EDGE MAXIMAL GRAPHS CONTAINING NO SPECIFIC WHEELS

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ABSTRACT. Let $k \geq 4$ be a positive integer. Let $\mathcal{G}(n; W_k)$ denote the class of graphs on n vertices containing no wheel W_k as a subgraph. In this paper, we study the following: (1) Edge maximal graphs containing no odd wheels. Furthermore, we characterize the extremal graphs. (2) The edge maximal graph containing no even wheels. (3) The edge maximal graph containing no specific even wheels.

1. INTRODUCTION.

Unless otherwise specified a graph G is finite, undirected, and has no loops or multiple edges. We denote the vertex set of G by $V(G)$, the edge set of G by $E(G)$ and the number of edges of G by $\mathcal{E}(G)$. The cycle on n vertices is denoted by C_n. A wheel graph $W_n, n \geq 4$ is defined to be a cycle C_{n-1} to which we add a new vertex that is adjacent each vertex of C_{n-1}. Let G be a graph, and $u \in V(G)$. The degree of a vertex u in G, denoted by $d_G(u)$, is the number of edges of G incident to u. $\Delta(G)$ stands for the maximum degree in G. The neighbor set of a vertex u of G is the set of adjacent vertices to u, and denoted by $N_G(u)$. For vertex disjoint subgraphs

H_1 and H_2 of G we let $E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $\mathcal{E}(H_1, H_2) = |E(H_1, H_2)|$.

For a positive integer n and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n; \mathcal{F})$ denote the class of non-bipartite $\mathcal{F}$-free graphs on n vertices, and

$$f(n; \mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; \mathcal{F})\}.$$

An important problem in extremal graph theory is to determine the value of the function $f(n; \mathcal{F})$ and to characterize the extremal graphs of $\mathcal{G}(n; \mathcal{F})$ in which $f(n; \mathcal{F})$ is attained. This problem has been studied by many authors, see [2, 3, 5 , 13]. We state the following powerful result which determines the asymptotic behavior of the maximal graphs in many situations:

Theorem 1.1. (Erdos-Stone-Simonovits) Let $\mathcal{F}$ be any finite set of graphs and r be the minimum chromatic number of $F \in \mathcal{F}$. Then

$$f(n; \mathcal{F}) = (1 - \frac{1}{r-1}) \binom{n}{2} + o(n^2).$$

One can notice that if $r = 2$ (i.e., if any of the subgraphs of $\mathcal{F}$ is bipartite), then this theorem does not tell us much.

Moon [13] proved that, if G is a graph on n vertices containing no wheels, then $\mathcal{E}(G) \le \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{4} \rfloor$. Furthermore, he characterized the extremal graphs. Alrhayyel and et al [1] proved that (1) $f(n; W_5) = \lfloor \frac{n-2}{4} \rfloor + \lfloor \frac{s}{4} \rfloor$ for $n \ge 3$ where $s = n$ if $n \ne 4k + 2$ and $s = n - 1$ if $n = 4k + 2$ and (2) $f(n; W_6) = \lfloor \frac{n^2}{3} \rfloor$ for $n \ge 6$.

The results cited above trigger off the following question: Can we determine the exact value of $f(n; \mathcal{F})$, and characterize the extremal graphs for the following cases?:

(1) $\mathcal{F} = \{W_5, W_7, \ldots, W_{2k+1}, \ldots\}$ which is the family of all odd wheels.

(2) $\mathcal{F} = \{W_4, W_6, \ldots, W_{2k}, \ldots\}$ which is the family of all even wheels.

(3) $\mathcal{F} = \{W_{2k}\}$ which consists of a specific even wheel where $k \geq 2$ is a positive integer.

(4) $\mathcal{F} = \{W_{2k+1}\}$ which consists of specific even wheel where $k \geq 2$ is a positive integer.

In this paper we determine the extremal function for the class of graphs that contains no odd wheels and characterize the extremal graphs as well which is the answer for question 1. Furthermore, we determine the edge maximal graphs containing no even wheels and graphs containing no specific even wheels which answers the first part of question 2 and 3. We pose the other problems as future ideas to explore. The following theorem will be used in proving our results which can be found in [6].

Lemma 1.2. (Bondy) Let G be a graph on n vertices with $\mathcal{E}(G) > \left\lfloor \frac{n^2}{4} \right\rfloor$. Then G contains cycles of every length l for $3 \leq l \leq \left\lfloor \frac{n+3}{2} \right\rfloor$.

2. Edge maximal W_k free graphs

We begin with some constructions which are similar to the constructions made in [13]. Let G be a graph with $n \geq 5$ vertices, let H_n denote the class of graphs obtained by splitting the vertices of G into two sets, P and Q, with $\left\lfloor \frac{n+1}{2} \right\rfloor$ and $\left\lfloor \frac{n}{2} \right\rfloor$, respectively. There are as many edges joining pairs of vertices in P (and analogously in Q) as are consistent with the requirement that no two of these edges have a vertex in common. In addition, each vertex in P is adjacent to each vertex in Q. If $n \equiv 2 \, mod 4$, let L_n denote the class of graphs obtained as above except that P and Q have $\frac{n}{2} + 1$ and $\frac{n}{2} - 1$ vertices. Let l be a positive integer, then the following properties of these graph can readily be verified: (2.1) H_n and L_n each have $\left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ edges if $n \neq 4l + 2$ and $\left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$ edges if $n = 4l + 2$.

(2.2) Neither H_n nor L_n contain any odd wheels.

(2.3) At least one odd wheel is formed when any new edge is added to H_n or L_n.

Theorem 2.1. Let

$$\varnothing(n) = \begin{cases} 4l^2 + 2l, & \text{if } n = 4l \\ 4l^2 + 4l, & \text{if } n = 4l + 1 \\ 4l^2 + 6l + 1, & \text{if } n = 4l + 2 \\ 4l^2 + 8l + 3, & \text{if } n = 4l + 3. \end{cases}$$

Then the only graphs with $n \geq 5$ vertices and at least $\varnothing(n)$ edges which contain no odd wheels are H_n or L_n. Furthermore, $f(n, \mathcal{F}) = \varnothing(n)$ where $\mathcal{F} = \{W_5, W_7, \ldots, W_{2k+1}, \ldots\}$ which is the family of all odd wheels.

Proof : We use induction on n to prove the theorem:

n = 5. *One can easily see that the only graph G on 5 vertices and 8 edges containing no odd wheels is K_4 plus a vertex v adjacent to two vertices of K_4 (see Figure 1). Thus, $G = H_5$.*
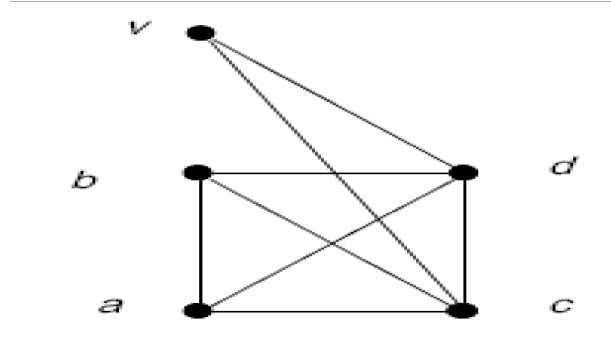

FIGURE 1. Represents the situation when $n = 5$

n = 6. *Let G be a graph on 6 vertices, 11 edges and contains no odd wheels. Since $(4)(6) > (2)(11)$, by Handshaking Lemma, G must have a vertex with degree less than or equal 3, say f. If the degree of f is equal to 1 or 2, then $\mathcal{E}(G - f) = 11 - \deg(f) \geq 9$. And so, by the Case $\mathbf{n = 5}$, $G - f$ has a W_5 as a subgraph. To this end, $d_G(f) = 3$.*

Let H be the graph obtained by removing f. Using the case $\mathbf{n} = \mathbf{5}$ and (2.3) one can see that $H = H_5$. Let $\{a, b, c, d, v\}$ be the vertex set of H_5 as in Figure 1. Note that f can not be adjacent to three vertices of $\{a, b, c, d\}$ as otherwise a W_5 is produced. Also, if f is adjacent to v and adjacent to one vertex of $\{a, b\}$, say a, and one vertex of $\{c, d\}$, say c, then f is adjacent to every vertex of the 4-cycle $vcabv$. Thus, W_5 is produced. Hence f is adjacent to v and adjacent to either one of the following (1) a and b and so G must be H_6 (see Figure 2.A) or (2) c and d and so G must be L_6 (see Figure 2.B).
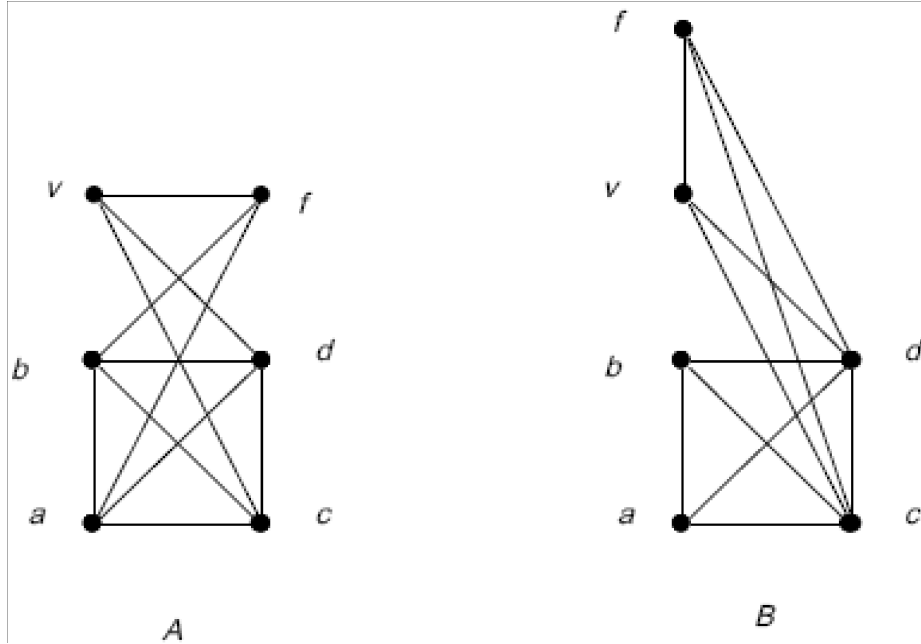

FIGURE 2. Represents the situation when $n = 6$

$\mathbf{n} = \mathbf{7}$. Let G be a graph on 7 vertices, 15 edges and contains no odd wheels. Note that $(5)(7) > (2)(15)$. Then G contains a vertex with degree at most 4, say x. If degree of x is less than or equal to 3, then $\mathcal{E}(G - x) = 15 - d_G(x) \geq 12$. And so, by the Case $\mathbf{n} = \mathbf{6}$, $G - x$ has a W_5 as a subgraph. To this end, $d_G(x) = 4$. Let H be the graph obtained by removing x. Using the case $\mathbf{n} = \mathbf{6}$ and (2.3), we get that H must be either H_6 (as in Figure 2.A) or L_6 (as in Figure 2.B).

As in above, x can not be adjacent to three vertices of $\{a, b, c, d\}$. So x is adjacent to v and f and to two vertices of $\{a, b, c, d\}$. Now we consider the case $H = H_6$. It is easy to see that if x is adjacent to a vertex of $\{a, b\}$ and a vertex of $\{c, d\}$, then W_5 is produced. Therefore, x is adjacent to either one of the following (1) both a and b or (2) both c and d. Note that either (1) or (2) implies that $G = H_7$ (see Figure 3A and B). To end this, we consider that $H = L_6$. As in above, one can see that if x is adjacent to one vertex of $\{a, b\}$ and a vertex of $\{c, d\}$ or is adjacent to both c and d, then W_5 is produced. Thus, x must be adjacent to both a and b which implies that $G = H_7$ (see Figure 3C).
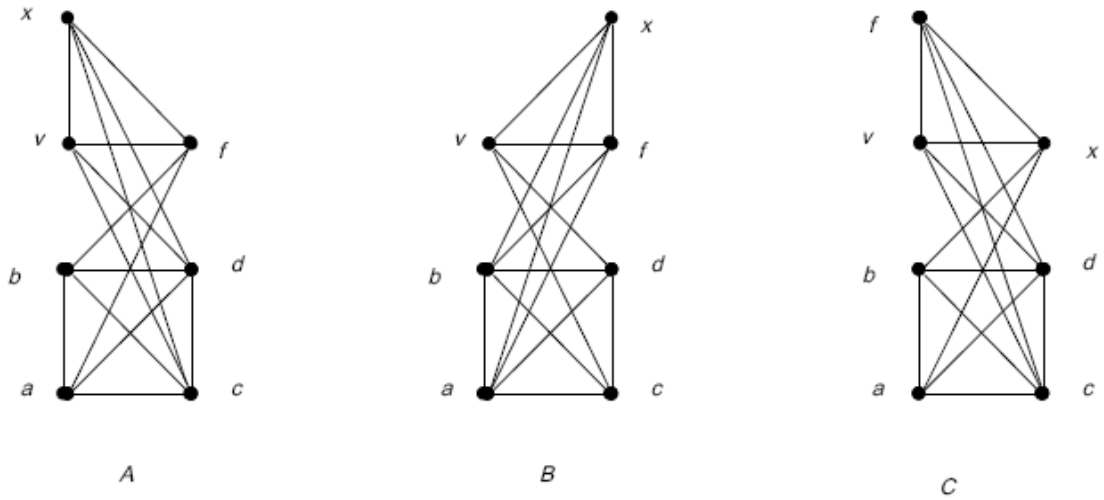

FIGURE 3. Represents the situation when $n = 7$

n = 8. *Let G be a graph on 8 vertices and 20 edges. A similar argument to arguments in case **n = 7** will show that there needs to be a vertex of degree 5 in G, say z. Consider H to be the graph obtained by removing z . Then, by similar arguments to above, $H = H_7$ (we may assume it as in Figure 3A) . Also, z is adjacent to v, f, x and to two vertices of $\{a, b, c, d\}$. Furthermore, if z is adjacent to one vertex of $\{a, b\}$ and one vertex of $\{c, d\}$ or to both vertices c and d, then W_5 is produced. Thus, z must be adjacent only to both vertices a and b and hence, $G = H_8$.*

Now, suppose that the result holds for $m \geq 8$. Let G be a graph with $m+1$ vertices and at least $\varnothing(m+1)$ edges which contains no odd wheels. First we assume that G has exactly $\varnothing(m+1)$ edges. Now, we show that $G = H_{m+1}$ or L_{m+1}. We consider four cases according to m.

   Case 1: $m = 4l$. *Then G has $4l+1$ vertices and $4l^2 + 4l$ edges. Since $(4l+1)(2l+2) > 2(4l^2+4l)$, there is a vertex x in G with degree d where $d \leq 2l+1$. If $d < 2l$, then $\mathcal{E}(G-x) = 4l^2 + 4l - d > 4l^2 + 2l$. And so, by the induction step, $G-x$ has a W_5 as a subgraph. To this end, $d \in \{2l, 2l+1\}$. Let G' be the graph obtained by removing x . Then G' has $4l$ vertices. We now consider the following two subcases :*

   Subcase 1.1: $d = 2l$. *Then G' has no odd wheels, $4l$ vertices and $\varnothing(4l+1) - d = \varnothing(4l+1) - 2l = \varnothing(4l)$. It follows from the induction hypothesis and (2.1) that $G' = H_m$ and $d = 2l$. Hence G consists of H_m and the vertex x which adjacent to precisely $2l$ vertices of H_m. Clearly, each of P and Q contains only $2l$ vertices.*

   Now, observe that x can not be adjacent to more than one vertex of P and to one vertex of Q simultaneously, for if x is adjacent to two vertices of P, say p_1 and p_2, and to one vertex of Q say q_1, then q_1 is adjacent to every vertex of the cycle $p_1 x p_2 q_2 p_1$ which forms W_5 as a subgraph of G, where q_2 is the neighbor of q_1 in Q, a contradiction. Analogously, x can not be adjacent to more than one vertex of Q and to one vertex of P simultaneously.

   This observation and the fact that both P and Q have $2l$ vertices, leave the following alternatives: The vertex x is adjacent to every vertex of P and to no vertex of Q or x is adjacent to every vertex of Q and to no vertex of P, which implies that $G = H_{m+1}$.

   Subcase 1.2: $d = 2l+1$. *Then G' has $4l$ vertices and $4l^2 + 4l - 2l - 1$ edges. Since $4l(2l+1) > 2(4l^2 + 2l - 1)$, as a result G' has a vertex, say y, such that $d_{G'}(y) \leq 2l$. By the same argument as in the above, we need only to consider $d_{G'}(y) = 2l$, so we consider this case only. Let $G'' = G' - y$. Then G'' has $4l - 1$ vertices and $4l^2 - 1$*

edges, so by induction step, G'' must be H_{4l-1}. Moreover, P contains only $2l$ and Q contains only $2l-1$ vertices. Now, G' consists of H_{4l-1} and the vertex y which is adjacent to precisely $2l$ vertices of H_{4l-1}. Clearly, as in Subcase 1.1, y can not be adjacent to more than one vertex of both P and Q simultaneously. This observation and the fact that y must be adjacent to at least one vertex of P, since Q contains only $2l-1$ vertices, leaves the following alternatives:

(a) The vertex y is adjacent to every vertex of P and to no vertex of Q.

(b) The vertex y is adjacent to $2l-1$ vertices of P and to one vertex of Q.

(c) The vertex y is adjacent to every vertex of Q and to one vertex of P.

We know that G consists of G' plus the vertex x which is adjacent to precisely $2l+1$ vertices of G'. So we have the following possibilities according to the alternative (a), (b) and (c):

(I) Alternative (a) implies that G' is a graph obtained by splitting the vertices of G' into two sets, P_1 and Q_1 each of which contains only $2l$ vertices. There are two nonadjacent vertices, say q_1 and q_2, in Q_1 (one of them is y) each of which is adjacent to no vertex of Q_1. Further, there are as many edges joining pairs of vertices in P_1 (and analogously Q_1) as are consistent with the requirement that no two of these edges have a vertex in common. In addition, each vertex in P_1 is adjacent to each vertex in Q_1. As in the observation of Subcase 1.1, x can not be adjacent to more than one vertex of both P_1 and Q_1 simultaneously. This observation and the fact that x should be adjacent to at least one vertex of P or Q, since both P and Q have $2l$ vertices, leaves the following possibilities:

(i) x is adjacent to $2l$ vertices of P_1 and to one vertex, say q, of Q_1. If this is the case, then $q \in \{q_1, q_2\}$ and so $G = H_{m+1}$ since otherwise q is adjacent to every vertex of the cycle $p_1 x p_2 q^* p_1$ where q^* is the neighbor of q in Q_1 and p_1, p_2 are two vertices of P_1 which forms W_5 as a subgraph of G, a contradiction.

(ii) x *is adjacent to* $2l$ *vertices of* Q_1 *and to one vertex* p_1 *of* P_1. *If this is the case, then by interchanging* Q_1 *and* P_1 *in (i) we get the same contradiction.*

(II) *Alternative (b) and (2.3) imply that* G' *is a graph obtained by splitting the vertices of* G' *into two sets,* P_1 *and* Q_1 *each of which contains only* $2l$ *vertices. There are as many edges joining pairs of vertices in* P_1 *(and analogously* Q_1*) as are consistent with the requirement that no two of these edges have a vertex in common. In addition, each vertex in* P_1 *is adjacent to each vertex in* Q_1 *except that there is a vertex* $q = y$ *in* Q_1 *which is not adjacent to a vertex* p *in* P_1. *As above,* x *can not be adjacent to more than one vertex of both* P_1 *and* Q_1 *simultaneously. Similarly, this observation and the fact that* x *should be adjacent to at least one vertex of* P *or* Q, *since both* P *and* Q *have* $2l$ *vertices, leaves the following possibilities:*

(i) x *is adjacent to* $2l$ *vertices of* P_1 *and to one vertex* q *of* Q_1. *If this is the case, then the vertex* q *is adjacent to every vertex of the cycle* $p^* x p_2 q^* p^*$ *where* q^* *is the neighbor of* q *in* Q_1 *and* p^*, p_2 *are two vertices of* P_1 *which forms* W_5 *as a subgraph of* G, *so this case is impossible.*

(ii) x *is adjacent to* $2l$ *vertices of* Q_1 *and to one vertex of* P_1, *then as in the above (i),* W_5 *is produced.*

(III) *By using observation of Subcase 1.1, the alternative (c) is impossible, and so this case is impossible.*

Case 2: $m = 4l + 1$. *Then* G *has* $4l + 2$ *vertices and* $4l^2 + 6l + 1$ *edges. Since* $(4l + 2)(2l + 2) > 2(4l^2 + 6l + 1)$, *as in Case 1, we need only to consider that there is a vertex, say* x, *in* G *of degree* d *where* $d \leq 2l + 1$. *Let* G' *be the graph obtained by removing* x. *Then* G' *has* $4l + 1$ *vertices and contains no odd wheels. Further* $\varnothing(4l + 2) - d \geq \varnothing(4l + 2) - (2l + 1) = \varnothing(4l + 1)$. *It follows from the induction hypothesis, (2.1) and a similar argument to as in Case 1, we get that* $G' = H_m$ *and* $d = 2l + 1$. *Hence* G *consists of* H_m *and the vertex* x *which joins to precisely* $2l + 1$ *vertices of* H_m. *We now consider the possibilities. First, suppose that* $G' = H_m$.

Clearly, P contains only $2l + 1$ vertices and Q contains only $2l$ vertices. As in Case 1, x can not be adjacent to more than one vertex of both P and Q simultaneously. This observation and the fact that x must be adjacent to at least one vertex of P, since Q contains $2l$ vertices, leaves the following alternatives:

a) The vertex x is adjacent to every vertex of P and to no vertex of Q.

b) The vertex x is adjacent to $2l$ vertices of P and to one vertex q_1 of Q.

c) The vertex x is adjacent to one vertex p_1 of P and to every vertex of Q.

Alternative (a) implies that $G = H_{m+1}$, by definition. Alternative (b) is impossible for the same reason as in the observation of Subcase 1.1. In alternative (c) if p_1 is adjacent to another vertex p_2 in P, then p_1 is adjacent to every vertex of the cycle $q_2 x q_1 p_2 q_2$ where q_1, q_2 are two vertices in Q which forms W_5 as a subgraph. Hence p_1 is the only vertex that is adjacent to no other vertices of P. This implies $G = L_{m+1}$.

Case 3: *$m = 4l + 2$. Then G has $4l + 3$ vertices and $4l^2 + 8l + 3$ edges. By using Handshaking lemma, there is some vertex x in G of degree d where $d \leq 2l + 2$. As in the argument in Case 1 in which we excluded the case $d < 2l$, it is easy to show that if $d < 2l - 1$, then G has W_5. So, we consider that $d = 2l + 2$. Let G' be the graph obtained by removing x and its d incident edges from G. Then by using the same argument as in Case 1 and by the inductive hypothesis G' must be either H_m or L_m. Hence G consists of H_m (or L_m) and the vertex x which joins to precisely $2l + 2$ vertices of H_m (or L_m). We now consider the possibilities. First, suppose that $G' = H_m$. Clearly, each of P and Q contains only $2l + 1$ vertices. As in the observation of Subcase 1.1, x can not be adjacent to more than one vertex of both P and Q simultaneously. This observation and the fact that x must be adjacent to at least one vertex in P or Q, since both P and Q contain only $2l + 1$ vertices, leaves the following alternatives:*

a) The vertex x is adjacent to every vertex of P and one vertex q_1 of Q.

b) The vertex x is adjacent to one vertex p_1 of P and to every vertex of Q.

If the alternative (a) is the case, then by using the same argument as in the alternative (c) of Case 2, we can show that q_1 is the only vertex adjacent to no other vertices of Q. This implies $G = H_{m+1}$. Similarly, if alternative (b) is the case, then $G = H_{m+1}$.

Now we consider the case $G' = L_m$. By the definition of L_m, P contains only $2l+2$ and Q contains only $2l$ vertices. As above x can not be adjacent to two vertices of P and two vertices of Q, simultaneously. This observation and the fact that x must be adjacent to at least one vertex of P, since Q contains $2l$ vertices, leave the following alternatives:

(a) x is adjacent to $2l + 2$ vertices of P.

(b) x is adjacent to $2l + 1$ vertices of P and one vertex of Q. If alternative (a) is the case, then $G = H_{m+1}$. By observation of Subcase 1.1, alternative (b) is impossible to happen.

Case 4: *$m = 4l - 1$. Similarly to the other cases, we see that there is a vertex x of degree at most $2l + 1$. However, it is impossible for the degree to be less than or equal to $2l$ because otherwise as in Case 1, W_5 is a subgraph of G. Further, the subgraph G' that is obtained by removing x is H_m. Hence G consists of $G' = H_m$ plus the vertex x which joins to precisely $2l + 1$ vertices of H_m. Clearly P contains only $2l$ vertices and Q contains only $2l - 1$ vertices. As above, it is clear that x can not be adjacent to more than one vertex of both P and Q simultaneously. This observation and the fact that x must be adjacent to at least 2 vertices of P, and Q contains $2l - 1$ vertices, leave the following alternatives: The vertex x is adjacent to every vertex of P and one vertex q_1 of Q. By a similar argument as in alternative (c) of Case 2, q_1 can only be the vertex adjacent to no other vertices of Q. This implies that $G = H_{m+1}$.*

For the case where G has $\varnothing(m+1) + \alpha$ edges where $\alpha \geq 1$, consider H' be a graph obtained from G by deleting any α edges. Then H' has $\varnothing(m + 1)$ edges. By the

above cases H' is either H_m or L_m and so by (2.3) G has an odd wheel. The proof is completed.

Now, we consider edge maximal graphs without W_{2k}. Let $\mathcal{W}(n)$ be the class of complete tripartite graph K_{n_1,n_2,n_3} where $n = n_1 + n_2 + n_3$ is a partition of n into three parts which are as equal as possible. Note that if $G \in \mathcal{W}(n)$, then G is a W_{2k}-free graph and $\mathcal{E}(G) = \lfloor \frac{n^2}{3} \rfloor$. Thus, $f(n; W_{2k}) \geq \lfloor \frac{n^2}{3} \rfloor$. In the following theorem, we determine the edge maximal graphs containing no even wheel.

Theorem 2.2. Let $k \geq 2$ be a positive integer and G be a graph containing no even wheel of order $2k$. Then for $n \geq 6(k-1)$

$$\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{3} \right\rfloor.$$

Furthermore, the bound is best possible.

Proof: Let G be a graph containing no even wheel of order $2k$. Let $u \in V(G)$ such that $\Delta(G) = d_G(u)$, say $d_G(u) = m$ for some positive integer m. If $m < \lceil \frac{2n}{3} \rceil$, then

$$2\mathcal{E}(G) = \sum_{v \in V(G)} d_G(v) < \left\lceil \frac{2n}{3} \right\rceil n.$$

Hence, $\mathcal{E}(G) < \lfloor \frac{n^2}{3} \rfloor$. So we need to consider the case when $m \geq \lceil \frac{2n}{3} \rceil$. Let $N_G(u) = \{v_1, v_2, ..., v_m\}$ be the neighbors of u in G. Define $H_1 = G[v_1, v_2, ..., v_m]$ and $H_2 = G - (H_1 \cup \{u\})$. Observe that H_1 contains no odd cycle of length $2k - 1$ as otherwise G would have W_{2k} as a subgraph. Thus, by Lemma 1.2, $\mathcal{E}(H_1) \leq \left\lfloor \frac{m^2}{4} \right\rfloor$. Further, $d_G(w) \leq m$ for every $w \in V(H_2)$, and hence $\mathcal{E}(H_2) + \mathcal{E}(H_1, H_2) \leq m(n - m - 1)$.

So we have

$$
\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(u, H_1 \cup H_2) + \mathcal{E}(H_1) + \mathcal{E}(H_2) + \mathcal{E}(H_1, H_2) \\
&\leq m + \frac{m^2}{4} + m(n - m - 1) \\
&= m + \frac{m^2}{4} + nm - m^2 - m \\
&= nm - \frac{3m^2}{4}.
\end{aligned}
$$

Define $g(m) = nm - \frac{3m^2}{4}$. Observe that g has the maximum value at $m = \frac{2n}{3}$. So we get that $\mathcal{E}(G) \leq \lfloor \frac{n^2}{3} \rfloor$. One notes that the bound is achievable by $G \in \mathcal{W}(n)$. The proof is completed.

Corollary 2.1 *Let G be a graph on n vertices containing no even wheels. Then*

$$
\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{3} \right\rfloor.
$$

Furthermore, the bound is best possible.

We conclude this work by posing the following problem: Determine the extremal function for the class of graphs that contains no specific odd wheel and characterize the extremal graphs as well.

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