Jordan Journal of Mathematics and Statistics (JJMS) 8(3), 2015, pp 195 - 207

CHARACTERIZATIONS OF SOME NEW CLASSES OF FUZZY SETS IN GENERALIZED FUZZY TOPOLOGY

D. MANDAL [(1)], SUMITA DAS(BASU) [(2)] AND M. N. MUKHERJEE [(3)]

Abstract. In the present paper we introduce the concepts of maximal μf-open sets, minimal μf-closed sets, local minimal μf-open sets etc. in a generalized fuzzy topological space. We study their fundamental properties and discuss relations among these different μf-open like sets.

1. Introduction and Preliminaries

After the foundation of fuzzy sets by L. A. Zadeh [10], its multidirectional applications in different branches of modern science inspired Chang [1] to introduce the concept of fuzzy topology which is a generalization of classical set topology. Further generalization was contemplated by Chetty [2], who introduced generalized fuzzy topology. In this paper, we introduce a few new classes of fuzzy sets, termed maximal μf-open sets, minimal μf-closed sets in Section 2, and discuss their behaviors in different situations in a generalized fuzzy topological space. In Section 3, we define local minimal μf-open sets at some point x of a non-empty set X as well as at some fuzzy point x_λ defined on X. Several results are obtained while discussing their properties and inter-relations.

2000 *Mathematics Subject Classification.* Primary 54A40, secondary 54D10, 54D15.

Key words and phrases. Generalized fuzzy topology, maximal μf-open set, minimal μf-open set, local minimal μf-open set, fuzzy (μ, λ)-continuous function.

Received: Feb. 19, 2015 Accepted: Aug. 9, 2015 .

A *fuzzy set* A in X is characterized by a membership function in the sense of Zadeh [10]. The basic fuzzy sets are the *zero set*, the *whole set* and *the class of all fuzzy sets* in X, to be denoted by 0_X and 1_X and I^X respectively. According to Chetty [2], a subcollection μ of I^X is called a *generalized fuzzy topology* (GFT, for short) if $0_X \in \mu$ and μ is closed under arbitrary unions of the members of μ. The structure (X, μ), where X is a non-empty set and μ is a generalized fuzzy topology defined on X, is said to be a generalized fuzzy topological space(GFTS, for short). In what follows, by (X, μ) or simply X we will mean a GFTS. The members of μ are called *μf-open sets* and their complements are said to be *μf-closed sets*. For any $A \in I^X$, the *μ-closure* of A and *μ-interior* of A are denoted by $c_\mu(A)$ and $i_\mu(A)$ respectively and are defined by $c_\mu(A) = \bigwedge\{F : A \le F,\ F \text{ is } \mu f\text{-closed}\}$ and $i_\mu(A) = \bigvee\{V \in \mu : V \le A\}$. For any two fuzzy sets A, B in X, we write $A \le B$ if $A(x) \le B(x)$, for each $x \in X$ whereas if $A \le B$ and $A(x) \ne B(x)$ for some $x \in X$ we write $A < B$. The notation AqB means that A is *quasi-coincident* [7] with B, i.e., AqB, if $A(x) + B(x) > 1$ for some $x \in X$. The negation of this statement is denoted by $A\overline{q}B$. For a fuzzy set A in X, the support of A, denoted by $S(A)$, is defined by $S(A) = \{x \in X : A(x) > 0\}$[10]. The union $\bigvee A_\alpha$ and intersection $\bigwedge A_\alpha$ of a family $\{A_\alpha : \alpha \in \Lambda\}$ of fuzzy sets A_α are defined in the usual way (see [10]). A *fuzzy singleton or a fuzzy point* [7] with support x and value α $(0 < \alpha \le 1)$ is denoted by x_α. The *fuzzy complement* of a fuzzy set A in an GFTS X, is written as $1 - A$. For a crisp set A of X, χ_A will stand for the characteristic function of A, and the cardinality of any set Y will be denoted by $|Y|$.

2. Maximal μf-open sets and minimal μf-closed sets

In this section, we introduce and investigate maximal μf-open sets and minimal μf-closed sets.

Definition 2.1. Let (X, μ) be a GFTS. A fuzzy μ-open set A in X $(A \neq 1_X)$ is said to be a fuzzy maximal μ-open set (maximal μf-open set, for short) in X if for any $B \in \mu$, ($A \leq B \Rightarrow$ either $B = A$ or $B = 1_X$). The set of all maximal μf-open sets in (X, μ) is denoted by $\max(X, \mu)$.

Example 2.1. *Let $X = \{a, b, c\}$ and $\mu = \{0_X, A, B, A \vee B\}$ be a GFT on X, where $A(a) = 0.2$, $A(b) = 0.8$, $A(c) = 0.4$ and $B(a) = 0.6$, $B(b) = 0.5$, $B(c) = 0.3$. Here $A \vee B$ is a maximal μf-open set in (X, μ).*

Definition 2.2. A fuzzy μ-closed set F in a GFTS (X, μ) with $F \neq 0_X$, is called a fuzzy minimal μ-closed set or a minimal μf-closed set in X if there is no μf-closed set lying strictly between 0_X and F.

Example 2.2. *Let $X = \{a, b\}$ and $\mu = \{0_X, A, B, A \vee B\}$, where $A(a) = 0.3$, $A(b) = 0.5$; $B(a) = 0.2$ and $B(b) = 0.6$, be a GFT on X. It is clear that $(1 - A \vee B)$ is a minimal μf-closed set in (X, μ).*

By definition of maximal μf-open set, it is clear that maximal μf-open sets are all μf-open, although the converse is not true, in general. We show this by the following example:

Example 2.3. *Let (X, μ) be a GFTS, where $X = \{a, b\}$, $\mu = \{0_X, A, B, A \vee B\}$ such that $A(a) = 0.4$, $A(b) = 0.6$; $B(a) = 0.2$, $B(b) = 0.8$. Clearly A and B are both μf-open sets but they are not maximal μf-open*

The following result gives a relation between maximal μf-open set and minimal μf-closed set.

Theorem 2.1. *A non-null fuzzy set $U (\neq 1_X)$ in a GFTS (X, μ) is maximal μf-open iff $(1 - U)$ is minimal μf-closed.*

Proof. *Let U be a maximal μf-open set in (X, μ) and let F be a μf-closed set such*

that $F \leq (1-U)$. *Then* $U \leq (1-F) \in \mu$. *Now* U *being maximal* μf*-open,* $(1-F)$ *is either* U *or* 1_X.
If $(1-F) = U$ *then* $F = (1-U)$, *and if* $(1-F) = 1_X$ *then* $F = 0_X$. *Thus we conclude that* $(1-U)$ *is a minimal* μf*-closed set.*
Conversely, let B *be a minimal* μf*-closed set and* G *be any* μf*-open set such that* $(1-B) \leq G$. *Since* B *is minimal* μf*-closed,* $(1-G)$ *is either* 0_X *or* B. *Now* $(1-G) = 0_X \Rightarrow G = 1_X$, *and* $(1-G) = B \Rightarrow G = (1-B)$. *Thus* $(1-B)$ *is a maximal* μf*-open set.*

Theorem 2.2. *Let* (X, μ) *be a GFTS and* $A \in \max(X, \mu)$. *If* B *is a non-zero* μf*-open set such that* $A \wedge B = 0_X$, *then* $A = \chi_{S(A)}$ *and* $B = 1 - A$.
Proof. *Let* $y \in S(B)$. *Since* $A \wedge B = 0_X$, $A(y) = 0$. *Thus* $(A \vee B)(y) = B(y) \neq A(y)$. *Since* $A \in \max(X, \mu)$, $A \vee B = 1_X$. *Again since* $A \wedge B = 0_X$, *it follows that* $A = \chi_{S(A)}$ *and* $B = 1 - A$.

Theorem 2.3. *Let* (X, μ) *be a GFTS. If* $A = \chi_{S(A)} \in \max(X, \mu)$ *then either* $c_\mu(A) = A$ *or* $c_\mu(A) = 1_X$.
Proof. *If* $c_\mu(A) = 1_X$ *then there is nothing to prove. So let* $c_\mu(A) \neq 1_X$. *Then there exists* $y \in X$ *such that* $c_\mu(A)(y) < 1$. *Let* $B = 1 - c_\mu(A)$. *Then* $B \in \mu$, $B \neq 0_X$ *and* $A \wedge B = 0_X$. *Hence by Theorem 2.2,* $B = 1 - A$ *which implies that* $c_\mu(A) = A$.

Theorem 2.4. *Let* (X, μ) *be a GFTS and* $A \in \max(X, \mu)$. *If* B *is a non-null fuzzy set in* X *with* $B \leq 1 - A$ *then* $c_\mu(B) = 1 - A$.
Proof. *If possible, let* $c_\mu(B) \neq 1 - A$. *Since* $B \leq (1-A)$ *where* $A \in \mu$, *we have* $c_\mu(B) \leq (1-A)$. *Again since* $c_\mu(B) \neq (1-A)$, *it follows that there exists* $x_0 \in X$ *such that* $c_\mu(B)(x_0) < (1-A)(x_0)$. *Now,* $A \in \max(X, \mu)$ *and* $A(x_0) < 1 - c_\mu(B)(x_0) \Rightarrow (1 - c_\mu(B)) = 1_X \Rightarrow c_\mu(B) = 0_X$, *a contradiction.*

Theorem 2.5. *Let* (X, μ) *be a GFTS and* $A = \chi_{S(A)} \in \max(X, \mu)$. *If* B *is a* μf*-closed set in* X *such that* $A < B$, *then* $B = 1_X$.

Proof. *If possible, let there exist* $y \in X$ *such that* $B(y) < 1$. *Then* $[(1-B) \vee A](y) \neq 0 = A(y)$ (*since* $y \in X \setminus S(A)$ *as* $A < B$) $\Rightarrow (1-B) \vee A \neq A \Rightarrow (1-B) \vee A = 1_X$. *As* $A \neq 1_X$ *and* $A = \chi_{S(A)}$, $S(A) \neq X$. *Thus we can choose* $x_1 \in X \setminus S(A)$ *and then* $(1-B)(x_1) = 1$. *Since* $A < B$, $A(x_1) < B(x_1)$. *Now* $(1-B)(x_1) = 1$ $\Rightarrow B(x_1) = 0 > A(x_1)$ *which is a contradiction.*

Corollary 2.1. *Let* (X, μ) *be a GFTS and* $A = \chi_{S(A)} \in \max(X, \mu)$. *If* $B \in I^X$ *such that* $A < B$, *then* $c_\mu(B) = 1_X$.

Theorem 2.6. *Let* (X, μ) *be a GFTS and* $A \in I^X$ *such that* $A \neq (1-A)$. *Then the following are equivalent:*

(*a*) $\{A, (1-A)\} \subseteq \max(X, \mu)$.

(*b*) $A = \chi_{S(A)}$ *and* $\mu = \{0_X, A, (1-A), 1_X\}$.

Proof. **(a)** $\Rightarrow$ **(b) :** *Since* $A \neq (1-A)$, *we choose* $x_0 \in X$ *such that* $A(x_0) \neq (1-A)(x_0)$. *Then* $(A \vee (1-A))(x_0) \neq A(x_0)$ *or* $(A \vee (1-A))(x_0) \neq (1-A)(x_0)$. *So by* (*a*), $(A \vee (1-A)) = 1_X$. *Hence for every* $x \in X$ *with* $A(x) < 1$, *we must have* $(1-A)(x) = 1$ *and it follows that* $A(x) = 0 \Rightarrow A = \chi_{S(A)}$. *Next let* $B \in \mu \setminus \{0_X\}$. *If* $B \leq A$, *then* $(1-A) \wedge B = 0_X$ *and hence by Theorem 2.2,* $B = 1-(1-A) = A$. *If* $B \not\leq A$, *i.e., there exists* $x_0 \in X \setminus S(A)$ *such that* $B(x_0) > A(x_0) = 0$; *then* $A \vee B = 1_X$, *as* $A \in \max(X, \mu)$. *Thus* $B(x) = 1$, *for all* $x \in X \setminus S(A)$ ... (*i*)

Now three cases arise:

Case(*I*): *Let* $B(x) = 1$, *for all* $x \in S(A)$; *then* $B = 1_X$.

Case(*II*): *Let* $B(x) = 0$, *for all* $x \in S(A)$; *then* $B = 1-A$.

Case(*III*): *Let* $B(x) = t$, *where* $0 < t < 1$, *for some* $x \in S(A)$. *Then* $(1-A) \vee B > 1-A$ *and hence by maximality of* $(1-A)$, $(1-A) \vee B = 1_X$. *Thus* $B(x) = 1 = t$, *a contradiction. Hence case*(*III*) *is not tenable, so that* $B = 1_X$ *or* $1-A$.

(b) $\Rightarrow$ **(a):** *Obvious.*

Theorem 2.7. *Let (X, μ) be a GFTS. Then the following statements are true:*
(a) If U is a maximal μf-open set and V is any μf-open set then either $U \vee V = 1_X$ or $V \leq U$.
(b) For any two maximal μf-open sets U and V, either $U \vee V = 1_X$ or $U = V$.

Proof. (a) Here two cases arise:
Case-I: $U \vee V = 1_X$. In this case we get the result.
Case-II: $U \vee V \neq 1_X$. Then $U \vee V$ is a μf-open set for which $U \leq U \vee V$. But U being maximal μf-open, $U \vee V = U$ (since $U \vee V \neq 1_X$) $\Rightarrow V \leq U$.
(b) For two maximal μf-open sets U and V, either $U \vee V = 1_X$ or $U \vee V \neq 1_X$. If $U \vee V \neq 1_X$ then $U \vee V$ is a μf-open set such that $U, V \leq U \vee V \Rightarrow U \vee V = U = V$.

Corollary 2.2. *Let (X, μ) be a GFTS and $A \in \max(X, \mu)$ with $I(A) = \phi$, where $I(A) = \{x \in X : A(x) = 1\}$. Then for every $B \in \mu \setminus \{1_X\}$, $B \leq A$.*
Proof. Follows from Theorem 2.7(a).

Corollary 2.3. *Let (X, μ) be a GFTS and $A \in \max(X, \mu)$ with $I(A) = \phi$. Then $\max(X, \mu) = \{A\}$.*

Corollary 2.4. *Let (X, μ) be a GFTS. If $|\max(X, \mu)| > 1$, then for every $A \in \max(X, \mu)$, $I(A) \neq \phi$.*

Theorem 2.8. *Let (X, μ) be a GFTS. Then the following statements are true:*
(a) If F is a minimal μf-closed set and G is any μf-closed set then either $F \wedge G = 0_X$ or $F \leq G$.
(b) For any two minimal μf-closed sets F and G, either $F \wedge G = 0_X$ or $F = G$.
Proof. The proof is similar to that of Theorem 2.7.

Corollary 2.5. *Let (X, μ) be a GFTS in which U is a maximal μf-open set and x_λ a fuzzy point such that $x_\lambda q(1 - U)$. Then for any μf-open set V in X containing x_λ, $(1 - U) \leq V$.*

Proof. *Since* $x_\lambda q(1-U)$, *then* $x_\lambda \not\leq U$. *Thus for any* μf*-open set* V *in* X *containing* x_λ, $V \not\leq U$. *Hence by Theorem 2.7(a),* $U \vee V = 1_X \Rightarrow (1-U) \leq V$.

Corollary 2.6. *For any maximal* μf*-open set* U *in any GFTS* (X, μ), *only one of the following statements (a) and (b) holds:*

(a) For each fuzzy point x_λ *in* X, *if* $x_\lambda q(1-U)$ *then for each* μf*-open set* V *in* X *containing* x_λ, $V = 1_X$.

(b) There exists a fuzzy point x_λ *with* $x_\lambda q(1-U)$ *and there exists a* μf*-open set* V *in* X *containing* x_λ *such that* $(1-U) \leq V$ *and* $V \neq 1_X$.

Proof. *If (a) holds, then we are done. On the other hand, if (a) does not hold then there exist a fuzzy point* x_λ *in* X *and a* μf*-open set* V *containing* x_λ *such that* $x_\lambda q(1-U)$ *and* $V \neq 1_X$. *Clearly* $V < 1_X$. *Then by Theorem 2.7(a),* $U \vee V = 1_X$ *or* $V \leq U$. *But* $x_\lambda q(1-U) \Rightarrow V \not\leq U$. *Thus* $U \vee V = 1_X$ *and so* $(1-U) \leq V$.

Theorem 2.9. *Let* C *be a minimal* μ*-closed set in a GFTS* (X, μ) *and* x_λ *be any fuzzy point in* X *such that* $x_\lambda \leq C$. *Then*

(i) $C \leq F$ *for any* μf*-closed set* F *containing* x_λ.

(ii) $C = \bigwedge\{F : x_\lambda \leq F$ *and* F *is* μf*-closed* $\}$.

Proof. *(i) Let* $x_\lambda \leq C$ *and* F *be a* μf*-closed set such that* $x_\lambda \leq F$. *Then* $C \wedge F \neq 0_X$. *By Theorem 2.8(a),* $C \leq F$.

(ii) By (i) above, $C \leq \bigwedge\{F : x_\lambda \leq F$ *and* F *is* μf*-closed* $\}$. *On the other hand,* $y_\beta \leq \bigwedge\{F : x_\lambda \leq F$ *and* F *is* μf*-closed* $\} \Rightarrow y_\beta \leq F$, *for all* μf*-closed set* F *containing* $x_\lambda \Rightarrow y_\beta \leq C \Rightarrow \bigwedge\{F : x_\lambda \leq F$ *and* F *is* μf*-closed* $\} = C$.

Theorem 2.10. *Let* $\{F_\alpha : \alpha \in \Lambda\}$ *be a family of minimal* μf*-closed sets in a GFTS* (X, μ) *and* F *be a minimal* μf*-closed set in* X.

(a) If $F q \bigvee_{\alpha \in \Lambda} F_\alpha$, *then there exists some* $\alpha_0 \in \Lambda$ *such that* $F = F_{\alpha_0}$.

(b) If $F \neq F_\alpha$ *for any* $\alpha \in \Lambda$, *then* $(\bigvee_{\alpha \in \Lambda} F_\alpha) \wedge F = 0_X$.

Proof. (*a*) *We first show that* $F \wedge F_{\alpha_0} \neq 0_X$ *for at least one* $\alpha_0 \in \Lambda$. *If possible, let* $F \wedge F_\alpha = 0_X$ *for each* $\alpha \in \Lambda$. *Then* $F\overline{q}F_\alpha$ *for each* $\alpha \in \Lambda \Rightarrow F_\alpha \leq (1 - F)$ *for each* $\alpha \in \Lambda \Rightarrow \bigvee_{\alpha\in\Lambda} F_\alpha \leq (1 - F) \Rightarrow F\overline{q}(\bigvee_{\alpha\in\Lambda} F_\alpha)$, *a contradiction. Thus* $F \wedge F_{\alpha_0} \neq 0_X$ *for some* $\alpha_0 \in \Lambda$. *Since* F *and* F_{α_0}, *are both minimal* μf*-closed sets in* X, *by Theorem 2.8*(*b*), $F = F_{\alpha_0}$.
(*b*) *If possible, let* $(\bigvee_{\alpha\in\Lambda} F_\alpha) \wedge F \neq 0_X$. *Then there exists* $\alpha \in \Lambda$ *such that* $F_\alpha \wedge F \neq 0_X$. *Now by Theorem 2.8*(*b*), $F = F_\alpha$ *for that* α, *which contradicts our assumption.*

Let us recall the definitions of fuzzy (μ, λ)-continuous and fuzzy (μ, λ)-open functions which are defined in [5]. Our goal is to look for the behavior of maximal μf-open sets under these functions.

Definition 2.3. Let (X, μ) and (Y, λ) be two GFTS's. A mapping $f : (X, \mu) \to (Y, \lambda)$ is said to be
(*i*) fuzzy (μ, λ)-continuous if $f^{-1}(F)$ is μf-closed for every λf-closed set F in Y.
(*ii*) fuzzy (μ, λ)-open if for every μf-open set U in X, $f(U)$ is λf-open in Y.

Theorem 2.11. *Let* (X, μ) *and* (Y, λ) *be two GFTS's and* $f : (X, \mu) \to (Y, \lambda)$ *be a fuzzy* (μ, λ)*-continuous and fuzzy* (μ, λ)*-open surjection. If* $A \in \max(X, \mu)$ *then either* $f(A) = 1_Y$ *or* $f(A) \in \max(Y, \lambda)$.
Proof. If $f(A) = 1_Y$ *then there is nothing to prove. So let* $f(A) \neq 1_Y$. *Since* f *is fuzzy* (μ, λ)*-open and* A *is a* μf*-open set in* X, $f(A)$ *is* λf*-open in* Y. *Again since* $A \neq 0_X$, *there exists* $x_0 \in X$ *such that* $A(x_0) > 0$ *and so* $f(A)(f(x_0)) = \sup\{A(x) : f(x) = f(x_0)\} \geq A(x_0) > 0$ *and hence* $f(A) \neq 0_Y$. *Let* $B \in \lambda$ *such that* $f(A) < B$. *It is sufficient to show that* $B = 1_Y$. *Let us choose* $y_0 \in Y$ *such that* $f(A)(y_0) < B(y_0)$. *Since* f *is surjective, there exists* $x_0 \in X$ *such that* $f(x_0) = y_0$. *Thus* $A(x_0) \leq f(A)(y_0) < B(y_0)$. *Now* f *being fuzzy* (μ, λ)*-continuous,* $f^{-1}(B) \in \mu$. *Hence we get* $f^{-1}(B) \vee A \in \mu$, $A \leq f^{-1}(B) \vee A$ *and*

$(f^{-1}(B) \vee A)(x_0) = \max\{A(x_0), B(y_0)\} = B(y_0) > A(x_0)$. *Since* $A \in \max(X, \mu)$, $f^{-1}(B) \vee A = 1_X$. *Next let* $y \in Y$. *Then there exists* $x \in X$ *such that* $f(x) = y$. *Thus* $1 = \max\{A(x), B(y)\} \leq \max\{f(A)(y), B(y)\} = B(y)$ *and so* $B(y) = 1$. *Hence* $B = 1_Y$.

3. Local minimal μf-open sets

In this section, we develop the notion of locally minimal μf-open sets at some point of a non-empty set X as well as at some fuzzy point x_λ, defined on a GFTS X and study their basic properties.

Definition 3.1. Let (X, μ) be a GFTS, $x \in X$ and $A \in \mu$ such that $x \in S(A)$. Then A is called a locally minimal μf-open set at x if for each $B \in \mu$ with $x \in S(B)$ one has $A \leq B$. The set of all locally minimal μf-open sets at a point $x \in X$ is denoted by $\min(X, \mu, x)$.

Definition 3.2. Let (X, μ) be a GFTS, p_λ a fuzzy point in X and $A \in \mu$ such that $p_\lambda \leq A$. Then A is called a locally minimal μf-open set at p_λ if for each $B \in \mu$ with $p_\lambda \leq B$, $A \leq B$ holds. The set of all locally minimal μf-open sets at p_λ will be denoted by $\min(X, \mu, p_\lambda)$.

Example 3.1. *Let* $X = \{a, b, c\}$, $\mu = \{0_X, P, Q\}$ *be a GFT on* X, *where* $P(a) = 0.2$, $P(b) = 0.4$, $P(c) = 0.6$ *and* $Q(a) = 0.3$, $Q(b) = 0.6$, $Q(c) = 1$. *Here* $P \in \mu$ *is a locally minimal* μf*-open set at* $a \in X$. *Now we consider a fuzzy point* $c_{0.7}$ *in* X. *Then* $c_{0.7} \leq Q \in \mu$ *and clearly* Q *is a locally minimal* μf*-open set at* $c_{0.7}$.

Theorem 3.1. *Let* (X, μ) *be a GFTS,* $x \in X$ *and* p_λ *be any fuzzy point in* X. *Then* $|\min(X, \mu, x)| \leq 1$ *and* $|\min(X, \mu, p_\lambda)| \leq 1$.

Proof. *Let* A, $B \in \min(X, \mu, x)$. *Then by definition, we have* $A \leq B$ *and* $B \leq A$ *and hence* $A = B$. *Therefore* $|\min(X, \mu, x)| \leq 1$. *Similarly we can show that* $|\min(X, \mu, p_\lambda)| \leq 1$.

Theorem 3.2. *Let (X, μ) be a GFTS, $A \in \mu$ and $x \in X$. Then the following are equivalent:*

(a) $\min(X, \mu, x) = \{A\}$.

(b) *For each fuzzy point $x_\lambda \le A$, $\min(X, \mu, x_\lambda) = \{A\}$.*

Proof. **(a)⇒ (b):** *Let x_λ be a fuzzy point in X such that $x_\lambda \le A$. Let $B \in \mu$ such that $x_\lambda \le B$. Then $x \in S(B)$. Since $\min(X, \mu, x) = \{A\}$, we have $A \le B$. So $\min(X, \mu, x_\lambda) = \{A\}$.*

(b)⇒ (a): *Let $B \in \mu$ such that $x \in S(B)$. Let us consider a fuzzy point x_λ in X where $\lambda = \min\{\frac{A(x)}{2}, \frac{B(x)}{2}\}$. Then $x_\lambda \le A \wedge B$. Since by (b), $\min(X, \mu, x_\lambda) = \{A\}$, we have $A \le B$. So $\min(X, \mu, x) = \{A\}$.*

Theorem 3.3. *Let (X, μ) be a GFTS, $A \in \mu$ and p_λ be any fuzzy point in X with $p_\lambda \le A$. Then the following are equivalent:*

(a) $\min(X, \mu, p_\lambda) = \{A\}$.

(b) $\min(X, \mu, p_\beta) = \{A\}$, *for every fuzzy point $p_\beta \le A$ with $\lambda \le \beta$.*

Proof. **(a)⇒ (b):** *Suppose that p_β is a fuzzy point with $p_\beta \le A$ and $\lambda \le \beta$ hold. Let $B \in \mu$ with $p_\beta \le B$. Then $\beta \le B(p)$ and $\lambda \le \beta \le B(p) \Rightarrow p_\lambda \le B$. Then by (a), it follows that $A \le B$ and hence $\min(X, \mu, p_\beta) = \{A\}$.*

(b)⇒ (a): *Clear.*

Theorem 3.4. *Let (X, μ) be a GFTS and p_λ be any fuzzy point in X. Then the following are equivalent:*

(a) $\min(X, \mu, p_\lambda) \ne \phi$.

(b) $\bigwedge\{B \in \mu : p_\lambda \le B\} \in \mu$.

Proof. **(a)⇒ (b):** *In view of Theorem 3.1 and (a), we have $min(X, \mu, p_\lambda) = \{A\}$ for some fuzzy set A. Now, for each $B \in \mu$ with $p_\lambda \le B$ we have $A \le B$. Thus $A \le \bigwedge\{B \in \mu : p_\lambda \le B\}$. Also by definition of $min(X, \mu, p_\lambda)$, $p_\lambda \le A(\in \mu)$. Hence $\bigwedge\{B \in \mu : p_\lambda \le B\} \le A$. Therefore $\bigwedge\{B \in \mu : p_\lambda \le B\} = A \in \mu$.*

(b)⇒ (a): *Let* $A = \bigwedge\{B \in \mu : p_\lambda \leq B\}$. *Clearly,* $p_\lambda \leq A \in \mu$. *Also, for each* $B \in \mu$ *with* $p_\lambda \leq B$, *we have* $A \leq B$. *Thus* $min(X, \mu, p_\lambda) = \{A\} \neq \phi$.

Remark 1. The counterpart of the above theorem for the locally minimal μf-open set $min(X, \mu, x)$, viz ‘ for any $x \in X$, where (X, μ) is a GFTS, $min(X, \mu, x) \neq \phi$ iff $\bigwedge\{B \in \mu : x \in S(B)\} \in \mu$’ is false. In fact, let $X = \{a, b\}$ and $\mu = \{0_X, A_r : 0 < r \leq \frac{1}{3}\}$, where $A_r(a) = A_r(b) = r$. Then (X, μ) is a GFTS. It is easy to see that $\bigwedge\{B \in \mu : a \in S(B)\} = \bigwedge\{B \in \mu : b \in S(B)\} = 0_X \in \mu$, but $min(X, \mu, a) = min(X, \mu, b) = \phi$. The desired result for $min(X, \mu, x)$ corresponding to that in the above theorem goes as follows.

Theorem 3.5. *Let* (X, μ) *be a GFTS and* $x \in X$. *Then the following are equivalent:*
(*a*) $\min(X, \mu, x) \neq \phi$.
(*b*) $(\bigwedge\{B \in \mu : x \in S(B)\})(x) \neq 0$ *and* $\bigwedge\{B \in \mu : x \in S(B)\} \in \mu$.

Proof. **(a)⇒ (b):** *Suppose* $\min(X, \mu, x) = \{A\}$. *Then for each* $B \in \mu$ *with* $x \in S(B)$ *we have* $A \leq B$. *Thus* $A \leq \bigwedge\{B \in \mu : x \in S(B)\}$. *Again* $x \in S(A) \Rightarrow \bigwedge\{B \in \mu : x \in S(B)\} \leq A$. *Thus* $A = \bigwedge\{B \in \mu : x \in S(B)\}$. *Hence* (*b*) *follows as* $x \in S(A)$ *and* $A = \bigwedge\{B \in \mu : x \in S(B)\} \in \mu$.

(b)⇒ (a): *Let* $F = \bigwedge\{B \in \mu : x \in S(B)\}$. *Then by* (*b*), $F \in \mu$. *Also by the first condition of* (*b*), $x \in S(F)$. *Now for any* $G \in \mu$ *with* $x \in S(G)$, *we have* $F \leq G$. *Thus* $F \in \min(X, \mu, x)$ *and hence* $\min(X, \mu, x) \neq \phi$.

If a fuzzy set A is locally minimal at some point in a GFTS, then it is not true in general that each point of $S(A)$ must have a locally minimal μf-open set. We show this by the following example:

Example 3.2. *Let* (X, μ) *be a GFTS, where* $X = \{a, b\}$, *and* μ *consist of* 0_X, 1_X *and all those fuzzy sets* P *such that* $0 < P(a) \leq \frac{1}{2}$ *and* $P(b) = 0$. *Then* 1_X *is a locally minimal* μf*-open set at* b, *i.e.,* $\min(X, \mu, b) = \{1_X\}$. *But* $a \in S(1_X)$, $\min(X, \mu, a) = \phi$.

We have already defined minimal μf-closed set. In an analogous way we define minimal μf-open sets as follows:

Definition 3.3. A non-null μf-open set U in a GFTS (X, μ) is called a minimal μf-open set in X if there is no μf-open set strictly lying between 0_X and U.

Theorem 3.6. *Let (X, μ) be a GFTS, $A \in \mu$ and p_λ be any fuzzy point in X such that $p_\lambda \leq A$. Then the following are equivalent:*
(a) A is a minimal μf-open set in X, and $min(X, \mu, p_\lambda) \neq \phi$.
(b) $min(X, \mu, p_\lambda) = \{A\}$.

Proof. **(a)$\Rightarrow$ (b):** *Let A be a minimal μf-open set in X and $p_\lambda \leq A$. Since $min(X, \mu, p_\lambda) \neq \phi$, let $B \in min(X, \mu, p_\lambda)$. Then by definition of $min(X, \mu, p_\lambda)$, $B \leq A$. Again, A is minimal μf-open set in X implies $A = B$. Thus $min(X, \mu, p_\lambda) = \{A\}$.*
(b)$\Rightarrow$ (a): *Let us take $B \in \mu \setminus \{0_X\}$ such that $B \leq A$. Let us choose some $y \in X$ such that $B(y) > 0$. Then $A(y) \geq B(y) > 0$. We consider the fuzzy point y_α where $\alpha = \frac{B(y)}{2}$. Then $y_\alpha \leq A \wedge B$. Now by (b), $\min(X, \mu, y_\alpha) = \{A\} \Rightarrow A \leq B$. So $A = B$ and hence A is a minimal μf-open set in X.*
The second condition of (a) is clear from (b).

Remark 2. The implication '$(a) \Rightarrow (b)$' of the above theorem fails if the second condition of (a) is dropped. In fact, let $X = \{a, b, c\}$, $\mu = \{0_X, A, B, A \vee B\}$, where $A(a) = 0.1$, $A(b) = 0.4$, $A(c) = 0.2$; $B(a) = 0.2$, $B(b) = 0.3$ and $B(c) = 0.5$. Then both A and B are minimal μf-open sets in the GFTS (X, μ) and the fuzzy point $a_{0.1} \leq A \bigwedge B$. But $min(X, \mu, a_{0.1}) = \phi$ ($\neq \{A\}$ or $\{B\}$).

Acknowledgement

The authors are grateful to the referee for his/her meticulous reading of the manuscript and making critical comments, which have gone significantly towards marked improvement of the paper.

References

[1] C.L. Chang; *Fuzzy topological spaces*, J. Math. Anal. Appl. 24 (1968), 182-190.

[2] G. P. Chetty; *Generalized fuzzy topology*, Ital. J. Pure Appl. Math. 24(2008), 91-96.

[3] G. Choquet; *Sur les notions de filter et de grille*, C. R. Acad. Sci. Paris 224 (1947), 171-173.

[4] Á. Császár; *Generalized topology, generalized continuity*, Acta Math. Hangar.96(2002), 351-357.

[5] D. Mandal, M.N.Mukherjee; *Some classes of fuzzy sets in a generalized fuzzy topological spaces and certain unifications*, Annals of Fuzzy Mathematics and Informatics, 7(6)(2014), 949-957.

[6] Samer Al. Ghour; *Some generalizations of minimal fuzzy open sets*, Acta Math. Univ. Comenianac. LXXV(1)(2006), 107-117.

[7] Pao Ming Pu, Ying Ming Liu; *Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. 76 (1980), 571-599.

[8] Pao Ming Pu, Ying Ming Li; *Fuzzy topology II. Product and quotient spaces*, J. Math. Anal. Appl. 77 (1980), 20-37.

[9] C. K. Wong; *Fuzzy topology; product and quotient theorems*, J. Math. Anal. Appl. 45(1974), 512-521.

[10] L.A. Zadeh; *Fuzzy sets*, Inform. Control 8 (1965), 338-353.

(1) Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata-700 019, India

E-mail address: dmandal.cu@gmail.com

(2) Department of Mathematics, Sammilani Mahavidyalaya,, E. M. Bypass, Kolkata 700 075, India

E-mail address: das.sumita752@gmail.com

(3) Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata-700 019, India

E-mail address: mukherjeemn@yahoo.co.in