Jordan Journal of Mathematics and Statistics (JJMS) 8(4), 2015, pp 293 - 308

PARITY COMBINATION CORDIAL LABELING OF GRAPHS

R.PONRAJ[1], S.SATHISH NARAYANAN[2] AND A.M.S.RAMASAMY[3]

Abstract. In this paper we define a new graph labeling called parity combination cordial labeling. Let G be a (p, q) graph. Let f be an injective map from $V(G)$ to $\{1, 2, \ldots, p\}$. For each edge xy, assign the label $\binom{x}{y}$ or $\binom{y}{x}$ according as $x > y$ or $y > x$. f is called a parity combination cordial labeling (PCC-labeling) if f is a one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with an even number and odd number, respectively. A graph with a parity combination cordial labeling is called a parity combination cordial graph (PCC-graph). Also we investigate the PCC-labeling behavior of path, cycle, fan, comb, complete graph, wheel, crown, star. A conjecture is stated at the end.

1. Introduction

Graphs considered here are finite, undirected and simple. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The number of vertices in G is denoted by p and that of edges we denote q. Most graph labeling methods trace their origin to one introduced by Rosa [7] in year 1967. A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Labeled graphs serve as a useful mathematical model for a broad range of applications such as coding theory, X-ray crystallography analysis, communication network addressing systems, astronomy, radar, circuit design

1991 *Mathematics Subject Classification.* 05C78.

Key words and phrases. path, cycle, comb, star, wheel.

Received: June 10, 2015 Accepted : Dec. 28 , 2015 .

and database management [3]. The join of two graphs G_1 and G_2 is denoted by $G_1 + G_2$ and whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. In 1980, Cahit [1] introduced the cordial labeling of graphs. Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. Call f a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Several authors studied the cordial graphs and some of them are M.A. Seoud and A.E.I.Abdel Maqsoud [8], S.C.Shee and Y.S.Ho [9]. In [5], Hegde et al. introduced the concept of combination labeling of graphs. Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \ldots, p\}$ be a bijection such that the induced map $g : E(G) \to \mathbb{N}$ defined as $\binom{f(u)}{f(v)}$ if $f(u) > f(v)$ or $\binom{f(v)}{f(u)}$ if $f(v) > f(u)$ is injective. Such a labeling f is called combination labeling of G. A graph G which admits a combination labeling is called a combination graph. Motivated by these two labelings, we introduce a new type of labeling called parity combination cordial labeling. In this paper we investigate the PCC-labeling behavior of path, cycle, fan, comb. For graph theoretic terminology we refer Harary [4] and for number theoretic we refer [2].

2. PCC-labeling

Definition 2.1. Let G be a (p, q) graph. Let f be a one to one map from $V(G)$ to $\{1, 2, \ldots, p\}$. For each edge xy, assign the label $\binom{x}{y}$ if $x > y$ or the label $\binom{y}{x}$ if $y > x$. f is called a parity combination cordial labeling (PCC-labeling) if f is one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with an even number and odd number respectively. A graph with a parity combination cordial labeling is called a parity combination cordial graph (PCC-graph).

Theorem 2.1. Any path P_n is a PCC-graph.

Proof. Let $P_n : u_1u_2 \dots u_n$ be a path. Assign the labels $1, 2, \dots, n$ consecutively to the vertices $u_1, u_2, \dots, u_n$. Since $\binom{n}{n-1} = \binom{n}{1} = n$, it is easy to verify that the edges $u_{2i-1}u_{2i}$ received even labels whereas the edges $u_{2i}u_{2i+1}$ received the odd labels. Hence we have the following table 1.

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 2$	$\frac{n}{2}$	$\frac{n}{2} - 1$
$n \equiv 1 \pmod 2$	$\frac{n-1}{2}$	$\frac{n-1}{2}$

TABLE 1

Table 1 establishes that P_n is a PCC-graph. □

Theorem 2.2. Any cycle C_n is a PCC-graph.

Proof. Let $C_n : u_1u_2 \dots u_nu_1$ be a cycle.

Case 1. n is odd.

Assign the labels to the vertices of C_n as in theorem 2.1. Here $e_f(0) = \frac{n-1}{2}$ and $e_f(1) = \frac{n+1}{2}$.

Case 2. n is even.

Assign the labels to the vertices of C_n as in theorem 2.1 then interchange the labels of u_2 and u_3. As in theorem 2.1, $e_f(0) = e_f(1) = \frac{n}{2}$ and hence C_n is PCC. □

Theorem 2.3. All stars $K_{1,n}$ are PCC-graphs.

Proof. Assign the label 1 to the central vertex of the star and assign the remaining labels from 2 to $n + 1$ to the leaves. Since $\binom{n}{1} = n$, the edges with even labeled pendent vertex received the even label. Also the other edges received the odd label. This forces the following table 2.

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 2$	$\frac{n}{2}$	$\frac{n}{2}$
$n \equiv 1 \pmod 2$	$\frac{n+1}{2}$	$\frac{n-1}{2}$

Table 2

From table 2, we conclude that f is a parity combination cordial labeling of the star $K_{1,n}$. □

Now we investigate the PCC-labeling behavior of triangular snakes. a triangular snake ΔT_n is obtained from a path $u_1u_2\dots u_n$ by joining u_i and u_{i+1} to a new vertex v_i for $i = 1, 2, 3 \dots, n-1$. Let P_n be the path $u_1u_2\dots u_n$. Let $V(\Delta T_n) = V(P_n) \cup \{v_i : 1 \le i \le n-1\}$ and $E(\Delta T_n) = E(P_n) \cup \{u_iv_i, v_iu_{i+1} : 1 \le i \le n-1\}$.

Theorem 2.4. The triangular snake ΔT_n is a PCC-graph.

Proof. Define an injective map $f : V(\Delta T_n) \to \{1, 2, \dots, 2n-1\}$ by

$$\begin{aligned} f(u_i) &= 2i-1, \quad 1 \le i \le n \\ f(v_i) &= 2i, \qquad\; 1 \le i \le n-1. \end{aligned}$$

Since $\binom{n}{2}$ is even if $n \equiv 0, 1 \pmod 4$ and odd if $n \equiv 2, 3 \pmod 4$, the number of 0's and 1's in the edges of the path are balanced. Consider the other edges. Note that $\binom{f(v_i)}{f(u_i)}$ is even and $\binom{f(u_i)}{f(v_i)}$ is odd. So the edges u_iv_i are labeled with even numbers and the edges v_iu_{i+1} are labeled with odd numbers. So, here the number of 0's and 1's are balanced. Hence f is a PCC-labeling of ΔT_n. □

Next we consider the alternate triangular snake. An alternate triangular snake $A(T_n)$ is obtained from a path $u_1u_2\dots u_n$ by joining u_{2i} and u_{2i+1} to new vertex v_i, $1 \le i \le \frac{n-2}{2}$ if the first triangle starts from u_2 and the last triangle ends with u_{n-1}, by joining u_{2i-1} and u_{2i} to new vertex v_i, $1 \leq i \leq \frac{n}{2}$ if the first triangle starts from u_1 and the last triangle ends with u_n, by joining u_{2i} and u_{2i+1} to new vertex v_i, $1 \leq i \leq \frac{n-1}{2}$ if the first triangle starts from u_2 and the last triangle ends with u_n. That is every alternate edge of a path is replaced by C_3.

Theorem 2.5. Alternate triangular snakes are PCC-graphs.

Proof. According to the parity of n, we consider the following two cases:

Case 1. n is even.

Consider the case that the first triangle starts from u_2 and the last triangle ends with u_{n-1}.

In this case, $|V(A(T_n))| = \frac{3n-2}{2}$ and $|E(A(T_n))| = 2n-3$. Define an injective map $f : V(A(T_n)) \to \{1, 2, \ldots, \frac{3n-2}{2}\}$ by

$$\begin{aligned} f(u_{2i-1}) &= 3i-2, \quad 1 \leq i \leq \tfrac{n}{2} \\ f(u_{2i}) &= 3i-1, \quad 1 \leq i \leq \tfrac{n}{2} \\ f(v_i) &= 3i, \qquad\ \ 1 \leq i \leq \tfrac{n-2}{2}. \end{aligned}$$

Here

$$e_f(0) = \begin{cases} n-2 & \text{if} \quad n \equiv 0 \pmod 8 \\ n-1 & \text{otherwise} \end{cases}$$

and

$$e_f(1) = \begin{cases} n-1 & \text{if} \quad n \equiv 0 \pmod 8 \\ n-2 & \text{otherwise} \end{cases}$$

If the first triangle starts from u_1 and the last triangle ends with u_n then $|V(A(T_n))| = \frac{3n}{2}$ and $|E(A(T_n))| = 2n-1$. Define a one to one map $f : V(A(T_n)) \to \left\{1, 2, \ldots, \frac{3n}{2}\right\}$ by

$$\begin{aligned} f(u_{2i-1}) &= 3i-2, \quad 1 \leq i \leq \tfrac{n}{2} \\ f(u_{2i}) &= 3i, \qquad\ \ 1 \leq i \leq \tfrac{n}{2} \\ f(v_i) &= 3i-1, \quad 1 \leq i \leq \tfrac{n}{2}. \end{aligned}$$

In this case

$$e_f(0) = \begin{cases} n & \text{if} \quad n \equiv 0 \pmod 8 \\ n-1 & \text{otherwise} \end{cases}$$

and

$$e_f(1) = \begin{cases} n-1 & \text{if} \quad n \equiv 0 \pmod 8 \\ n & \text{otherwise} \end{cases}$$

Case 2. n is odd.

In this case, the first triangle start from u_2 and the last triangle ends with u_n. Note that in this case, $|V(A(T_n))| = \frac{3n-1}{2}$ and $|E(A(T_n))| = 2n-2$. Define an injective map $f : V(A(T_n)) \to \left\{1, 2, \ldots, \frac{3n-1}{2}\right\}$ by

$$\begin{aligned} f(u_{2i-1}) &= 3i-2, \quad 1 \le i \le \tfrac{n+1}{2} \\ f(u_{2i}) &= 3i-1, \quad 1 \le i \le \tfrac{n-1}{2} \\ f(v_i) &= 3i, \quad\quad 1 \le i \le \tfrac{n-1}{2}. \end{aligned}$$

Then if $n \equiv 3 \pmod 8$ relabel the vertices u_2, v_1 by 3, 2 respectively. Here $e_f(0) = e_f(1) = n-1$. Hence all alternate triangular snakes are PCC. □

A rooted tree consisting of k branches, where the i^{th} branch is a path of length i, is called an olive tree.

Theorem 2.6. All olive trees are PCC-graphs.

Proof. Assign the label 1 to the root vertex. Next consider the path of highest order. Let n be the order of this path. Assign the label 2 to the vertex on this path which is neighbor of the root vertex. Then assign 3 to the vertex which is adjacent to the vertex with label 2. (other than root vertex). Thereafter label to the next vertex by 4 and so on. In this process the last vertex of the path receives the label $n+1$. Next consider the path of order $n-1$. Assign the label $n+2$ to the vertex of this path which is adjacent to the root vertex. Next assign $n+3, n+4, \ldots$ to the successive vertices of this path. Subsequently consider the path of length $n-2, n-3, \ldots$ and proceed as before. Note that the vertex in the path of order 2 receives the label $\binom{n}{2}+1$. Using the results $\binom{n}{1}=n$ and $\binom{n}{n-1}=\binom{n}{1}=n$, we get

$$e_f(0)-e_f(1)=\begin{cases}0 & \text{if} \quad n \text{ is odd}\\ 1 & \text{if} \quad n \text{ is even}\end{cases}$$

For illustration we consider the olive tree with order 29 as given in figure 1.

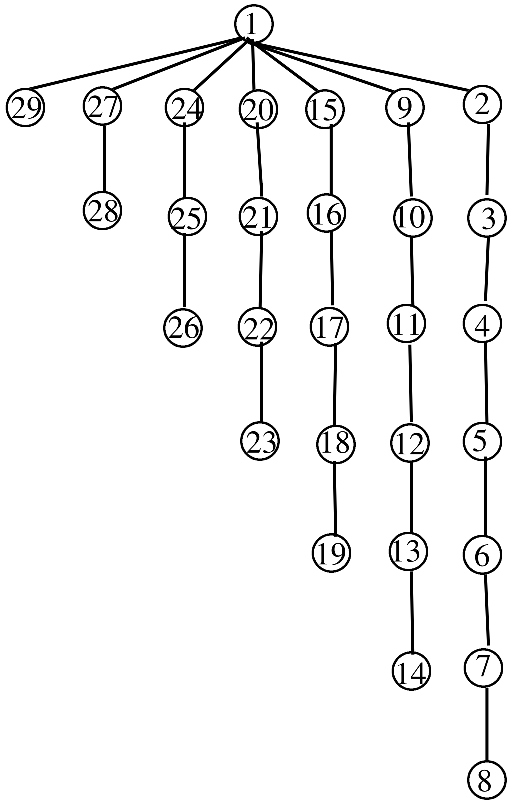

FIGURE 1

□

Next we investigate the parity combination cordial labeling behavior of comb $P_n \odot K_1$. Let $P_n : u_1u_2 \ldots u_n$ be a path. Let $V(P_n \odot K_1) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = E(P_n) \cup \{u_iv_i : 1 \leq i \leq n\}$.

Theorem 2.7. The comb $P_n \odot K_1$ is a PCC-graph.

Proof. Here $p = 2n$ and $q = 2n - 1$.

Case 1. n is odd.

Subcase 1a. $n \equiv 1 \pmod 4$.

Let $n = 4t+1$ and $t > 0$. Define an injective map $f : V(P_n \odot K_1) \to \{1, 2, \ldots, 8t+2\}$ by $f(v_{4t+1}) = 1$,

$$\begin{aligned} f(u_i) &= 2i, & 1 \leq i \leq 4t+1 \\ f(v_i) &= 2i+1, & 1 \leq i \leq 2t \\ f(v_{2t+2i-1}) &= 4t+4i+1, & 1 \leq i \leq t \\ f(v_{2t+2i}) &= 4t+4i-1, & 1 \leq i \leq t. \end{aligned}$$

Here $e_f(0) = 4t+1$ and $e_f(1) = 4t$.

Subcase 1b. $n \equiv 3 \pmod 4$.

Let $n = 4t-1$ and $t > 0$. Define an injective map $f : V(P_n \odot K_1) \to \{1, 2, \ldots, 8t-2\}$ by $f(v_{4t-1}) = 1$,

$$\begin{aligned} f(u_i) &= 2i, & 1 \leq i \leq 4t-1 \\ f(v_i) &= 2i+1, & 1 \leq i \leq 2t \\ f(v_{2t+2i-1}) &= 4t+4i+1, & 1 \leq i \leq t-1 \\ f(v_{2t+2i}) &= 4t+4i-1, & 1 \leq i \leq t-1. \end{aligned}$$

Here $e_f(0) = 4t-2$ and $e_f(1) = 4t-1$.

Case 2. n is even.

Let $n = 2t$ and $t > 0$. Define an injective map $f : V(P_n \odot K_1) \to \{1, 2, \ldots, 4t\}$ by $f(v_{2t}) = 1$, $f(v_1) = 3$, $f(v_2) = 5$, $f(v_3) = 7$,

$$\begin{aligned} f(u_i) &= 2i, & 1 \leq i \leq 2t \\ f(v_{2i+2}) &= 4i+7, & 1 \leq i \leq t-2 \\ f(v_{2i+3}) &= 4i+5, & 1 \leq i \leq t-2. \end{aligned}$$

Here $e_f(0) = 2t-1$ and $e_f(1) = 2t$. Hence $P_n \odot K_1$ is PCC. □

Theorem 2.8. The crown $C_n \odot K_1$ is a PCC-graph.

Proof. In each of the following cases, assign the labels as in Theorem 2.7.

Case 1. $n \equiv 0, 1, 2 \pmod 4$.

Here $e_f(0) = e_f(1) = n$.

Case 2. $n \equiv 3 \pmod 4$.

For $n = 3$ the following figure 2 shows that $C_3 \odot K_1$ is PCC.

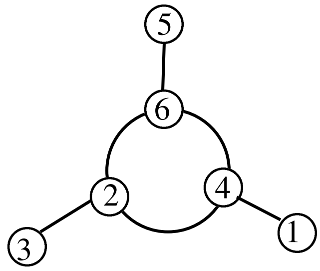

FIGURE 2

For $n > 3$, relabel the vertices v_2, v_3 by 7, 5 respectively. Here also $e_f(0) = e_f(1) = n$ and hence $C_n \odot K_1$ is PCC. □

The graph $P_n + K_1$ is called the fan F_n. Let $V(F_n) = \{u, u_i : 1 \le i \le n\}$ and $E(F_n) = \{uu_i : 1 \le i \le n\} \cup \{u_i u_{i+1} : 1 \le i \le n-1\}$.

Theorem 2.9. The fan F_n is a PCC-graph.

Proof. Define a map $f : V(F_n) \to \{1, 2, \ldots, n+1\}$ by $f(u) = 1$, $f(u_i) = i + 1$, $1 \le i \le n$. The following table 3 shows that F_n is PCC.

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 2$	$n-1$	n
$n \equiv 1 \pmod 2$	n	$n-1$

TABLE 3

□

The graph $C_n + K_1$ is called a Wheel W_n.

Theorem 2.10. The wheel W_n is a PCC-graph if and only if $n \geq 4$.

Proof. It is easy to check that the graph W_3 is not PCC. Suppose $n \geq 4$, then assign the labels to the vertices of W_n as in theorem 2.9. Now if $n \equiv 2 \pmod 4$, then relabel u_1, u_3 by 4, 2 respectively and if $n \equiv 3 \pmod 4$ then relabel u_3, u_4 by 5, 4 respectively. Then $e_f(0) = e_f(1) = n$. □

The umbrella $U_{n,m}$ is obtained from a fan F_n by appending a path $P_m : v_1v_2 \ldots v_m$ to the central vertex of the fan F_n. Take the vertex set and edge set of F_n as in theorem 2.9.

Theorem 2.11. The umbrella $U_{n,m}$, $m > 1$ is a PCC-graph.

Proof. Identify the vertices u and v_1. It is clear that $|E(U_{n,m})| = 2n+m-2$. Assign the labels $i+1$ to the vertices u_i $(1 \leq i \leq n)$ and put the label 1 to the vertex u. Then put the labels $n+j$ $(2 \leq j \leq m)$ to the vertices v_j $(2 \leq j \leq m)$. For $n \equiv 1 \pmod 2$, $m \equiv 0 \pmod 2$ and $n \neq 3$, relabel the vertices u_3, u_4 by 5, 4 respectively. If $n = 3$ and $m \equiv 0 \pmod 2$ then assign the label i $(1 \leq i \leq m)$ to v_i $(1 \leq i \leq m)$ respectively and then put the labels $m+1$, $m+2$, $m+3$ to the vertices u_1, u_2, u_3 respectively. The following table 4 shows that $U_{m,n}$ is PCC.

□

Our next investigation is about the complete graph K_n.
According to the paper of Karl Goldberg et al. [6], the following table 5 is derived. This proves that K_n $(4 \leq n \leq 100)$ is not PCC.

Values of n & m	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 2$ & $m \equiv 0 \pmod 2$	$n-1+\frac{m}{2}$	$n+\frac{m-2}{2}$
$n \equiv 0 \pmod 2$ & $m \equiv 1 \pmod 2$	$n-1+\frac{m-1}{2}$	$n+\frac{m-1}{2}$
$n \equiv 1 \pmod 2$, $m \equiv 0 \pmod 2$ & $n \neq 3$	$n-1+\frac{m}{2}$	$n+\frac{m-2}{2}$
$n = 3$ & $m \equiv 0 \pmod 2$	$\frac{m}{2}+2$	$\frac{m-2}{2}+3$
$n \equiv 1 \pmod 2$ & $m \equiv 1 \pmod 2$	$n+\frac{m-1}{2}$	$n-1+\frac{m-1}{2}$

TABLE 4

Values of n	$e_f(0)$	$e_f(1)$	$\|e_f(0)-e_f(1)\|$
2	1	0	1
3	1	2	1
4	4	2	2
5	6	4	2
6	9	6	3
7	9	12	3
8	16	12	4
9	22	14	8
10	29	16	13
11	33	22	11
12	42	24	18
13	48	30	18
14	55	36	19
15	55	50	5
16	70	50	20
17	84	52	32
18	99	54	45

19	111	60	51
20	128	62	66
21	142	68	74
22	157	74	83
23	165	88	77
24	186	90	96
25	204	96	108
26	223	102	121
27	235	116	119
28	256	122	134
29	270	136	134
30	285	150	135
31	285	180	105
32	316	180	136
33	346	182	164
34	377	184	193
35	405	190	215
36	438	192	246
37	468	198	270
38	499	204	295
39	523	218	305
40	560	220	340
41	594	226	368
42	629	232	397
43	657	246	411

44	694	252	442
45	724	266	458
46	755	280	475
47	771	310	461
48	816	312	504
49	858	318	540
50	901	324	577
51	937	338	599
52	982	344	638
53	1020	358	662
54	1057	374	683
55	1081	404	677
56	1130	410	720
57	1172	424	748
58	1215	438	777
59	1245	466	779
60	1290	480	810
61	1320	510	810
62	1351	540	811
63	1351	602	749
64	1414	602	812
65	1476	604	872
66	1539	606	933
67	1599	612	987
68	1664	614	1050

69	1726	620	1106
70	1789	626	1163
71	1845	640	1205
72	1914	642	1272
73	1980	648	1332
74	2047	654	1393
75	2107	668	1439
76	2176	674	1502
77	2238	688	1550
78	2301	702	1599
79	2349	732	1617
80	2426	734	1692
81	2500	740	1760
82	2575	746	1829
83	2643	760	1883
84	2720	766	1954
85	2790	780	2010
86	2861	794	2067
87	2917	824	2093
88	2998	830	2168
89	3073	844	2229
90	3148	858	2290
91	3208	888	2320
92	3285	902	2383
93	3347	932	2415

94	3410	962	2448
95	3442	1024	2418
96	3535	1026	2509
97	3625	1032	2593
98	3716	1038	2678
99	3800	1052	2748
100	3893	1058	2835

Table 5:

Based on the numerical evidence provided by table 5, we propose the following conjecture.

Conjecture 2.1. For $n \geq 4$, K_n is not a PCC-graph.

Acknowledgement

We wish to thank the referees for their thoughtful suggestions.

References

[1] I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, *Ars combin.*, **23** (1987) 201-207.

[2] David M. Burton, Elementary number theory, Tata McGraw Hill Edition, New Delhi (2010).

[3] J. A. Gallian, A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, **16** (2013) # Ds6.

[4] F. Harary, Graph theory, *Narosa Publishing house*, New Delhi (2001).

[5] S.M.Hegde and Sudhakar Shetty, Combinatorial Labelings Of Graphs, *Applied Mathematics E-Notes*, **6**(2006), 251-258.

[6] Karl Goldberg, Frank Thomson Leighton, Morris Newman, and Susan Lana Zuckerman, Tables of Binomial Coefficients and Sterling Numbers, *Journal of Research of the Notional Bureau of Standards - B. Mathematical Sciences*, Vol. BOB, No. **1**, January-March 1976.

[7] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966)*, Gordon and Breach, N. Y. and Dunod Paris (1967) 349-355.

[8] M. A. Seoud and A. E. I. Abdel Maqsoud, On cordial and balanced labelings of graphs, *J. Egyptian Math. Soc.,* **7**(1999) 127-135.

[9] S. C. Shee and Y. S. Ho, The cordiality of one-point union of n-copies of a graph, *Discrete Math.,* **117** (1993) 225-243.

(1, 2) Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627 412, India.

E-mail address: (1) ponrajmaths@gmail.com

E-mail address: (2) sathishrvss@gmail.com

(3) Department of Mathematics, Vel Tech Dr.R.R & Dr.S.R Technical University,, Chennai-600002, India.

E-mail address: (3) ramasamyams@veltechuniv.edu.in