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SOME RESULTS ON A CONE RECTANGULAR METRIC SPACE

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ABSTRACT. The notion of cone rectangular metric spaces was introduced by A. Azam, M. Arshad and I. Beg in [1] (Applicable Analysis and Discrete Mathematics, 2009). The object of this paper is to prove some common fixed point result for two weakly compatible self maps satisfying a generalized contractive condition in a cone rectangular metric space. Our result generalizes the said result of [1]. All the results presented in this paper are new.

1. INTRODUCTION

There has been a number of generalizations of metric space. One such generalization was initiated by Huang and Zhang [4] in the name of cone metric space. In this space they replaced the set of real numbers of a metric space by an ordered Banach space and gave some fundamental results for a self map satisfying some contractive conditions. These results were generalized in Abbas and Jungck [6]. Papers [6] and [4] to [13] represent a comprehensive work in cone metric space. In [8] authors define compatibility in a cone metric space establishing some results which are known to be true in a metric space. A. Azam, M. Arshad and I. Beg in [1] introduced the concept of cone rectangular metric space proving Banach contraction principle with a simple contraction for one self map assuming the normality of cone associated with

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the space. In this paper, we establish the existence of a unique common fixed point through weak compatibility, of two self maps satisfying a more general contractive condition than one adopted in [1] without assuming the normality of cone associated with the space. Our results generalize, extend and unify several well known results in this space.

2. PRELIMINARIES

Definition 2.1 ([4]). *Let E be a real Banach space and P be a subset of E and θ is the zero vector of E. Then P is called a cone if:*

(i) P is a closed, nonempty;

(ii) $a, b \in R, a, b \geq 0, x, y \in P$ imply $ax + by \in P$;

(iii) $x \in P$, $-x \in P$ imply $x = \theta$.

Given a cone $P \subseteq E$, we define a partial ordering " $\preceq$ " in E by $x \preceq y$ if $y - x \in P$. We write $x \prec y$ to denote $x \preceq y$ but $x \neq y$ and $x \ll y$ to denote $y - x \in P^\circ$, where P° stands for the interior of P.

The cone P is called normal if there exists some $M > 0$ such that for

$$x, y \in E, \theta \preceq x \preceq y \implies \|x\| \leq \ M|y\|.$$

Proposition 2.1. *Let P be a cone in a real Banach space E . If for $a \in P$ and $a \preceq ka$, for some $k \in [0, 1)$ then $a = \theta$.*

Proof. For $a \in P, k \in [0, 1)$ and $a \preceq ka$ gives $(k - 1)a \in P$ implies $-(1 - k)a \in P$. Therefore by (ii) we have $-a \in P$, as $1/(1 - k) > 0$. Hence $a = \theta$, by (iii). □

Proposition 2.2 ([13]). *Let P be a cone is a real Banach space E with non-empty interior. If for $a \in E$ and $a \ll c$, for all $c \in P^\circ$, then $a = \theta$.*

Remark 2.1 ([13]). *$\lambda P^\circ \subseteq P^\circ$, for $\lambda > 0$ and $P^\circ + P^\circ \subseteq P^\circ$.*

Definition 2.2 ([4]). *Let X be a nonempty set and P be a cone in a real Banach space E. Suppose the mapping $d : X \times X \to E$ satisfies:*

(a) $\theta \preceq d(x,y)$, for all $x,y \in X$ and $d(x,y) = \theta$, if and only if $x = y$;

(b) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(c) $d(x,y) \preceq d(x,z) + d(z,y)$, for all $x,y,z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space. If P is normal, then (X,d) is said to be a normal cone metric space.

For examples of cone metric spaces we refer Huang et al. [5].

Proposition 2.3 ([3]). *Let (X,d) be a cone metric space and P be a cone in a real Banach space E. If $u \preceq v, v \ll w$ then $u \ll w$.*

Definition 2.3 ([1]). *Let X be a nonempty set and P be a cone in a real Banach space E. Suppose the mapping $d : X \times X \to E$ satisfies:*

(a) $\theta \preceq d(x,y)$, for all $x,y \in X$ and $d(x,y) = \theta$, if and only if $x = y$;

(b) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(c) $d(x,y) \preceq d(x,w) + d(w,z) + d(z,y)$, for all $x,y \in X$ and for all distinct points $w,z \in X \setminus \{x,y\}$.

Then d is called a cone rectangular metric on X, and (X,d) is called a cone rectangular metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is a positive integer N_c such that for all $n > N_c, d(x_n,x) \ll c$, then the sequence $\{x_n\}$ is said to converges to x, and x is called the limit of $\{x_n\}$. We write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, as $n \to \infty$. If for every $c \in E$ with $\theta \ll c$ there is a positive integer N_c such that for all $n,m > N_c, d(x_n,x_m) \ll c$, then the sequence $\{x_n\}$ is said to be a Cauchy sequence in X. If every Cauchy sequence in X is convergent in X then X is called a complete cone rectangular metric space.

In the following (X,d) will stand for a cone metric space with respect to a cone P with $P^\circ \neq \phi$ in a real Banach space E and $\preceq$ is partial ordering in E with respect to P.

Example 2.1 ([1]). *Let* $X = \mathbb{N}, E = \mathbb{R}^2$ *and* $P = \{(x,y) : x,y \geq 0\}$. *Define* $d : X \times X \to E$ *as follows:*

$$d(x,y) = \begin{cases} (0,0), & \text{if } x = y; \\ (3,9), & \text{if } x,y \in \{1,2\}, x \neq y; \\ (1,3), & \text{otherwise.} \end{cases}$$

Then (X,d) *is a cone rectangular metric space but not a cone metric space because it lacks the triangular property:*

$(3,9) = d(1,2) \npreceq d(1,3) + d(3,2) = (1,3) + (1,3) = (2,6)$

as $(3,9) - (2,6) = (1,3) \in P$.

Lemma 2.1. *Let* (X,d) *be a cone rectangular metric space and* P *be a cone in a real Banach space* E *and* $k_1, k_2, k_3, k > 0$ *are some fixed real numbers. If* $x_n \to x, y_n \to y$ *and* $z_n \to z$ *in* X *and for some* $a \in E$

(1.1) $ka \preceq k_1 d(x_n,x) + k_2 d(y_n,y) + k_3(z_n,z)$, *for all* $n > N$, *for some integer* N, *then* $a = 0$.

Proof. As $x_n \to x$, and $y_n \to y$ and $z_n \to z$ for $c \in P^0$ there exists a positive integer N_c such that

$$\frac{c}{(k_1+k_2+k_3)} - d(x_n,x), \frac{c}{(k_1+k_2+k_3)} - d(y_n,y), \frac{c}{(k_1+k_2+k_3)} - d(z_n,z) \in P^\circ$$

for all $n > \max\{N, N_c\}$. Therefore by Remark 2.1, we have

$$\frac{k_1 c}{(k_1+k_2+k_3)} - k_1 d(x_n,x), \frac{k_2 c}{(k_1+k_2+k_3)} - k_2 d(y_n,y), \frac{k_1 c}{(k_1+k_2+k_3)} - k_3 d(z_n,z) \in P.$$

Again by the above and Proposition 2.3, we have

$$c - k_1 d(x_n,x) - k_2 d(y_n,y) - k_3 d(z_n,z) \in P^\circ \text{ for all } n > \max\{N, N_c\}.$$

From (1.1), we have $ka \ll c$, for each $c \in P^\circ$. By Proposition 2.2 , we have $a = \theta$, as $k > 0$. □

Definition 2.4 ([6]). *Let A and S be self maps of a set X . If $w = Ax = Sx$, for some $x \in X$, then x is called a coincidence point and w is called the point of coincidence of A and S corresponding to x.*

Definition 2.5 ([10]). *Let X be a nonempty any set. A pair (A, S) of self maps of X is said to be weakly compatible if $u \in X, Au = Su$ imply $SAu = ASu$.*

Proposition 2.4 ([6]). *Let (f, g) be a pair of weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

3. MAIN RESULTS

Lemma 3.1. *Let (X, d) be a cone rectangular metric space with respect to a cone P with $P^\circ \neq \emptyset$ in a real Banach space E. Let A and S be self mappings on X satisfying:*

(3.1.1) $A(X) \subseteq S(X)$;

(3.1.2) there exist $\lambda, \mu, \delta, \beta \in [0, 1)$ such that $0 < \lambda + \mu + \delta + \beta < 1$ and the following inequality is satisfied:

$$d(Ax, Ay) \preceq \lambda d(Ax, Sx) + \mu d(Ay, Sy) + \delta d(Sx, Sy) + \beta d(Ax, Sy)$$

for all $x, y \in X$.

For some $x_0 \in X$, define sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_n = Sx_{n+1} = y_n,\ n \in \mathbb{N}.$$

Suppose $d_n = d(y_n, y_{n+1})$, for all $n \geq 0$. Then:

(I) If the pair (A, S) has a coincidence point then it is unique.

(II) If $y_n = y_{n+1}$, *for some* $n \in \mathbb{N}$, *then* $Sx_{n+1} = Ax_{n+1} = u$ *(say) and thus the pair* (A, S) *has a unique point of coincidence* u.

(III) (a) there exists $h < 1$ *such that* $d_n \preceq hd_{n-1}$, *and so,* $d_n \preceq h^n d_0$ *for all* $n \in \mathbb{N}$, *and if* $m > 1$ *then* $d_{n+m} \prec d_n$.

(b) If $y_n = y_{n+p}$ *for some* $p \geq 1$, *then* $p = 1$. *Thus, if two terms of* $\{y_n\}$ *are equal, then they are consecutive.*

(c) If the sequence $\{y_n\}$ *consist of all distinct terms then,* $d(y_{n+2}, y_n) \preceq kh^{n-1}d_0$, *for some* $0 \leq h < 1, k > 0$.

Proof. (of I) Suppose u and v are two coincidence points of the maps A and S, i.e., $Au = Su$ and $Av = Sv$, for some $u, v \in X$. Taking $x = u, y = v$ in (3.1.2) we have

$$d(Au, Av) \preceq \lambda d(Au, Su) + \mu d(Av, Sv) + \delta d(Su, Sv) + \beta d(Au, Sv)$$

i.e., $d(Su, Sv) \preceq (\delta + \beta) d(Su, Sv)$.
As $\delta + \beta < 1$, by Proposition 2.2, $d(Su, Sv) = \theta$. Hence, $Su = Sv$. This proves the uniqueness of point of coincidence of the pair (A, S). □

Proof. (of II) By definition of y_n and y_{n+1}, the result follows. □

Proof. (of III (a)) : Taking $x = x_n, y = x_{n+1}$ in (3.1.2) we get,

$$\begin{aligned} d(Ax_n, Ax_{n+1}) &\preceq \lambda d(Ax_n, Sx_n) + \mu d(Ax_{n+1}, Sx_{n+1}) + \delta d(Sx_n, Sx_{n+1}) \\ &+ \beta d(Ax_n, Sx_{n+1}) \end{aligned}$$

i.e.,

$$\begin{aligned} d(y_n, y_{n+1}) &\preceq \lambda d(y_n, y_{n-1}) + \mu d(y_{n+1}, y_n) + \delta d(y_{n-1}, y_n) + \beta d(y_n, y_n) \\ &= \lambda d(y_n, y_{n-1}) + \mu d(y_{n+1}, y_n) + \delta d(y_{n-1}, y_n) \\ &\preceq \lambda d(y_n, y_{n-1}) + \mu d(y_{n+1}, y_n) + \delta d(y_{n-1}, y_n). \end{aligned}$$

Writing $d(y_n, y_{n+1}) = d_n$, we have $d_n \preceq \lambda d_{n-1} + \mu d_n + \delta d_{n-1}$, i.e., $(1-\mu)d_n \preceq (\lambda+\delta)d_{n-1}$. Thus,

$$d_n \preceq h d_{n-1} \quad \text{for all} \quad n \in \mathbb{N} \tag{3.1}$$

where $h = \frac{\lambda+\delta}{1-\mu}$. In view of (3.1.2) we have $0 \le h < 1$. □

Proof. (of III (b)) Suppose $y_n = y_{n+p}$ for some $p > 1$ Then,

$$d(y_n, y_{n+1}) = d(y_{n+p}, y_{n+1}) = d(Ax_{n+p}, Ax_{n+1}).$$

Taking $x = x_{n+p}, y = x_{n+1}$ in (3.1.2) we get,

$$\begin{aligned} d(Ax_{n+p}, Ax_{n+1}) &\preceq \lambda d(Ax_{n+p}, Sx_{n+p}) + \mu d(Ax_{n+1}, Sx_{n+1}) + \delta d(Sx_{n+p}, Sx_{n+1}) \\ &+ \beta d(Ax_{n+p}, Sx_{n+1}). \end{aligned}$$

Thus,

$$\begin{aligned} d_n &\preceq \lambda d_{n+p-1} + \mu d_n + \delta d(y_{n+p-1}, y_n) + \beta d(y_{n+p}, y_n) \\ &= \lambda d_{n+p-1} + \mu d_n + \delta d(y_{n+p-1}, y_{n+p}) \\ &= (\lambda+\delta) d_{n+p-1} + \mu d_n. \end{aligned}$$

Hence, $d_n \preceq \frac{\lambda+\delta}{1-\mu} d_{n+p-1} = h d_{n+p-1}$. This gives $d_n \prec d_{n+p-1}$. Thus $d_n \prec d_{n+m}$, for some $m > 1$. This contradicts III(a). Hence $p = 1$ and the result follows. □

Proof. (of III (c))Taking $x = x_n, y = x_{n+2}$ in (3.1.2) we have

$$\begin{aligned} d(Ax_n, Ax_{n+2}) &\preceq \lambda d(Ax_n, Sx_n) + \mu d(Ax_{n+2}, Sx_{n+2}) + \delta d(Sx_n, Sx_{n+2}) \\ &+ \beta d(Ax_n, Sx_{n+2}) \end{aligned}$$

which implies that

$$\begin{aligned} d(y_n, y_{n+2}) &\preceq \lambda d(y_n, y_{n-1}) + \mu d(y_{n+2}, y_{n+1}) + \delta d(y_{n-1}, y_{n+1}) + \beta d(y_n, y_{n+1}) \\ &\preceq \lambda d_{n-1} + \mu d_{n+1} + \delta[d(y_{n-1}, y_n) + d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+1})] + \beta d_n \\ &= \lambda d_{n-1} + \mu d_{n+1} + \delta[d_{n-1} + d(y_n, y_{n+2}) + d_{n+1}] + \beta d_n. \end{aligned}$$

Thus,

$$(1-\delta)d(y_n, y_{n+2}) \preceq (\lambda+\delta)d_{n-1} + (\mu+\delta)d_{n+1} + \beta d_n$$

which implies that

$$\begin{aligned} d(y_n, y_{n+2}) &\preceq \frac{\lambda+\delta}{1-\delta}d_{n-1} + \frac{\mu+\delta}{1-\delta}d_{n+1} + \frac{\beta}{1-\delta}d_n. \\ &\preceq (\frac{\lambda+\delta}{1-\delta}h^{n-1} + \frac{\mu+\delta}{1-\delta}h^{n+1} + \frac{\beta}{1-\delta}h^n)d_0, \\ &= \frac{\lambda+\mu+2\delta+\beta}{1-\delta}h^{n-1}d_0, \\ &= kh^{n-1}d_0, \end{aligned}$$

where $k = \dfrac{\lambda+\mu+2\delta+\beta}{1-\delta} > 0$, in view of (3.1.2). □

Theorem 3.1. *Let (X, d) be a cone rectangular metric space with respect to a cone P with $P^\circ \neq \phi$ in a real Banach space E. Let A and S be self mappings on X satisfying* $(3.1.1), (3.1.2)$ *and*

(3.2.1) Either $A(X)$ or $S(X)$ is complete.

Then the mappings A and S have a unique point of coincidence in X. Moreover, if the pair (A, S) is weakly compatible, then this coincidence point is the unique common fixed point of the maps A and S in X.

Proof. We construct two sequences $\{x_n\}$ and $\{y_n\}$ in X as defined in Lemma 3.1. If $y_n = y_{n+1}$ for some n, then from then from Lemma 3.1 the maps A and S have a unique point of coincidence in X and that the elements of the sequence $\{y_n\}$ are all distinct or else some consecutive terms are equal . Now, it remains to prove the existence of unique point of coincidence when the terms of $\{y_n\}$ are all distinct. First, we show that $\{y_n\}$ is a Cauchy sequence in X, by considering $d(y_{n+p}, y_n)$ in two cases when p is odd and p is even.

By using rectangular inequality and Lemma 3.1 III (a), we have

$$\begin{aligned}
d(y_{n+2m+1}, y_n) &\preceq d(y_{n+2m+1}, y_{n+2m}) + d(y_{n+2m}, y_{n+2m-1}) + d(y_{n+2m-1}, y_n) \\
&= d_{n+2m} + d_{n+2m-1} + d(y_{n+2m-1}, y_n), \\
&\preceq d_{n+2m} + d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + \cdots + d_n, \\
&\preceq h^{n+2m} d_0 + h^{n+2m-1} d_0 + \cdots + h^n d_0, \\
&= h^n(1 + h + h^2 + \cdots + h^{2m}) d_0, \\
&\preceq \frac{h^n}{1-h} d_0,
\end{aligned}$$

as $h < 1$ and P is closed. Thus

$$d(y_{n+2m+1}, y_n) \preceq \frac{h^n}{1-h} d_0. \tag{3.2}$$

Again, from Lemma 3.1 (III (c)), we have

$$\begin{aligned}
d(y_{n+2m}, y_n) &\preceq d(y_{n+2m}, y_{n+2m-1}) + d(y_{n+2m-1}, y_{n+2m-2}) + d(y_{n+2m-2}, y_n) \\
&= d_{n+2m-1} + d_{n+2m-2} + d(y_{n+2m-2}, y_n) \\
&\preceq d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + d_{n+2m-4} + \cdots \\
&\quad + d_{n+2} + d(y_{n+2}, y_n) \\
&= d(y_{n+2}, y_n) + d_{n+2} + d_{n+3} + \cdots + d_{n+2m-1} \\
&\preceq kh^{n-1} d_0 + h^{n+2}(1 + h + h^2 + \cdots + h^{2m-3}) d_0 \\
&\preceq kh^{n-1} d_0 + h^{n+2} \frac{d_0}{1-h},
\end{aligned}$$

as $h < 1$ and P is closed. Thus

$$d(y_{n+2m}, y_n) \preceq kh^{n-1}d_0 + h^{n+2}\frac{d_0}{1-h}. \tag{3.3}$$

Now for $c \in P^\circ$, there exists $r > 0$ such that $c/2 - y \in P^\circ$, if $||y|| < r$. Choose a positive integer N_c such that for all $n \geq N_c$, $||(h^n d_0)/(1-h)|| < r$, $\|kh^{n-1}d_0\| < r$, and $\|\frac{h^{n+2}}{1-h}d_0\| < r$, for all $n > N_c$. This implies $\frac{c}{2} - \frac{h^{n+2}}{1-h}d_0, \frac{c}{2} - kh^{n-1}d_0, c - \frac{h^n}{1-h}d_0 \in P^\circ$. Hence by remark 2.1, we have $c - \frac{h^{n+2}}{1-h}d_0 - kh^{n-1}d_0 \in P^\circ$, and $c - \frac{h^n}{1-h}d_0 \in P^\circ$, for all $n > N_c$. Also from (3.2) and (3.3), $\frac{h^{n+2}}{1-h}d_0 + kh^{n-1}d_0 - d(y_{n+2m}, y_n) \in P$ and $\frac{h^n}{1-h}d_0 - d(y_{n+2m+1}, y_n) \in P$. Hence by Proposition 2.3, $c - d(y_{n+2m}, y_n) \in P^\circ$ and $c - d(y_{n+2m+1}, y_n) \in P^\circ$, for all $n > N_c$. Thus $\{y_n\}$ is a Cauchy sequence in $S(X) \bigcap A(X)$. Now, we show that the mappings A and S have a unique point of coincidence.

Case I: S(X) is complete

In this case $y_n = Sx_{n+1}$ is a Cauchy sequence in $S(X)$, which is complete. So $\{y_n\} \to z \in S(X)$. Hence there exist $u \in X$ such that $z = Su$

Now,

$$\begin{aligned}
d(Au, Su) &\preceq d(Au, Ax_n) + d(Ax_n, Ax_{n+1}) + d(Ax_{n+1}, Su), \\
&= d(Au, Ax_n) + d(y_n, y_{n+1}) + d(y_{n+1}, Su), \\
&= d_n + d(y_{n+1}, Su) + d(Ax_n, Au) \\
&\preceq d_n + d(y_{n+1}, Su) + \mu d(Au, Su) + \lambda d(Ax_n, Sx_n) \\
&\quad + \delta d(Su, Sx_n) + \beta d(Au, Su) \\
&= d_n + d(y_{n+1}, Su) + \mu d(Au, Su) + \lambda d(y_n, y_{n-1}) \\
&\quad + \delta d(Su, y_{n-1}) + \beta d(y_n, Su).
\end{aligned}$$

Writing $d_n = d(x_n, x_{n+1})$ we obtain

$$\begin{aligned} d(Au, Su) &\preceq d_n + d(y_{n+1}, Su) + \mu d(Au, Su) + \lambda d_{n-1} \\ &\quad + \delta d(Su, y_{n-1}) + \beta d(y_n, Su). \end{aligned}$$

Rearranging the terms we obtain:

$$\begin{aligned} (1-\mu)d(Au, Su) &\preceq d_n + d(y_{n+1}, Su) + \lambda d_{n-1} + \delta d(Su, y_{n-1}) + \beta d(Su, y_n) \\ &\preceq h^n d_0 + \lambda h^{n-1} d_0 + d(y_{n+1}, Su) + \delta d(Su, y_{n-1}) + \beta d(Su, y_n) \\ &\preceq (1+\lambda)h^n d_0 + d(Su, y_{n+1}) + \delta d(Su, y_{n-1}) + \beta d(Su, y_n). \end{aligned}$$

Thus,

$$(3.4) \quad (1-\mu)d(Au, Su) - (1+\lambda)h^n d_0 \preceq d(Su, y_{n+1}) + \delta d(Su, y_{n-1}) + \beta d(Su, y_n).$$

As $\beta \geq 0, \delta \geq 0$ and $y_{n-1}, y_n, y_{n+1} \to Su$, by Lemma 2.1 for $c \in P^\circ$ there exists a positive integer N_c such that

$$c - d(Su, y_{n+1}) - \delta d(Su, y_{n-1}) - \beta d(Su, y_{n-1}) \in P^\circ \text{ for all } n > N_c.$$

This implies that

$$d(Su, y_{n+1}) + \delta d(Su, y_{n-1}) + \beta d(Su, y_n) \ll c \text{ for all } n > N_c.$$

Using Proposition 2.3 and equation 3.4, we get

$$(1-\mu)d(Au, Su) - (1+\lambda)h^n d_0 \ll c \text{ for all } n > N_c.$$

As $h < 1$, we have $(1-\mu)d(Au, Su) \ll c$ for all $c \in P^\circ$. Using Proposition 2.2 it follows that $d(Au, Su) = \theta$ and we get $Au = Su$. Thus, the pair (A, S) has a point of coincidence $z = Au = Su$.

Case II: $A(X)$ is complete.

In this case $y_n = Ax_n$ is a Cauchy sequence in $A(X)$ which is complete. So $\{y_n\} \to$

$z = Aw$, for some $w \in X$. As $A(X) \subseteq S(X)$ there exists $v \in X$ such that $Aw = Sv$. Thus $\{y_n\} \to Sv$. It follows from case I that $Av = Sv$. Thus in both the cases the pair (A, S) has a point of coincidence, which is unique in view of Lemma 2.2 As (A, S) is weakly compatible from Proposition 2.4, it follows that the point of coincidence of A and S is their unique common fixed point in X. □

Theorem 3.2. *Let (X, d) be a cone rectangular metric space with respect to a cone P with $P^\circ \neq \phi$ in a real Banach space E. Let A and S be self mappings on X satisfying* $(3.1.1), (3.1.2)$ *and*

(3.3.1) there exists $\lambda, \mu, \delta, \alpha \in [0, 1)$ such that $0 < \lambda + \mu + \delta + \alpha < 1$ and the following inequality is satisfied:

$$d(Ax, Ay) \preceq \lambda d(Ax, Sx) + \mu d(Ay, Sy) + \delta d(Sx, Sy) + \alpha d(Ay, Sx)$$

for all $x, y \in X$.

Then the mappings A and S have a unique coincidence point in X. Moreover, if the pair (A, S) is weakly compatible, then A and S have a unique common fixed point in X.

Proof. we have,

$$d(Ax, Ay) = d(Ay, Ax) \preceq \lambda d(Ay, Sy) + \mu d(Ax, Sx) + \delta d(Sx, Sy) + \alpha d(Ax, Sy).$$

Interchanging λ and μ and writing α as β, we get

$$d(Ax, Ay) \preceq \lambda d(Ax, Sx) + \mu d(Ay, Sy) + \delta d(Sx, Sy) + \beta d(Ax, Sy).$$

Rest follows from Theorem 3.1 □

Example 3.1. *(of Theorem 3.2) Let $X = \{1, 2, 3, 4\}, E = \mathbb{R}^2, P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\} \subset \mathbb{R}^2$, be a cone in E. Define $d : X \times X \to E$ as follows:*
$d(x, y) = 0$ if $x = y$,

$d(1,2) = d(2,1) = (3,6)$
$d(2,3) = d(3,2) = d(1,3) = d(3,1) = (1,2)$
$d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = (2,4)$
Then, (X,d) is a complete rectangular cone metric space. In fact it is not a cone metric space as $d(1,3) + d(3,2) < d(1,2)$. Define the mappings A and S on X as follows:
$A(1) = 3, A(2) = 3, A(3) = 3, A(4) = 1$
$S(1) = 2, S(2) = 1, S(3) = 3, S(4) = 2$. *Then the maps A and S are weakly compatible and $A(X) \subset S(X)$. The conditions (3.2.1) and (3.1.2) are satisfied with $\lambda = \mu = 1/4$ and $\beta = \delta = 1/8$ and the pair (A,S) has a unique common fixed point $x = 3 \in X$.*

Taking $S = I$, the identity mapping in Theorems 3.2 and 3.3, we have the following results:

Theorem 3.3. *Let (X,d) be a cone rectangular metric space with respect to a cone P with $P^\circ \neq \phi$ in a real Banach space E. Let A be self mappings on X and there exist $\lambda, \mu, \delta, \beta \in [0,1)$ such that $0 < \lambda + \mu + \delta + \beta < 1$ and the following inequality is satisfied:*

$$d(Ax, Ay) \preceq \lambda d(Ax, x) + \mu d(Ay, y) + \delta d(x,y) + \beta d(Ax, y)$$

for all $x, y \in X$. Then the mapping A has a unique fixed point in X.

Theorem 3.4. *Let (X,d) be a cone rectangular metric space with respect to a cone P with $P^\circ \neq \phi$ in a real Banach space E. Let A be self mapping on X and there exist $\lambda, \mu, \delta, \alpha \in [0,1)$ such that $\lambda + \mu + \delta + \beta < 1$ and the following inequality is satisfied:*

$$d(Ax, Ay) \preceq \lambda d(x, Ax) + \mu d(y, Ay) + \delta d(x,y) + \alpha d(x, Ay)$$

for all $x, y \in X$. Then the mapping A has a unique fixed point in X.

Again, taking $\lambda = \mu = \alpha = 0$ and $\delta = k$ in Theorem 3.3, we have the following result:

Corollary 3.1. *Let* (X, d) *be a cone rectangular metric space with respect to a cone* P *with* $P^\circ \neq \phi$ *in a real Banach space* E*. Let* A *and* S *be self mappings on* X *satisfying (3.1.1) and for some* $k \in [0, 1)$ *and for all* $x, y \in X$,

$$d(Ax, Ay) \preceq kd(Sx, Sy).$$

If $A(X)$ *or* $S(X)$ *is complete and the pair* (A, S) *is weakly compatible then the mappings* A *and* S *have a unique common fixed point in* X*.*

Remark 3.1. *On taking* $S = I$*, the identity mapping in Corollary 3.1 the result of Azam et all* [1], *Theorem 2 follows.*

Again taking $\lambda = \mu = k$ and $\alpha = \delta = 0$ in Theorem 3.2, we have the following result:

Corollary 3.2. *Let* (X, d) *be a cone rectangular metric space with respect to a cone* P *with* $P^\circ \neq \phi$ *in a real Banach space* E*. Let* A *and* S *be self mappings on* X *satisfying (3.1.1) and for some* $k \in (0, 1/2)$ *and for all* $x, y \in X$,

$$d(Ax, Ay) \preceq k[d(Ax, Sx) + d(Ay, Sy)]$$

If $A(X)$ *or* $S(X)$ *is complete and the pair* (A, S) *is weakly compatible then the mappings* A *and* S *have a unique common fixed point in* X*.*

Finally, taking $\beta = 0$ in Theorem 3.3, we have the following result

Corollary 3.3. *Let* (X, d) *be a cone rectangular metric space with respect to a cone* P *with* $P^\circ \neq \phi$ *in a real Banach space* E*. Let* A *and* S *be self mappings on* X *satisfying*

(3.1.1),(3.2.1) and there exist $\lambda, \mu, \delta \in [0, 1)$ *such that* $0 < \lambda + \mu + \delta < 1$ *and the following inequality is satisfied:*

$$d(Ax, Ay) \preceq \lambda d(Sx, Ax) + \mu d(Sy, Ay) + \delta d(Sx, Sy)$$

for all $x, y \in X$*. If the pair* (A, S) *is weakly compatible then the mappings* A *and* S *have a unique common fixed point in* X*.*

Remark 3.2. *Corollaries 3.1 and 3.2 are known to be true in a normal cone metric space from Abbas and Jungck* [6]*. Corollary 3.3 is known to be true in a normal cone metric space from P. Vetro* [7]*. Also, an ordered version of this corollary and a consequence of this can be seen in* [11, 12]*.*

Keeping one of the constants $\alpha, \beta, \gamma, \delta, \mu$ non-zero and all others equal to zero in Theorems 3.1 and 3.2, we have the following result:

Corollary 3.4. *Let* (X, d) *be a cone rectangular metric space with respect to a cone* P *with* $P^\circ \neq \phi$ *in a real Banach space* E*. Let* A *and* S *be two weakly compatible self mappings on* X *such that* $A(X)$ *or* $S(X)$ *is complete. Suppose that the following inequality is satisfied:*

$$d(Ax, Ay) \leq \lambda u, \text{ where } u \in \{d(Ax, Sx), d(Ay, Sy), d(Sx, Sy), d(Ax, Sy)\}$$

for all $x, y \in X$*, where* $\lambda \in (0, 1)$*. Then* A *and* S *have a unique common fixed point in* X*.*

The following definition and Theorem appears in D. Ilic, V. Rakocevic [2].

Definition 3.1 (Quasi contraction [2])**.** *A self-map* f *on a cone metric space* (X, d) *is said to be a quasi contraction if for a fixed* $\lambda \in (0, 1)$ *and for all* $x, y \in X$

$$d(fx, fy) \leq \lambda u, \text{ where } u \in \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}.$$

Theorem 3.5 ([2]). *Let (X, d) be a complete cone metric space and P be normal cone. Then a Quasi contraction f has a unique fixed point in X, and for each $x \in X$, the iterative sequence $\{f^n(x)\}$ converges to this fixed point.*

Remark 3.3. *Taking $S = I$, the identity mapping in Corollary 3.4 it follows that the above result of* [2] *is true even for non-normal complete cone rectangular metric spaces. The above result has been established in* [5] *for a complete metric space.*

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