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THE NORM OF CERTAIN MATRIX OPERATORS ON NEW DIFFERENCE SEQUENCE SPACES

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ABSTRACT. The purpose of the present study is to introduce the sequence space

$$l_p(\Delta, E) = \left\{ x = (x_n)_{n=1}^{\infty} \ : \ \sum_{n=1}^{\infty} \left| x_{\min E_n} - x_{\min E_{n+1}} \right|^p < \infty \right\},$$

where $E = (E_n)$ is a partition of finite subsets of the positive integers and $p \geq 1$. The topological properties and inclusion relations of this space are studied. Moreover, the problem of finding the norm of certain matrix operators such as Copson and Hilbert from l_p into $l_p(\Delta, E)$ is investigated.

1. INTRODUCTION

Let ω denote the space of all real-valued sequences. Any vector subspace of ω is called a sequence space. Let $E = (E_n)$ be a partition of finite subsets of the positive integers such that

$$\max E_n < \min E_{n+1}, \tag{1.1}$$

for $n = 1, 2, \cdots$. We define the sequence spaces $l_p(E)$ by

$$l_p(E) = \left\{ x = (x_n) \in \omega \ : \ \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p < \infty \right\}, \qquad (1 \leq p < \infty),$$

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with the semi-norm $\|.\|_{p,E}$, which is defined the following way:

$$\|x\|_{p,E} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p \right)^{1/p}. \tag{1.2}$$

It is significant that in the special case $E_n = \{n\}$ for $n = 1, 2, \cdots$, we have $l_p(E) = l_p$ and $\|x\|_{p,E} = \|x\|_p$. For more details on the sequence space $l_p(E)$, the reader may refer to [5].

The idea of difference sequence spaces was introduced by Kizmaz [10]. The difference sequence space $l_p(\Delta)$ is defined by

$$l_p(\Delta) = \{x = (x_n) : \sum_{n=1}^{\infty} |x_n - x_{n+1}|^p < \infty\}, \qquad (1 \leq p < \infty),$$

with norm

$$\|x\|_{p,\Delta} = \left(\sum_{n=1}^{\infty} |x_n - x_{n+1}|^p \right)^{\frac{1}{p}}.$$

Let X, Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \{1, 2, \cdots\}$. We say that A defines a matrix mapping from X into Y, and denote by $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}_{n=1}^{\infty}$ exists and is in Y, where $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$ for $n = 1, 2, \cdots$.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined by

$$X_A = \{x = (x_n) \in \omega \ : \ Ax \in X\}, \tag{1.3}$$

which is a sequence space. The new sequence space X_A generated by the limitation matrix A from a sequence space X can be the expansion or the contraction and or the overlap of the original space X. A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. If A is triangle, then one can easily observe that the sequence spaces X_A and X are linearly isomorphic, i.e., $X_A \cong X$.

In the past, several authors studied matrix transformations on sequence spaces that are the matrix domains of triangle matrices in classical spaces l_p, l_∞, c and c_0. For instance, some matrix domains of the difference operator were studied in [1, 3, 4, 10, 12]. In these studies the matrix domains are obtained by triangle matrices, hence these spaces are normed sequence spaces. For more details on the domain of triangle matrices in some sequence spaces, the reader may refer to Chapter 4 of [2]. The matrix domains given in this paper specify by a certain non-triangle matrix, so we should not expect that related spaces are normed sequence spaces.

In the study, the normed sequence space $l_p(\Delta)$ is extended to semi-normed space $l_p(\Delta, E)$. We investigate some topological properties of this space and derive inclusion relations concerning with its. Moreover, we shall consider the inequality of the form

$$\|Ax\|_{p,\Delta,E} \leq U\|x\|_p,$$

for all sequence $x \in l_p$. The constant U is not depending on x, and we seek the smallest possible value of U. We write $\|A\|_{p,\Delta,E}$ for the norm of A as an operator from l_p into $l_p(\Delta, E)$, and $\|A\|_{p,\Delta}$ for the norm of A as an operator from l_p into $l_p(\Delta)$. More recently, the problem of finding the upper bound of certain matrix operators on the sequence spaces $l_p(w)$, $d(w,p)$ and $l_p(\Delta)$ are studied in [6, 9, 11]. In the study, we examine this problem for matrix operators from l_p into $l_p(\Delta, E)$ and we consider certain matrix operators such as Copson and Hilbert.

In a similar way, the Authors have introduced the sequence space $l_p(E, \Delta)$ and investigated the norm of certain matrix operators on this space [7].

2. The sequence space $l_p(\Delta, E)$ of non-absolute type

Suppose $p \geq 1$ and $E = (E_n)$ is a sequence of finite subsets of the positive integers that satisfies condition (1.1). We define the sequence spaces $l_p(\Delta, E)$ by

$$l_p(\Delta, E) = \left\{ x = (x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} \left| x_{\min E_n} - x_{\min E_{n+1}} \right|^p < \infty \right\},$$

with the semi-norm $\|.\|_{p,\Delta,E}$, which is defined the following way:

$$\|x\|_{p,\Delta,E} = \left(\sum_{n=1}^{\infty} \left| x_{\min E_n} - x_{\min E_{n+1}} \right|^p \right)^{1/p}. \tag{2.1}$$

It should be noted that the function $\|.\|_{p,\Delta,E}$ cannot be the norm, since if $x = (1,1,1,\cdots)$ and $E_n = \{2n-1, 2n\}$ for all n, then $\|x\|_{p,\Delta,E} = 0$ while $x \neq 0$. It is also significant that in the special case $E_n = \{n\}$ for $n = 1,2,\cdots$, we have $l_p(\Delta, E) = l_p(\Delta)$ and $\|x\|_{p,\Delta,E} = \|x\|_{p,\Delta}$.

If the infinite matrix $A = (a_{nk})$ is defined by

$$a_{nk} = \begin{cases} 1 & if\ k = \min E_n \\ -1 & if\ k = \min E_{n+1} \\ 0 & otherwise, \end{cases} \tag{2.2}$$

with the notation of (1.3), we can redefine the space $l_p(\Delta, E)$ as follows:

$$l_p(\Delta, E) = (l_p)_A.$$

Throughout this article, the cardinal number of the set E_k is denoted by $|E_k|$. The purpose of this section is to consider some properties of the sequence space $l_p(\Delta, E)$ and is to derive some inclusion relations related to them. Now, we may begin with the following theorem which is essential in the study.

Theorem 2.1. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies condition (1.1). The set $l_p(\Delta, E)$ becomes a vector space with coordinatewise addition and scalar multiplication, which is the complete semi-normed space by $\|.\|_{p,\Delta,E}$ defined by (2.1).*

Proof. This is a routine verification and so we omit the detail. □

It can easily be checked that the absolute property does not hold on the space $l_p(\Delta, E)$, that is $\|x\|_{p,\Delta,E} \neq \||x|\|_{p,\Delta,E}$ for at least one sequence in the space $l_p(\Delta, E)$, and this says that $l_p(\Delta, E)$ is a sequence space of nonabsolute type, where $|x| = (|x_k|)$. Suppose that $E_n = \{2n-1, 2n\}$ for $n = 1, 2, ...$ If $x = (1, 0, 1, 0, \cdots)$ and $y = (1, -1, 0, 0, 1, -1, 0, 0, \cdots)$, then $x \in l_p(\Delta, E) - l_p(E)$ and $y \in l_p(E) - l_p(\Delta, E)$. So neither of the spaces $l_p(E)$ and $l_p(\Delta, E)$ include the other one.

Theorem 2.2. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies condition (1.1). We have the following statements:*
(i) If $M_1 = \{x = (x_n) \ : \ x_n = x_{n+1}, \ \forall n\}$, then $l_p(\Delta, E)/M_1 \simeq l_p(E)$.
(ii) If $M_2 = \{x = (x_n) \ : \ x_{\min E_n} = x_{\min E_{n+1}}, \ \forall n\}$, then $l_p(\Delta, E)/M_2 \simeq l_p$.

Proof. (i) If the map $T : l_p(\Delta, E) \longrightarrow l_p(E)$ is defined by $(Tx)_n = x_n - x_{n+1}$ for all $x \in l_p(\Delta, E)$ and for all n, then T is well-defined and linear. Let $y \in l_p(E)$, we define the sequence $x = (x_k)$ by

$$x = (0, -y_1, -y_1 - y_2, -y_1 - y_2 - y_3, \cdots,).$$

It is clear that $x \in l_p(\Delta, E)$ and $Tx = y$, so the map T is surjective. The remaining proof of part (i) is obvious.
(ii) Consider the map $T : l_p(\Delta, E) \longrightarrow l_p$ defined by $(Tx)_n = x_{\min E_n} - x_{\min E_{n+1}}$ for all $x \in l_p(\Delta, E)$ and for all n. T is well-defined and linearity of T is trivial. Let $y \in l_p$, we define the sequence $x = (x_k)$ by

$$x = (\underbrace{0, 0, \cdots, 0}_{E_1}, \underbrace{-y_1, 0, \cdots, 0}_{E_2}, \underbrace{-y_1 - y_2, 0, \cdots, 0}_{E_3}, \cdots).$$

It is clear that $x \in l_p(\Delta, E)$ and $Tx = y$, so the map T is surjective. By applying the first isomorphism theorem we deduce the desired result. □

One may expect a similar result for the space $l_p(\Delta, E)$ as was observed for the space l_p, and ask the following natural question: Is the space $l_p(\Delta, E)$ a semi-inner product space for $p = 2$? The answer is positive and is given by the following theorem:

Theorem 2.3. *Except the case p =2, the space $l_p(\Delta, E)$ is not a semi-inner product space.*

Proof. If we define $< x, y >= \sum_{n=1}^{\infty}(x_{\min E_n} - x_{\min E_{n+1}})(y_{\min E_n} - y_{\min E_{n+1}})$, then it is a semi-inner product on the space $l_2(\Delta, E)$ and $\|x\|^2_{2,\Delta,E} =< x, x >$. Now consider the sequences x and y such that

$$x = (\underbrace{1, 0, \cdots, 0}_{E_1}, \underbrace{1, 0, \cdots, 0}_{E_2}, 0, 0, \cdots)$$
$$y = (\underbrace{1, 1, \cdots, 1}_{E_1}, 0, 0, \cdots)$$

we see that

$$\|x + y\|^2_{p,\Delta,E} + \|x - y\|^2_{p,\Delta,E} \neq 2 \left(\|x\|^2_{p,\Delta,E} + \|y\|^2_{p,\Delta,E}\right) \quad (p \neq 2).$$

Since the equation $2 = 2^{\frac{2}{p}}$ has only one root $p = 2$, the semi-norm of the space $l_p(\Delta, E)$ does not satisfy the parallelogram equality, which means that the semi-norm cannot be obtained from the semi-inner product. Hence the space $l_p(\Delta, E)$ with $p \neq 2$ is not a semi-inner product space. □

Let X be a semi-normed space with a semi-norm g. A sequence (b_n) of elements of the semi-normed space X is called a Schauder basis (or briefly a basis) for X iff, for each $x \in X$ there exists a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} g\left(x - \sum_{k=1}^{n} \alpha_k b_k\right) = 0.$$

The series $\sum_{k=1}^{\infty} \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum_{k=1}^{\infty} \alpha_k b_k$.
In the following, we will give a sequence of the points of the space $l_p(\Delta, E)$ which forms a basis for the space $l_p(\Delta, E)$. Here and in the sequel, we shall use the convention that any term with a zero subscript is equal to zero.

Theorem 2.4. *If the sequence* $b^{(k)} = \{b_j^{(k)}\}_{j=1}^{\infty}$ *is defined such that*

$$b_j^{(k)} = \begin{cases} 0 & for\ j < \min E_k \\ 1 & for\ j \geq \min E_k. \end{cases}$$

Then the sequence $\{b^{(k)}\}_{k=1}^{\infty}$ *is a basis for the space* $l_p(\Delta, E)$*, and any* $x \in l_p(\Delta, E)$ *has a unique representation of the form*

$$x = \sum_{k=1}^{\infty} \alpha_k b^{(k)},$$

where $\alpha_k = x_{\min E_k} - x_{\min E_{k-1}}$ *for* $k = 1, 2, \cdots$.

Proof. This is a routine verification and so we omit the detail. □

Definition 2.1. Let $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies condition (1.1), and $s = (s_n)$ be a strictly increasing sequence of the positive integers. The generated partition $H = (H_n)$ is defined by E and s, as follows

$$H_n = \cup_{j=s_{n-1}+1}^{s_n} E_j,$$

for $n = 1, 2, \cdots$.

Note that any arbitrary partition $H = (H_n)$ that satisfies condition (1.1) generated by the partition $E = (E_n)$ and the sequence $s = (s_n)$, where $E_n = \{n\}$ and $s_n = \max H_n$ for all n. It is also important to know $s_n - s_{n-1} = |H_n|$.
In the following, the inclusion relation between the spaces $l_p(\Delta, E)$ and $l_p(\Delta, H)$ is examined. Obviously if $s_n - s_{n-1} > 1$ only for a finite number of n, then

$$l_p(\Delta, E) = l_p(\Delta, H).$$

Theorem 2.5. *Let* E*,* s *and* H *be as in Definition 2.1, and*

$$\sup_n (s_n - s_{n-1}) < \infty. \tag{2.3}$$

We have $l_p(\Delta, E) \subset l_p(\Delta, H)$ for all $p \geq 1$. Moreover if $s_n - s_{n-1} > 1$ for an infinite number of n, then these inclusions are strict.

Proof. Let $p \geq 1$ and $\zeta = \sup_n (s_n - s_{n-1})$. To prove the validity of the inclusion $l_p(\Delta, E) \subset l_p(\Delta, H)$, we show that

$$\|x\|_{p,\Delta,H} \leq \zeta^{\frac{p-1}{p}} \|x\|_{p,\Delta,E},$$

for each $x \in l_p(\Delta, E)$. Note that $\zeta = 1$, when $p = 1$. Suppose that $x = (x_n) \in l_p(\Delta, E)$ is an arbitrary sequence. By using Definition 2.5, we have $\min H_n = \min E_{s_{n-1}+1}$ and $\min H_{n+1} = \min E_{s_n+1}$, so

$$\begin{aligned} x_{\min H_n} - x_{\min H_{n+1}} &= x_{\min E_{s_{n-1}+1}} - x_{\min E_{s_n+1}} \\ &= \sum_{k=s_{n-1}+1}^{s_n} (x_{\min E_k} - x_{\min E_{k+1}}). \end{aligned} \tag{2.4}$$

By apply the triangular inequality and Hölder's inequality, we deduce that

$$\begin{aligned} \left| \sum_{k=s_{n-1}+1}^{s_n} (x_{\min E_k} - x_{\min E_{k+1}}) \right| &\leq \sum_{k=s_{n-1}+1}^{s_n} |x_{\min E_k} - x_{\min E_{k+1}}| \\ &\leq \left(\sum_{k=s_{n-1}+1}^{s_n} 1^{p^*} \right)^{1/p^*} \left(\sum_{k=s_{n-1}+1}^{s_n} \left| x_{\min E_k} - x_{\min E_{k+1}} \right|^p \right)^{1/p} \\ &= (s_n - s_{n-1})^{1/p^*} \left(\sum_{k=s_{n-1}+1}^{s_n} \left| x_{\min E_k} - x_{\min E_{k+1}} \right|^p \right)^{1/p}, \end{aligned} \tag{2.5}$$

where $p^* = p/(p-1)$. The relations (2.4) and (2.5) imply that

$$\left| x_{\min H_n} - x_{\min H_{n+1}} \right|^p \leq (s_n - s_{n-1})^{p-1} \sum_{k=s_{n-1}+1}^{s_n} \left| x_{\min E_k} - x_{\min E_{k+1}} \right|^p,$$

so

$$\begin{aligned}\sum_{n=1}^{\infty}\left|x_{\min H_n}-x_{\min H_{n+1}}\right|^p &\leq \sum_{n=1}^{\infty}(s_n-s_{n-1})^{p-1}\sum_{k=s_{n-1}+1}^{s_n}\left|x_{\min E_k}-x_{\min E_{k+1}}\right|^p\\ &\leq \sup_n (s_n-s_{n-1})^{p-1}\sum_{n=1}^{\infty}\sum_{k=s_{n-1}+1}^{s_n}\left|x_{\min E_k}-x_{\min E_{k+1}}\right|^p\\ &= \zeta^{p-1}\sum_{n=1}^{\infty}\left|x_{\min E_n}-x_{\min E_{n+1}}\right|^p .\end{aligned}$$

This means that

$$\|x\|_{p,\Delta,H}^p \leq \zeta^{p-1}\|x\|_{p,\Delta,E}^p.$$

Moreover, let $s_n - s_{n-1} > 1$ for an infinite number of n. There is a sequence (n_j) which $s_{n_j} - s_{n_{j-1}} > 1$ for $j = 1, 2, \cdots$. We define the sequence $x = (x_k)$ such that

$$x_{\min E_k} - x_{\min E_{k+1}} = \begin{cases} 1 & if\ k = s_{n_{j-1}} + 1 \\ -1 & if\ k = s_{n_{j-1}} + 2 \\ 0 & otherwise, \end{cases} \tag{2.6}$$

for $k = 1, 2, \cdots$. It is obvious that $x \in l_p(\Delta, H) - l_p(\Delta, E)$, and the inclusion $l_p(\Delta, E) \subset l_p(\Delta, H)$ strictly holds. □

Corollary 2.1. *Let $H = (H_n)$ be a partition of finite subsets of the positive integers that satisfies condition (1.1). If $p \geq 1$ and $\sup_n |H_n| < \infty$, then we have*
(i) $l_p(\Delta) \subset l_p(\Delta, H)$
(ii) $l_p \subset l_p(\Delta, H)$
Moreover if $|H_n| > 1$ for an infinite number of n, then these inclusion are strict.

Proof. (i) If $E_n = \{n\}$ and $s_n = \min H_n$ for all n, then the partition $H = (H_n)$ is generated by $E = (E_n)$ and $s = (s_n)$. The desired result follows from Theorem 2.5.
(ii) Since $l_p \subset l_p(\Delta)$ the proof will be finished by part (i). □

Corollary 2.2. *Let $p \geq 1$, and M and N be two positive integers. If we put $E_i = \{Mi - M + 1, Mi - M + 2, \cdots, Mi\}$ and $H_i = \{MNi - MN + 1, MNi - MN + 2, \cdots, MNi\}$ for all i, then $l_p(\Delta, E) \subset l_p(\Delta, H)$. Moreover if $N > 1$, then this inclusion strictly holds.*

Proof. If $s_i = Ni$ for all i, then the partition $H = (H_n)$ is generated by E and s. The desired result follows from Theorem 2.5. □

In the following, we consider the necessity of the condition (2.3) in Theorem (2.5).

Theorem 2.6. *Let E, s and H be as in Definition 2.1. If $p > 1$ and*

$$\sup_n (s_n - s_{n-1}) = \infty, \tag{2.7}$$

then neither of the spaces $l_p(\Delta, E)$ and $l_p(\Delta, H)$ includes the other one.

Proof. There exists a sequence $\{n_k\}$ such that $s_{n_k} - s_{n_k-1} \geq k$, by (2.7). Consider the sequence $y = (y_i)$ such that

$$y_{\min E_i} - y_{\min E_{i+1}} = \frac{1}{(s_{n_k} - s_{n_k-1})\, k^{1/p}},$$

for $s_{n_k-1} + 1 \leq i \leq s_{n_k}$. We conclude that $y \in l_p(\Delta, E) - l_p(\Delta, H)$. Also if the sequence $x = (x_n)$ is defined as (2.6), then $x \in l_p(\Delta, H) - l_p(\Delta, E)$. This completes the proof of the theorem. □

Corollary 2.3. *Let $H = (H_n)$ be a sequence of finite subsets of the positive integers that satisfies condition (1.1). If $p > 1$ and*

$$\sup_n |H_n| = \infty,$$

then neither of the spaces $l_p(\Delta, H)$ and $l_p(\Delta)$ includes the other one.

Proof. If $E_n = \{n\}$ and $s_n = \max H_n$ for all n, the desired result follows from Theorem 2.6. □

Theorem 2.7. *If $1 \leq p < s$, then $l_p(\Delta, H) \subset l_s(\Delta, H)$.*

Proof. Let $x \in l_p(\Delta, H)$, we have $\left(x_{\min H_n} - x_{\min H_{n+1}}\right)_{n=1}^{\infty} \in l_p$. So $\left(x_{\min H_n} - x_{\min H_{n+1}}\right)_{n=1}^{\infty} \in l_s$, by the inclusion $l_p \subset l_s$. Hence $x \in l_s(\Delta, H)$, this finishes the proof of the theorem. □

3. The norm of matrix operators from l_p into $l_p(\Delta, E)$

In this section the problem of finding the norm of certain matrix operators such as Copson and Hilbert from l_p into $l_p(\Delta, E)$ are considered, where $p \geq 1$. At the beginning, we tend to compute the norm of operators from l_1 into $l_1(\Delta, E)$.

Theorem 3.1. *Let $A = (a_{n,k})$ be a matrix operator and $E = (E_n)$ be a partition that satisfies condition (1.1). If*

$$M = \sup_k \sum_{n=1}^{\infty} \left|a_{\min E_n,k} - a_{\min E_{n+1},k}\right| < \infty,$$

then A is a bounded operator from l_1 into $l_1(\Delta, E)$ and $\|A\|_{1,\Delta,E} = M$. In particular if $a_{\min E_n,k} \geq a_{\min E_{n+1},k}$ for all n, k, then

$$\|A\|_{1,\Delta,E} = \sup_k |a_{1,k}| \,.$$

Proof. Let (x_n) be in l_1 and $u_k = \sum_{n=1}^{\infty} \left|a_{\min E_n,k} - a_{\min E_{n+1},k}\right|$ for all k. We have

$$\begin{aligned}
\|Ax\|_{1,\Delta,E} &\leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left|a_{\min E_n,k} - a_{\min E_{n+1},k}\right| |x_k| \\
&= \sum_{k=1}^{\infty} u_k \, |x_k| \\
&\leq M\|x\|_1.
\end{aligned}$$

which says that $\|A\|_{1,\Delta,E} \leqslant M$. Conversely, we take $x = e_n$ which e_n denotes the sequence having 1 in place n and 0 elsewhere, then $\|x\|_1 = 1$ and $\|Ax\|_{1,\Delta,E} = u_n$ which proves that $\|A\|_{1,\Delta,E} = M$. □

In the sequel we will compute the norms of Copson and Hilbert operators from sequence space l_1 into $l_1(\Delta, E)$.
The Copson operator C is defined by $y = Cx$, where

$$y_n = \sum_{k=n}^{\infty} \frac{x_k}{k}, \qquad (\forall n).$$

It is given by the Copson matrix:

$$c_{n,k} = \begin{cases} \frac{1}{k} & for\ n \leq k \\ 0 & for\ n > k. \end{cases}$$

Corollary 3.1. *Let C be the Copson operator and $E = (E_n)$ be a partition that satisfies condition (1.1), then C is a bounded operator from l_1 into $l_1(\Delta, E)$ and $\|C\|_{1,\Delta,E} = 1$.*

Proof. Since $M = \sup_k c_{1,k} = c_{1,1} = 1$, we obtain the desired result from Theorem 3.1. □

Corollary 3.2. *Suppose that C is the Copson operator and $E_n = \{n\}$ for all n, then C is a bounded operator from l_1 into $l_1(\Delta)$ and $\|C\|_{1,\Delta} = 1$.*

We recall the Hilbert operator H which is defined by the matrix:

$$h_{n,k} = \frac{1}{n+k}, \qquad (n, k = 1, 2, \cdots).$$

Corollary 3.3. *Let H be the Hilbert operator and $E = (E_n)$ be a partition that satisfies condition (1.1), then H is a bounded operator from l_1 into $l_1(\Delta, E)$ and*

$$\|H\|_{1,\Delta,E} = \frac{1}{2}.$$

Proof. According to the above notation $M = \sup_k h_{1,k} = \frac{1}{2}$, so $\|H\|_{1,\Delta,E} = \frac{1}{2}$ □

Corollary 3.4. *If H is the Hilbert operator, then H is a bounded operator from l_1 into $l_1(\Delta)$ and $\|H\|_{1,\Delta} = \frac{1}{2}$.*

Proof. By letting $E_n = \{n\}$ in Theorem 3.1, the proof is obvious. □

In the following, the problem of finding the norm of certain matrix operators such as Copson and Hilbert from l_p into $l_p(\Delta, E)$ are investigated for $p > 1$. For this purpose, we give the Schur's Theorem and a lemma which are needed to prove our main results.

Theorem 3.2. *([8], Theorem 275) Let $p > 1$ and $B = (b_{n,k})$ be a matrix operator with $b_{n,k} \geq 0$ for all n, k. Suppose that K, R are two strictly positive numbers such that*

$$\sum_{n=1}^{\infty} b_{n,k} \leq K \quad for\ all\ k, \qquad\qquad \sum_{k=1}^{\infty} b_{n,k} \leq R \quad for\ all\ n,$$

(bounds for column and row sums respectively). Then $\|B\|_p \leq R^{(p-1)/p} K^{1/p}$.

Lemma 3.1. *If $A = (a_{n,k})$ and $B = (b_{n,k})$ are two matrix operators such that $b_{n,k} = a_{\min E_n,k} - a_{\min E_{n+1},k}$, then*

$$\|A\|_{p,\Delta,E} = \|B\|_p.$$

Hence, if B is a bounded operator on l_p, then A will be a bounded operator from l_p into $l_p(\Delta, E)$.

Now we are ready to compute the norm of the Copson matrix operator when $p > 1$.

Theorem 3.3. *Suppose that $p > 1$ and N is a positive integer and $E_n = \{nN - N + 1, nN - N + 2, \cdots, nN\}$ for all n. If C is the Copson matrix operator, then it is a bounded operator from l_p into $l_p(\Delta, E)$ and*

$$\|C\|_{p,\Delta,E} \leq (1 + \frac{1}{2} + \cdots + \frac{1}{N})^{\frac{p-1}{p}}.$$

In particular if $N = 1$, then we have $\|C\|_{p,\Delta,E} = 1$.

Proof. By applying Lemma 3.1 we have $\|C\|_{p,\Delta,E} = \|B\|_p$, where $b_{n,k} = c_{\min E_n,k} - c_{\min E_{n+1},k}$. Let $C_n = \sum_{k=1}^{\infty} b_{k,n}$ and $R_n = \sum_{k=1}^{\infty} b_{n,k}$ for all n, by a simple calculation we deduce that $R_n \leq R_1$ and $C_n \leq 1$ for all n. Since

$$b_{1,k} = c_{1,k} - c_{N+1,k} = \begin{cases} \frac{1}{k} & for\ k \leq N \\ 0 & for\ k > N, \end{cases}$$

and $R_1 = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$, by using Theorem 3.2, we conclude that $\|C\|_{p,\Delta,E} \leq R_1{}^{(p-1)/p}$. In particular if $N = 1$, then $R_1 = 1$ so $\|C\|_{p,\Delta,E} \leq 1$. Now let $x = e_1$, we have $Cx = x$ and this completes the proof of the theorem. □

Finally, we try to solve the problem of finding the norm of the Hilbert matrix operator for $p > 1$.

Theorem 3.4. *Let H be the Hilbert operator and $p > 1$. If N is a positive integer and $E_n = \{nN - N + 1, nN - N + 2, \cdots, nN\}$ for all n, then H is a bounded operator from l_p into $l_p(\Delta, E)$ and*

$$\|H\|_{p,\Delta,E} \leq (\frac{1}{2} + \cdots + \frac{1}{N+1})^{\frac{p-1}{p}} (\frac{1}{2})^{\frac{1}{p}}.$$

In particular if $N = 1$, then we have $\|H\|_{p,\Delta,E} \leq 1/2$.

Proof. By applying Lemma 3.1 we have $\|H\|_{p,\Delta,E} = \|B\|_p$, where $b_{n,k} = h_{\min E_n,k} - h_{\min E_{n+1},k}$. Let C_n and R_n be defined as in Theorem 3.3. By a simple calculation we deduce that $R_n \leq R_1$ and $C_n \leq \frac{1}{2}$ for all n. But

$$b_{1,k} = h_{1,k} - h_{N+1,k} = \frac{1}{1+k} - \frac{1}{N+1+k},$$

so $R_1 = \sum_{k=1}^{\infty} b_{1,k} = \frac{1}{2} + \cdots + \frac{1}{N+1}$. This concludes the proof according to Theorem 3.2. □

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