(V) \subset U$. Now we have
$spCl_\theta(\{x\}) \subset spCl_\theta(V) = spCl(V) \subset U$.

Theorem 2.5. For a subset A of a space (X,τ), the following conditions are equivalent.
(1) A is $\beta\theta$-open and qspθ-closed;
(2) A is β-regular.

Proof. (1) $\Rightarrow$ (2): Since A is $\beta\theta$-open and qspθ-closed, then A is $\beta\theta$-closed. Hence A is β-regular (Lemma 1.1).
(2) $\Rightarrow$ (1): Follows from Lemma 1.1.

Remark 2.1. Union of two qspθ-closed sets need not to be qspθ-closed. Let
$X = \{a,b,c\}$ and let $\tau = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$. Set $A = \{b\}$ and $B = \{c\}$. Both A and B are qspθ-closed but their union is not a qspθ-closed set of (X,τ).

3. Additional Properties

We note that, every $\beta\theta$-closed set is a qspθ-closed set but not conversely as seen from the following example. Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Set $A = \{a, b\}$. Then $spCl(A) = X$ and so A is not β-closed. Hence A is not $\beta\theta$-closed. Since X is the only $\beta\theta$-open set containing A, A is qspθ-closed.

A topological space (X, τ) is said to be $\beta\theta$-$T_{\frac{1}{2}}$. if every qspθ-closed set is $\beta\theta$-closed.

Theorem 3.1. For a topological space (X, τ) the following conditions are equivalent:
(i) X is $\beta\theta$-$\mathrm{T}_{\frac{1}{2}}$,
(ii) For each $x \in X$, $\{x\}$ is $\beta\theta$-closed or $\beta\theta$-open.

Proof. $(i) \Rightarrow (ii)$: Suppose that for some $x \in X$, $\{x\}$ is not $\beta\theta$-closed. Then $\{x\}^c$ is not $\beta\theta$-open. Since X is the only $\beta\theta$-open containing $\{x\}^c$, the set $\{x\}^c$ is qspθ-closed and so it is $\beta\theta$-closed in the $\beta\theta$-$\mathrm{T}_{\frac{1}{2}}$-space (X, τ). Therefore $\{x\}^c$ is $\beta\theta$-closed or equivalently $\{x\}$ is $\beta\theta$-open.
$(ii) \Rightarrow (i)$: Let $A \subset X$ be qspθ-closed. Let $x \in \beta Cl_\theta(A)$. We will show that $x \in A$. By the hypothesis, the singleton $\{x\}$ is either $\beta\theta$-closed or $\beta\theta$-open. We consider these two cases.

Case 1. $\{x\}$ is $\beta\theta$-closed: Then if $x \notin A$, there exists a $\beta\theta$-closed set in $\beta Cl_\theta(A) \setminus A$. By Proposition 2.1 $x \in A$.

Case 2. $\{x\}$ is $\beta\theta$-open: Since $x \in \beta Cl_\theta(A)$, then $\{x\} \bigcap A \neq \emptyset$. Thus $x \in A$.

Hence in both cases, we have $x \in A$, i.e., $\beta Cl_\theta(A) \subset A$ or equivalently A is $\beta\theta$-closed since the inclusion $A \subset \beta Cl_\theta(A)$ is trivial.

Every $\beta\theta$-$T_{\frac{1}{2}}$ space is β-$T_{\frac{1}{2}}$ (i.e., for each $x \in X$, $\{x\}$ is β-closed or β-open). but the converse is false.

Example 3.1. *Let $X = \{a, b\}$, and $\tau = \{\emptyset, \{a\}, X\}$. Then (X, τ) is a β-$T_{\frac{1}{2}}$ space which is not $\beta\theta$-$T_{\frac{1}{2}}$.*

We shall recall definitions of some functions used in the sequel to obtain several preservation theorems.

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:
(i) qspθ-continuous if $f^{-1}(F)$ is qspθ-closed in (X, τ) for every $F \in \beta\theta C(Y, \sigma)$.
(ii) qspθ-irresolute if $f^{-1}(F)$ is qspθ-closed in (X, τ) for every F qspθ-closed in (Y, σ).
(iii) weakly β-irresolute [7] if $f^{-1}(V) \in \beta\theta C(X, \tau)$ (resp. $f^{-1}(V) \in \beta\theta O(X, \tau)$) for every $V \in \beta\theta C(Y\sigma)$. (resp. $V \in \beta\theta O(Y\sigma)$.)
(iv) pre-$\beta\theta$-open (resp. pre-$\beta\theta$-closed) [4] if $f(U) \in \beta\theta O(Y, \sigma)$ (resp. $f(U) \in \beta\theta C(Y, \sigma)$) for every $U \in \beta\theta O(X, \tau)$. (resp. $U \in \beta\theta C(X, \tau)$.)

Theorem 3.2. If $f : (X, \tau) \to (Y, \sigma)$ is a bijective pre- $\beta\theta$-open and qspθ-continuous function, then f is qspθ-irresolute.

Proof. Let F be a qspθ-closed set of (Y, σ). Assume that $f^{-1}(F) \subset O$, where $O \in \beta\theta O(X, \tau)$. Clearly $F \subset f(O)$. Since $f(O) \in \beta\theta O(Y, \sigma)$ and F is a qspθ-closed set, we have $spCl_\theta(F) \subset f(O)$ and therefore $f^{-1}(spCl_\theta(F)) \subset O$. By the fact that f is qspθ-continuous and $spCl_\theta(F)$ is $\beta\theta$-closed, it follows that $spCl_\theta(f^{-1}(spCl_\theta(F))) \subset O$ and thus $spCl_\theta(f^{-1}(F)) \subset O$. This means that $f^{-1}(F)$ is qspθ-closed in (X, τ) and therefore f is qspθ-irresolute.

Corollary 3.1. *If the domain of a bijective pre-$\beta\theta$-open and qspθ-continuous is a $\beta\theta$-$T_{\frac{1}{2}}$, then so is the codomain(=range).*

Proof. It suffices to apply Theorem 3.2.

Theorem 3.3. Let $f : (X, \tau) \to (Y, \sigma)$ is qspθ-irresolute. Then f is weakly β-irresolute if (X, τ) is $\beta\theta$-$T_{\frac{1}{2}}$.

Proof. Suppose that V is a $\beta\theta$-closed set in (Y, σ). Then, V is qspθ-closed in (Y, σ). Since f is qspθ-irresolute, $f^{-1}(V)$ is qspθ-closed in (X, τ). Since (X, τ) is $\beta\theta$-$T_{\frac{1}{2}}$, $f^{-1}(V)$ is $\beta\theta$-closed. This shows that f is weakly β-irresolute.

Theorem 3.4. If a function $f : (X,\tau) \to (Y,\sigma)$ is weakly β-irresolute and pre-$\beta\theta$-closed, then for every qspθ-closed set F of (Y,σ), $f^{-1}(F)$ is qspθ-closed set of (X,τ) (i.e., f is qspθ-irresolute).

Proof. Suppose that F is a qspθ-closed set of (Y,σ). Assume $f^{-1}(F) \subset O$, where $O \in \beta\theta O(X,\tau)$. By hypothesis, f is weakly β-irresolute and therefore $f(spCl_\theta(f^{-1}(F)) \cap O^c) \subset spCl_\theta(f(f^{-1}(F))) \cap f(f^{-1}(F^c)) \subset spCl_\theta(F) \setminus F$. By the fact that f is pre-$\beta\theta$-closed, it follows that $spCl_\theta(F) \setminus F$ contains a $\beta\theta$-closed subset $f(spCl_\theta(f^{-1}(F)) \cap O^c)$. Now we have $f(spCl_\theta(f^{-1}(F)) \cap O^c) = \emptyset$ by Proposition 2.1. This implies that $spCl_\theta(f^{-1}(F)) \subset O$. This shows that $f^{-1}(F)$ is qspθ-closed set of (X,τ).

Theorem 3.5. If a function $f : (X,\tau) \to (Y,\sigma)$ is weakly β-irresolute and pre-$\beta\theta$-closed, then for every qspθ-closed set D of (X,τ), $f(D)$ is -closed set of (Y,σ).

Proof. Suppose that D is a qspθ-closed set of (X,τ). Assume $f(D) \subset O$, where $O \in \beta\theta O(Y,\sigma)$. Now $D \subset f^{-1}(O)$ and since f is weakly β-irresolute $f^{-1}(O) \in \beta\theta O(X,\tau)$. But D is qspθ-closed and therefore $spCl_\theta(D) \subset f^{-1}(O)$. Thus $f(spCl_\theta(D) \subset O$.
Now we have $spCl_\theta(f(D)) \subset spCl_\theta(f(spCl_\theta(D))) = f(spCl_\theta(D)) \subset O$. This shows that $f(D)$ is a qspθ-closed set of (Y,σ).

Theorem 3.6. If a topological space (X,τ) is $\beta\theta$-$T_{\frac{1}{2}}$ and $f : (X,\tau) \to (Y,\sigma)$ is surjective, weakly β-irresolute and pre-$\beta\theta$-closed, then (Y,σ) is $\beta\theta$-$T_{\frac{1}{2}}$.

Proof. Assume that A is a qspθ-closed subset of (Y,σ). Then by Theorem 3.4 we have $f^{-1}(A)$ is a qspθ-closed subset of (X,τ) and hence $A = f(f^{-1}(A))$ is $\beta\theta$-closed. It follows that (Y,σ) is $\beta\theta$-$T_{\frac{1}{2}}$.

Recall that a topological space (X,τ) is said to be *$\beta\theta$-connected* [6] if X can not be written as a disjoint union of two non-empty $\beta\theta$-open sets.

Definition 3.2. A topological space (X,τ) is said to be $Q\beta\theta$-connected if X can not be written as a disjoint union of two non-empty qspθ-open sets.

Theorem 3.7. For a topological space (X, τ) the following conditions are equivalent:
(i) X is $Q\beta\theta$-connected;
(ii) the only subsets of X which are both qspθ-closed and qspθ-open are $\emptyset$ and X.

Proof. (i) $\Rightarrow$ (ii): Suppose that A is a qspθ-closed and qspθ-open subset of X. Then the complement of A, i.e. $X \setminus A$ is qspθ-closed and qspθ-open. By the fact that $X = A \cup (X \setminus A)$ and $A \cap (X \setminus A) = \emptyset$, either A is empty or A is X.

(ii)$\Rightarrow$ (i): Let $X = A \cup B$, where A and B are qspθ-open sets and $A \cap B = \emptyset$. Then A is qspθ-open and qspθ-closed. By hypothesis either A is empty or $A = X$. This implies that X is $Q\beta\theta$-connected.

Theorem 3.8. If (X, τ) is a $\beta\theta$-$T_{\frac{1}{2}}$ space, then the following conditions are equivalent:
(i) X is $Q\beta\theta$-connected;
(ii) X is $\beta\theta$-connected.

Theorem 3.9. (i) If $f : (X, \tau) \to (Y, \sigma)$ is a surjective qspθ-continuous function and X is $Q\beta\theta$-connected, then Y is $\beta\theta$-connected.
(ii) If $f : (X, \tau) \to (Y, \sigma)$ is a surjective qspθ-irresolute function and X is $Q\beta\theta$-connected, then Y is $Q\beta\theta$-connected.

Proof. (i) Suppose that Y is not $\beta\theta$-connected. Let $Y = A \cup B$ where A and B are disjoint non-empty $\beta\theta$-open set in Y. Since f is qspθ-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty and qspθ-open in X. This contradicts the fact that X is $Q\beta\theta$-connected. Hence Y is $\beta\theta$-connected.
(ii) The argument is a minor modification of the proof (i).

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