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MODIFIED NEWTON TYPE METHODS WITH HIGHER ORDER CONVERGENCE

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Abstract. Inspired by a recent result of McDougall and Wortherspoon, we obtain new iterative methods for solving nonlinear equations. Also we derive certain hybrid methods these methods and the standard secant method. The resulting methods turn out to be of higher order of convergence and are more efficient than the existing ones. The methods are compared with some of the recent existing methods.

1. Introduction

Nonlinear equations are encountered quite often in all fields of science and engineering but solving such equations analytically is not always possible. In those situations where an analytic solution cannot be obtained or it is difficult to obtain, numerical iterative methods are used. Two classical and standard methods for solving nonlinear equations numerically are the Newton method and the secant method. If $f(x) = 0$ is the given nonlinear equation then the Newton method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.1}$$

and the secant method is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n). \tag{1.2}$$

It is known that, for a simple root, the order of convergence of Newton method is 2 while for secant method it is 1.618.

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During the last decades, tremendous methods have appeared for solving nonlinear equations, each one is better than the other in some or the other aspect. Some of them to mention can be found in the papers [2]-[7].

Very recently in [7], McDougall and Wortherspoon obtained a method with a slight modification in the standard Newton method and achieved order of convergence $1+\sqrt{2}$. Their method is the following:

If x_0 is the initial approximation, then

$$x_0^* = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{1.3}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'[\frac{1}{2}(x_0 + x_0^*)]}. \tag{1.4}$$

Subsequently for $n \geq 1$, the iterations can be obtained as

$$x_n^* = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1} + x_{n-1}^*)]} \tag{1.5}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]}. \tag{1.6}$$

The above method is a predictor-corrector type method. The predictor step is obtained just as the Newton step whereas in the corrector step, an arithmetic average is obtained between the previous two points and derivative is calculated at the average value.

As the first aim of the paper, we provide two variants of the method (1.3)-(1.6) by replacing the arithmetic average with geometric average and harmonic average. The corresponding methods are shown to be of order $1+\sqrt{2}$ each.

Next, we construct a hybrid method by combining the iterations of the resulting method with the secant method. We show that the corresponding method is of order 3.5615. The motivation of combining two methods comes from the previous works of [2]-[6], where the authors successfully obtained higher order of convergence. We also compare the efficiency of the method (1.3)-(1.6) with that of McDougall and Wortherspoon [7].

2. The Methods with Harmonic and Geometric Averages

To begin with, we suggest the following method as a variant of (1.3)-(1.6) by replacing the arithmetic average with the harmonic average:

$$x_0^* = x_0 \tag{2.1}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'\left(\dfrac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.2}$$

followed by (for $n \geq 1$)

$$x_n^* = x_n - \frac{f(x_n)}{f'\left(\dfrac{2x_{n-1}x_{n-1}^*}{x_{n-1}+x_{n-1}^*}\right)} \tag{2.3}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\dfrac{2x_nx_n^*}{x_n+x_n^*}\right)}. \tag{2.4}$$

The convergence of the method has been discussed in the following:

Theorem 2.1. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighbourhood of α. Then for solving the nonlinear equation $f(x) = 0$, the method* (2.1)-(2.4) *is convergent with order of convergence* $1 + \sqrt{2}$.

Proof. Denote $c_j = \frac{1}{j!}.\frac{f^j(\alpha)}{f'(\alpha)}, j = 2, 3, 4....$ It is standard to work out that the error equation in the Newton method (1.1) is given by

$$e_{n+1} = c_2e_n^2, \tag{2.5}$$

where e_n denotes the error in the iterate x_n and the terms with higher powers of e_n are ignored.

Let us now proceed with convergence analysis of the method (2.1)-(2.4). Let e_n and e_n^* denote the errors in the iterates x_n and x_n^* respectively. Then obviously $e_0^* = e_0$ and in the view of (2.5), the error equation for (2.2) is given by

$$e_1 = c_2e_0^2 \tag{2.6}$$

using which, Taylor series expansion and binomial expansion, the error equation for (2.3) with $n = 1$, i.e., for x_1^* is given by

$$
\begin{aligned}
e_1^* &= e_1 - \frac{e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)}{1 + 2c_2e_0 + 3c_3e_0^2 + O(e_0^3)} \\
&= e_1 - \left(e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)\right)\left(1 - 2c_2e_0 - 3c_3e_0^2 + 4c_2^2e_0^2 + O(e_0^3)\right) \\
&= 2c_2e_0e_1 \\
&= 2c_2^2e_0^3,
\end{aligned}
\tag{2.7}
$$

neglecting the higher powers of e_0.

Next, we find that

$$
\begin{aligned}
\frac{2x_1x_1^*}{x_1 + x_1^*} &= \frac{2(\alpha + e_1)(\alpha + e_1^*)}{(\alpha + e_1) + (\alpha + e_1^*)} \\
&= \left(\alpha + (e_1 + e_1^*) + \frac{e_1e_1^*}{\alpha}\right)\left(1 + \frac{e_1 + e_1^*}{2\alpha}\right)^{-1} \\
&= \alpha + \frac{e_1 + e_1^*}{2},
\end{aligned}
$$

neglecting the higher powers of e_1 and e_1^*. Therefore, the error equation for (2.4) with $n = 1$, i.e., x_2 can be obtained as follows:

$$
\begin{aligned}
e_2 &= e_1 - \frac{f(\alpha + e_1)}{f'\left(\alpha + \frac{e_1 + e_1^*}{2}\right)} \\
&= e_1 - \left(1 + c_2e_1^2 + c_3e_1^3\right)\left(1 + 2c_2(\frac{e_1 + e_1^*}{2}) + 3c_3(\frac{e_1 + e_1^*}{2})^2\right)^{-1} \\
&= c_2e_1e_1^* \\
&= 2c_2^4e_0^5
\end{aligned}
$$

by using (2.6) and (2.7). It can be shown, in general, that for $n \geq 2$, the errors in respectively x_n^* and x_n can be obtained recursively by the relations

$$e_n^* = c_2e_ne_{n-1}$$

and

$$e_{n+1} = c_2e_ne_n^*.$$

Using the above information, the errors at each stage in x_n^* and x_{n+1} are obtained and are tabulated below:

n	e_n	e_n^*
0	e_0	e_0
1	$c_2e_0^2$	$2c_2e_0^3$
2	$2c_2^4e_0^5$	$2c_2^6e_0^7$
3	$2^2c_2^{11}e_0^{12}$	$2^3c_2^{16}e_0^{17}$
4	$2^5c_2^{28}e_0^{29}$	$2^7c_2^{40}e_0^{41}$
5	$2^{12}c_2^{69}e_0^{70}$	$2^{17}c_2^{98}e_0^{99}$
$\vdots$	$\vdots$	$\vdots$

Note that, we obtain the same sequences $\{e_n\}$ and $\{e_n^*\}$ as obtained in [7]. Consequently, the method (2.1)-(2.4) is convergent with order of convergence $1+\sqrt{2}$. $\square$

Next, we propose the following method that involves geometric average:

$$x_0^* = x_0 \tag{2.8}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(\sqrt{x_0x_0^*})} = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.9}$$

followed by (for $n \geq 1$)

$$x_n^* = x_n - \frac{f(x_n)}{f'(\sqrt{x_{n-1}x_{n-1}^*})} \tag{2.10}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(\sqrt{x_nx_n^*})}. \tag{2.11}$$

We prove the following:

Theorem 2.2. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighbourhood of α. Then for solving nonlinear equation $f(x)=0$, the method* (2.8)-(2.11) *is convergent with order of convergence $1+\sqrt{2}$.*

Proof. As in the proof of Theorem 2.1, $e_0^* = e_0$ and $e_1 = c_2 e_0^2$. For $n = 1$, (2.10) becomes

$$x_1^* = x_1 - \frac{f(x_1)}{f'(x_0)},$$

which is exactly the same as obtained from (2.3) for $n = 1$. Therefore, the error e_1^* in (2.10) is as given by (2.7), i.e.,

$$e_1^* = 2c_2^3 e_0^3.$$

We now calculate the error in (2.11) for $n = 1$. We have

$$\begin{aligned} f'\left(\sqrt{x_1 x_1^*}\right) &= f'\left(\sqrt{(\alpha + e_1)(\alpha + e_1^*)}\right) \\ &= f'\left[\alpha\left(1 + \frac{e_1 + e_1^*}{\alpha} + \frac{e_1 e_1^*}{\alpha^2}\right)^{\frac{1}{2}}\right] \\ &= f'\left(\alpha + \frac{e_1 + e_1^*}{2}\right) \\ &= f'(\alpha)[1 + c_2(e_1 + e_1^*)] \end{aligned}$$

using the binomial expansion for fractions, Taylor's expansion and neglecting higher power terms of e_1 and e_1^*. Using this, the error e_2 in (2.11) can be calculated as

$$\begin{aligned} e_2 &= e_1 - (e_1 + c_2 e_1^2 + c_3 e_1^3)[1 + c_2(e_1 + e_1^*)]^{-1} \\ &= c_2 e_1 e_1^* \\ &= 2c_2^4 e_0^5. \end{aligned}$$

It can be shown, in general, that for $n \geq 2$, the errors e_n and e_n^* can be calculated recursively by the relations

$$e_n^* = c_2 e_n e_{n-1}$$

$$e_{n+1} = c_2 e_n e_n^*.$$

These relations are exactly the same as obtained in Theorem 2.1. Consequently, the method (2.8)-(2.11) is convergent with order of convergence $1 + \sqrt{2}$.

Remark 2.3. For any $n = 0, 1, 2, ...$, if $x_n x_n^* < 0$, then in the method (2.8)-(2.11), $\sqrt{x_n x_n^*}$ will not be real and hence the method will not proceed further. To avoid such situation, one has to be a little cautious. Although, the exact root of the given nonlinear equation is not known, but it is not difficult to know the sign of the root, e.g., one can plot the corresponding curve. In the case of positive root, if we start with a positive initial approximation x_0, then since the method is convergent, all iterates will be positive and there will be no negative product. The case of negative root can be handled similarly.

□

3. Hybrid Methods with Faster Convergence

In this section, we provide a method by combining the iterations of the method (1.3)-(1.6) with secant method and show that the order of convergence of the resulting method is more than $1+\sqrt{2}$. Precisely, we propose the following method:

If x_0 is the initial approximation, then

$$x_0^* = x_0 \tag{3.1}$$

$$x_0^{**} = x_0 - \frac{f(x_0)}{f'[\frac{1}{2}(x_0 + x_0^*)]} = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{3.2}$$

$$x_1 = x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**}) \tag{3.3}$$

followed by (for $n \geq 1$)

$$x_n^* = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1} + x_{n-1}^*)]} \tag{3.4}$$

$$x_n^{**} = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]} \tag{3.5}$$

$$x_{n+1} = x_n^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**}). \tag{3.6}$$

For convergence of this method, we prove the following:

Theorem 3.1. *Let f be a function having sufficient number of continuous derivatives in a neighbourhood of α which is a simple root of the equation $f(x) = 0$. Then the method* (3.1)-(3.6) *to approximate the root α is convergent with order of convergence* 3.5615.

Proof. On the lines of the proofs of Theorems 2.1 and 2.2 and also the error equation of the standard secant method, it can be shown that the errors e_n^*, e_n^{**} and e_n respectively in x_n^*, x_n^{**} and x_n in the method (3.1)-(3.6) satisfy the following recursion formula:

$$e_n^* = c_2 e_{n-1} e_n$$
$$e_n^{**} = c_2 e_n e_n^*$$
$$e_{n+1} = c_2 e_n^* e_n^{**}.$$

The corresponding errors at each stage in x_n^*, x_n^{**} and x_n are obtained and tabulated as follows:

n	e_n	e_n^*	e_n^{**}
0	e_0	e_0	$c_2e_0^2$
1	$c_2^2e_0^3$	$2c_2^3e_0^4$	$2c_2^6e_0^7$
2	$2^2c_2^{10}e_0^{11}$	$2^2c_2^{13}e_0^{14}$	$2^4c_2^{24}e_0^{25}$
3	$2^6c_2^{38}e_0^{39}$	$2^8c_2^{49}e_0^{50}$	$2^{14}c_2^{88}e_0^{89}$
4	$2^{22}c_2^{138}e_0^{139}$	$2^{28}c_2^{177}e_0^{178}$	$2^{50}c_2^{316}e_0^{317}$
5	$2^{78}c_2^{494}e_0^{495}$	$2^{100}c_2^{633}e_0^{634}$	$2^{178}c_2^{1128}e_0^{1129}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

We make the analysis of the table as done in [7]. Note that the powers of e_0 in the error at each iterate form the sequence

$$3,\ 11,\ 39,\ 139,\ 495,\ 1763,\ 6279,\ 22363,\ \ldots$$

and the sequence of their successive ratios is

$$\frac{11}{3},\ \frac{39}{11},\ \frac{139}{495},\ \frac{495}{139},\ \frac{1763}{495},\ \frac{6279}{1763},\ \frac{22363}{6279},\ \ldots$$

or

$$3.67,\ 3.5454,\ 3.5641,\ 3.5611,\ 3.5616,\ 3.5615,\ 3.5615,\ \ldots$$

This sequence approaches to a fixed number which approximately can be taken as 3.5615 which is the order of convergence of the method (3.1)-(3.6). □

Remark 3.2. Amalgamation of methods already exists in literature. For example, Kasturiarachi [6] amalgamated standard Newton and Secant methods, Jain [5] mixed iterations of Steffensen and Secant methods, Jain [2], [3] also mixed several methods with secant as well as with modified secant methods. It is noticed that whenever a method is combined with secant method, the order of convergence of the method gets increased by 1. In the present situation, the method (1.3)-(1.6) of McDougall and Wortherspoon [7] is of order $1+\sqrt{2}$ but the increase in our method (3.1)-(3.6) is more than 1 when combines with the secant method.

Remark 3.3. It is known that the efficiency of numerical method for solving a nonlinear equation is defined to be $p^{\frac{1}{\theta}}$, where p is the order of convergence of the method and θ is the number of functions evaluation per iteration. Note that the efficiency of Newton method is $(2)^{\frac{1}{2}} \approx 1.4142$, that of McDougall and Wortherspoon method is $(\sqrt{2}+1)^{\frac{1}{2}} \approx 1.5538$ while the present method has efficiency $(3.5615)^{\frac{1}{3}} \approx 1.5271$. Comparing our method with McDougall and Wortherspoon's method, we point out that the efficiency of our method is quite close to their method. Moreover, the order of convergence of our method is much more than that of McDougall and Wortherspoon's method.

It is natural to consider the variants of methods (3.1)-(3.6), where in (3.2) and (3.5), the arithmetic mean is replaced by harmonic mean as well geometric mean as done in methods (2.1)-(2.4) and (2.8)-(2.11), respectively. Precisely, with harmonic mean, we propose the following method:

$$x_0^* = x_0 \tag{3.7}$$

$$x_0^{**} = x_0 - \frac{f(x_0)}{f'\left(\dfrac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{3.8}$$

$$x_1 = x_0^{**} - \frac{x_0^{**}-x_0^*}{f(x_0^{**})-f(x_0^*)} f(x_0^{**}) \tag{3.9}$$

followed by (for $n \geq 1$)

$$x_n^* = x_n - \frac{f(x_n)}{f'\left(\dfrac{2x_{n-1}x_{n-1}^*}{x_{n-1}+x_{n-1}^*}\right)} \tag{3.10}$$

$$x_n^{**} = x_n - \frac{f(x_n)}{f'\left(\dfrac{2x_n x_n^*}{x_n+x_n^*}\right)} \tag{3.11}$$

$$x_{n+1} = x_n^{**} - \frac{x_n^{**}-x_n^*}{f(x_n^{**})-f(x_n^*)}f(x_n^{**}) \tag{3.12}$$

and with the geometric mean, we propose the following:

$$x_0^* = x_0 \tag{3.13}$$

$$x_0^{**} = x_0 - \frac{f(x_0)}{f'(\sqrt{x_0x_0^*})} = x_0 - \frac{f(x_0)}{f'(x0)} \tag{3.14}$$

$$x_1 = x_0^{**} - \frac{x_0^{**}-x_0^*}{f(x_0^{**})-f(x_0^*)}f(x_0^{**}) \tag{3.15}$$

followed by (for $n \geq 1$)

$$x_n^* = x_n - \frac{f(x_n)}{f'\left(\sqrt{x_{n-1}x_{n-1}^*}\right)} \tag{3.16}$$

$$x_n^{**} = x_n - \frac{f(x_n)}{f'\left(\sqrt{x_n x_n^*}\right)} \tag{3.17}$$

$$x_{n+1} = x_n^{**} - \frac{x_n^{**}-x_n^*}{f(x_n^{**})-f(x_n^*)}f(x_n^{**}). \tag{3.18}$$

Using the arguments as used in the proofs of Theorems 2.1, 2.2 and 3.1, the following result can be proved. We omit the details for conciseness.

Theorem 3.4. *Let f be a function having sufficient number of continuous derivatives in a neighbourhood of α which is a simple root of the equation $f(x) = 0$. Then the methods* (3.7)-(3.12) *as well as* (3.13)-(3.18) *to approximate the root α are convergent with order of convergence 3.5615.*

4. Algorithms and Numerical Examples

We give below an algorithm in order to implement the method (3.1)-(3.6):

Algorithm 4.1. *Step* 1 : For the given tolerance $\varepsilon > 0$ and iteration N, choose the initial approximation x_0 and set $n = 0$.

Step 2 : Follow the following sequence of expressions:

$$x_0^* = x_0$$

$$x_0^{**} = x_0 - \frac{f(x_0)}{f'[\frac{1}{2}(x_0 + x_0^*)]} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**}).$$

Step 3 : For $n = 1, 2, 3, \ldots$, calculate $x_2, x_3, x_4, \ldots$ by the following sequence of expressions:

$$x_n^* = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1} + x_{n-1}^*)]}$$

$$x_n^{**} = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]}$$

$$x_{n+1} = x_n^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**}).$$

Step 4 : Stop if either $|x_{n+1} - x_n| < \varepsilon$ or $n > N$.

Step 5 : Set $n = n + 1$ and repeat Step 3.

Example 4.2. We apply Algorithm 4.1 on the nonlinear equation

$$\cos x - xe^x + x^2 = 0.$$

This equation has a simple root in the interval $(0, 1)$. Taking initial approximation as $x_0 = 1$, Table 1 shows the iterations of McDougall-Wortherspoon method (1.3)-(1.6), a third order method and our method (3.1)-(3.6).

Table 1

n	McDougall-Wotherspoon Method	A third order method [8]	Present method (3.1)-(3.6)
1.	0.64132328499316349	0.64599588437664313	0.63915520442184104
2.	0.6391544362117092	0.63915411336536088	0.6391541004893474
3.	0.63915409833960735	0.63915408672427509	division by zero
4.	0.63915411809538092	0.63915409327226524	
5.	0.63915407824650872	0.63915409982025551	
6.	0.63915409800228429	0.63915410636824566	
7.	0.63915411775805786	0.63915411291623581	
8.	0.63915407790918577	0.63915408627515002	
9.	0.63915409766496134	division by zero	
10.	0.63915411742073491		
11.	0.63915407757186271		
12.	0.63915409732763828		
13.	0.63915411708341185		
14.	0.63915407723453976		
15.	0.63915409699031522		
16.	0.63915411674608891		
17.	0.63915407689721671		
18.	0.63915409665299228		
19.	0.63915411640876585		
20.	0.63915407655989365		

Example 4.3. We refer to the problem of "Solving a Crime" from [1]. The problem is of estimating the time of death of a person. It was noticed that the core temperatures of the corpse were 90°F and 85°F at 8 PM and 9 PM, respectively. Also, it was noticed that due to the faliur of air conditioner, the room temperature increased at the rate of 1°F per hour.

Using the Newton's Law of Cooling, the problem reduces to solving the equations

$$\big(18+\frac{1}{k}\big)e^{-k}-\frac{1}{k}-12=0 \tag{4.1}$$

and

$$\big(18+\frac{1}{k}\big)e^{-kt}+t-\frac{1}{k}-26.6=0 \tag{4.2}$$

simultaneously, where k denotes the constant of proportionality and t denotes the time. The equations (4.1) and (4.2) are nonlinear and so precise values of k and t is difficult to find. The author in [1] used secant method to solve (4.1) with initial interval $(0.1, 1)$. After six iterations, the approximate value of k was obtained as 0.337114. Using $k = 0.337114$ in (4.2) and using secant method again with initial interval $(-2, 0)$, after six iterations, t was found to be -1.130939 which means that the man would have been dead approximately 1 hour 8 minutes before 8 PM. In Tables 2 and 3, we demonstrate that if instead of secant method, we apply our method (3.1)-(3.6), then we require much less than six iterations to reach the same conclusion.

Table 2. To calculate value of k in (4.1)

n	Mcdougall-Wortherspoon Method	A third order method [8]	Present method (3.1)-(3.6)
1.	0.39424536527674747	0.28399501628622575	0.33729218050164789
2.	0.33712186735193811	0.33711307862711176	0.33711438414127259
3.	0.33711437423853269	0.33711439449543812	division by zero
4.	0.33711439130907683	division by zero	
5.	0.33711437857729748		
6.	0.33711439564784168		
7.	0.33711438291606233		
8.	0.33711437018428397		
9.	0.33711438725482684		
10.	0.33711437452304749		

Table 3. To calculate value of t in (4.2)

n	McDougall-Wortherspoon Method	A third order method [8]	Present method (3.1)-(3.6)
1.	-0.43229389880795427	-1.4680241724237642	-1.1259217033754243
2.	-1.1245149717844931	-1.1310367658010889	-1.1309393994384249
3.	-1.1309384372848865	-1.13093937858347	division by zero
4.	-1.1309393943474491	division by zero	
5.	-1.1309393977356448		
6.	-1.1309394011238405		
7.	-1.1309394045120362		
8.	-1.130939407900232		
9.	-1.1309394112884277		
10.	-1.1309394146766234		

Remark 4.4. The examples in support of methods (3.7)-(3.12) as well as (3.13)-(3.18) have also been tested and verified. For the conciseness, we avoid the details.

5. Conclusion

In this paper, we have studied the method (1.3)-(1.6) given by Mcdougall and Wortherspoon [7] which is of order $1+\sqrt{2}$. We have obtained some variant of this method by replacing the arithmetic average by geometric average as well as harmonic average with the same order order of convergence. Then we derived new hybrid methods by combining these methods with the secant method. It is shown that the resulting methods are of order 3.5615 and moreover the efficiency of these methods is comparable with that of the method of Mcdougall and Wortherspoon.

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