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RINGS ON TORSION-FREE GROUPS OF RANK ONE AND TWO

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Abstract. This paper gives a survey about the possible rings which may be defined over torsion-free groups of rank one and two. In fact, we give a list of such rings which have been studied by some mathematicians over the past decades. In particular, we give a review of the authors' studies to determine such rings.

1. Introduction

The construction of the rings with given additive group and the related problem of characterizing the additive groups of rings satisfying various conditions has been considered by some mathematicians in the past decades. The results for rings over torsion-free Abelian groups are meager. The rings over a given torsion-free group of rank one have been determined, and every such ring is either a zero-ring or isomorphic to a subring of the field R of rational numbers (Redei and Szele [15], and Beaumont and Zuckerman [8]). Szele [18] has given a sufficient condition that an arbitrary torsion-free group be a nil group, and Ree and Wisner [16] have found necessary and sufficient condition that a completely reducible torsion-free group be a nil group. Beaumont [5] gave a construction which included all rings over free Abelian groups, and Fuchs [10] extended this construction to divisible torsion-free groups. Redei [14]

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generalized [5] to obtain algebras with given additive module. Beaumont and Wisner [7] considered torsion-free Abelian groups of rank two and gave a characterization of a group A in terms of groups of rank one. They found a necessary and sufficient condition that there exist a non-commutative ring over an Abelian group A and determined all such rings. They proved that every ring without zero-devisors over A is commutative. Moreover, they showed that a ring R is a ring without zero-divisors over A if and only if R is isomorphic to a subring of a quadratic field extension $R(\alpha)$ of the rationals R. Recently, the authors have studied torsion-free Abelian groups of rank two using their type set. They have considered homogeneous groups in [2] and non-homogeneous groups in [3] and listed all the possible rings over non-homogeneous groups. This paper, reviews those studies over torsion-free groups of rank one and two which give mainly a list of the possible rings over such groups. In order to make the paper self-contained, the proofs of the results related to rings constructions, have been provided with details. Our notations follow from [11].

2. Notations and Preliminaries

Let A be an Abelian group. If every element of A is of finite order, A is called a torsion group, while A is torsion-free if all its elements, except for 0, are of infinite order. Mixed groups contain both nonzero elements of finite order and elements of infinite order. A system $\{a_1, \cdots, a_k\}$ of nonzero elements of A is called linearly independent, or briefly independent, if $n_1a_1 + \cdots n_ka_k = 0$ $(n_i \in \mathbb{Z})$ implies $n_1a_1 = \cdots = n_ka_k = 0$. A system of elements is dependent if it is not independent. An infinite system $\{a_i\}_{i\in I}$ of elements of A is called independent, if every finite subsystem of L is independent. An independent system M of A is maximal if there is no independent system in A containing M properly. By the rank $r(A)$ of a group A is meant the cardinal number of a maximal independent system containing only elements of infinite and prime power orders. It is well-known that the rank of any group A is an invariant of A. A group A is divisible if and only if $nA = A$ for every positive n. A subgroup B of A is called a pure subgroup of A if $nB = B \cap nA$ for every $n \in \mathbb{Z}$. For a subset S of A, the symbol $< S >^*$ denotes the unique minimal pure subgroup of A generated by S. Indeed, $< S >^*$ is the intersection of all pure subgroups of A that contain S. A torsion-free group A is said to be completely decomposable if it is a direct sum of rank one groups. A set $\{A_i\}_{i\in I}$ of torsion-free groups $\neq 0$ is said to be a rigid system if $Hom(A_i, A_j)$ is isomorphic to a subgroup of rational numbers $\mathbb{Q}$ if $i = j$ and $Hom(A_i, A_j) = 0$ if $i \neq j$. A group A is rigid if the singleton $\{A\}$ is a rigid system.

Remark 1. Let A be a rigid group of rank ≥ 2. Then A is a nil group.

Proof. See [11, The comment proceeding Proposition 121.2]. □

Given a prime p, the largest integer k such that p^k divides x in the torsion-free group A ($x \in A$) is called the $p-$height , $h_p(x)$, of x; if no such maximal integer k exists, then we set $h_p^A(x) = \infty$. Now let $p_1, p_2, \cdots, p_n, \cdots$ be an increasing sequence of all primes. Then, the sequence

$$\chi_A(x) = (h_{p_1}^A(x), h_{p_2}^A(x), \cdots, h_{p_n}^A(x), \cdots),$$

is said to be the height-sequence of x. We omit the subscript A if no ambiguity arises. For any two height-sequences $\chi = (k_1, k_2, \cdots, k_n, \cdots)$ and $\mu = (l_1, l_2, \cdots, l_n, \cdots)$, we set $\chi \geq \mu$ if $k_n \geq l_n$ for all n. Moreover, χ and μ will be considered equivalent if $\sum_n |k_n - l_n|$ is finite [we set $\infty - \infty = 0$]. An equivalence class of height-sequences is called a type. If $\chi(x)$ belongs to the type $\mathbf{t}$, then we say that x is of type $\mathbf{t}$. If $\chi = (k_1, k_2, \cdots, k_n, \cdots)$ and $\mu = (l_1, l_2, \cdots, l_n, \cdots)$ are two height-sequences, then their product is defined as $\chi.\mu = (k_1 + l_1, k_2 + l_2, \cdots, k_n + l_n, \cdots)$ where, naturally, the sum of ∞ and anything is ∞. A height-sequence χ is idempotent (i.e., $\chi^2 = \chi$)

exactly if, for every n, either $k_n = 0$ or $k_n = \infty$. The multiplication of height-sequences is compatible with the equivalence relation defined above, so we may speak of the product $t.t_1$ of types t and t_1 and of an idempotent type ($t^2 = t$). For two types t_1, t_2 we have $t_1 \leq t_2$ if there exists $\chi \in t_1$ and $\mu \in t_2$ such that $\chi \leq \mu$. The type set of A is the partially ordered set of types, i.e.,

$$T(A) = \{t(x) \mid x \in A \setminus 0\}.$$

A torsion-free group A in which all non-zero elements are of the same type $\mathbf{t}$ is called homogeneous. In this situation, we use the symbol $t(A)$ to denote the type of the group which is indeed the type of any non-zero element of A. A torsion-free group of rank one is called of nil type if its type is not idempotent. Moreover, for a torsion-free Abelian group A and a type t, the elements a in A whose types are $\geq t$ form a pure subgroup of A which is denoted by $A(t)$.

Given an Abelian group A, we call R a ring over A if the group A is isomorphic to the additive group of R. In this situation we write $R = (A, *)$, where $*$ denotes the ring multiplication. This multiplication is not assumed to be associative. Every group may be turned into a ring in a trivial way, by setting all products equal to zero; such a ring is called a zero-ring. If this is the only multiplication over A, then A is said to be a nil group.

A function $\mu : A \times A \longrightarrow A$ is called a multiplication on A if it satisfies

(1) $\mu(a + b, c) = \mu(a, c) + \mu(b, c)$,

(2) $\mu(a, b + c) = \mu(a, b) + \mu(a, c)$,

for all $a, b, c \in A$. Thus, we may think of a ring R as a pair (A, μ). Now for any two multiplications μ and ν on A, we define

$$(\mu + \nu)(a, b) = \mu(a, b) + \nu(a, b),$$

which is a again multiplication on A. Under this rule of composition, the set of all multiplications on A forms an Abelian group which is denoted by $Mult(A)$.

Theorem 2.1. *Let A be an Abelian group. Then*

$$Mult(A) \cong Hom(A \otimes A, A) \cong Hom(A, End(A)).$$

Proof. See [11, Theorem 118.1]. □

Proposition 2.1. *Let A be a torsion-free group of finite rank. Then the length of every chain in $T(A)$ is at most equal to the rank of A.*

Proof. See [9, Propoition 1]. □

Theorem 2.2. *Let A be a torsion-free group of rank two. If A supports a non-zero ring, then $T(A)$ contains a unique minimal member and at most three elements.*

Proof. See [17, Theorem 3.3]. □

Remark 2. The following cases are realized according to the proof of Theorem 2.2:

(1) If $|T(A)| = 1$, then the type must be idempotent.
(2) If $|T(A)| = 2$, then one type is minimal and the other is maximal.
(3) If $|T(A)| = 3$, then one type is minimal and two are maximal. In this case at least one of the maximal types must be idempotent.

Proposition 2.2. *Let C be a pure subgroup of a torsion-free group A such that:*

(1) *A/C is completely decomposable and homogeneous of type t;*
(2) *all the elements in A but not in C are of type t.*

Then C is a direct summand of A.

Proof. See [11, Proposition 86.5]. □

Proposition 2.3. *(Aghdam* [1]*) If R is a finite rank, torsion-free ring without zero-divisors, then R^+ is homogeneous.*

Proof. Let $\{x_1, x_2, \cdots, x_r\}$ be a maximal independent subset of R^+. Let x be in R, $x \neq 0$. First we prove that $xx_1, xx_2, \cdots, xx_r$ are independent. Suppose not. Then there exist integers $a_1, a_2, \cdots, a_r$, not necessarily all equal to zero, such that

$$a_1xx_1 + a_2xx_2 + \cdots + a_rxx_r = 0.$$

Thus $x(a_1x_1 + a_2x_2 + \cdots + a_rx_r) = 0$; but R has no zero divisors, therefore

$$a_1x_1 + a_2x_2 + \cdots + a_rx_r = 0,$$

contradicting the independence of the set $\{x_1, x_2, \cdots, x_r\}$.

Hence if $x \neq 0 \neq y$ belong to R, then

$$my = m_1xx_1 + m_2xx_2 + \cdots + m_rxx_r = x(m_1x_1 + m_2x_2 + \cdots + m_rx_r),$$

which implies $t(x) \leq t(y)$. And similarly

$$nx = n_1yx_1 + n_2yx_2 + \cdots + n_ryx_r = y(n_1x_1 + n_2x_2 + \cdots + n_rx_r),$$

which implies $t(y) \leq t(x)$. Thus $t(x) = t(y)$, consequently R is homogeneous. □

Proposition 2.4. *(Aghdam* [1]*) If A is an indecomposable and homogeneous group of rank two, then any non-zero element of $End_{\mathbb{Z}}(A)$ is monic.*

Proof. Let $\varphi \in End_{\mathbb{Z}}(A)$, such that $0 \neq Ker(\varphi) \neq A$. Then $A/Ker(\varphi)$ is of rank one since A has rank two and $Ker(\varphi)$ is a pure subgroup of A. We have $A/Ker(\varphi) \cong Im(\varphi) \lneqq A$. Assume $\bar{a} = a + Ker(\varphi) \in A/Ker(\varphi)$ such that a does not belong to $Ker(\varphi)$. Then,

$$t(\bar{a}) = t(A/Ker(\varphi)) = t(Im(\varphi)) \leq t(A) = t(a).$$

On the other hand, $t(\bar{a}) \geq t(a)$, therefore $t(\bar{a}) = t(a)$. Hence by Proposition 2.2, $Ker(\varphi)$ is a summand of A. But A is indecomposable, so $Ker(\varphi) = 0$, and φ is monic. □

Corollary 2.1. *If A is an indecomposable and homogeneous group of rank two, then the endomorphism ring of A is associative and without zero-divisors.*

Proof. The associativity is clear. For the second claim, we suppose that ϕ_1 and ϕ_2 are non-zero elements of $End_{\mathbb{Z}}(A)$ such that $\phi_1.\phi_2 = 0$. Hence, $(\phi_1.\phi_2)(x) = 0$ for all $x \in A$. Now in view of Proposition 2.4, we get $\phi_2(x) = 0$ for all $x \in A$. This implies that $\phi_2 = 0$, a contradiction. □

Let $\{x,\ y\}$ be an independent set of a rank two torsion-free group A. We define $U,\ U_0,\ V$ and V_0 as subgroups of $\mathbb{Q}$ by:

$$U = \{u \in \mathbb{Q} : ux + vy \in A, \text{ for some } v \in \mathbb{Q}\},\ U_0 = \{u_0 \in \mathbb{Q}\ :\ u_0 x \in A\}.$$

$$V = \{v \in \mathbb{Q} : ux + vy \in A, \text{ for some } u \in \mathbb{Q}\},\ V_0 = \{v_0 \in \mathbb{Q}\ :\ v_0 y \in A\}.$$

Clearly, $U_0 \subseteq U,\ \ V_0 \subseteq V$. We call $U,\ U_0,\ V$ and V_0 the groups of rank one belonging to the independent set $\{x,\ y\}$.

Theorem 2.3. *If there exists an associative ring with no non-trivial zero divisors over the torsion-free group A of rank two, then A contains independent elements x and y such that the groups of rank one $U \supseteq U_0$ and $V \supseteq V_0$ belonging to $\{x, y\}$, satisfy the following conditions:*

(1) $U \cong V,\ U_0 \cong V_0$;

(2) *none of the groups U, U_0, V and V_0 are of nil type.*

Proof. See [7, Theorem 5]. □

3. Rank One Groups

The rings over torsion-free groups of rank one are determined in this section. We need the following propositions.

Proposition 3.1. *If A and C are torsion-free groups of rank one, then $A \otimes C$ is of rank one and $t(A \otimes C) = t(A).t(C)$.*

Proof. See [11, Proposition 85.3]. □

Proposition 3.2. *If A and C are torsion-free groups of rank one, then $Hom(A, C)$ is 0 if $t(A)$ is not $\leq t(C)$, and is a torsion-free group of rank one and of type $t(C) : t(A)$ if $t(A) \leq t(C)$.*

Proof. See [11, Proposition 85.4]. □

The following theorem gives the possible rings over torsion-free groups of rank one. The proof is from [11, Theorem 121.1].

Theorem 3.1. *(Redei and Szele* [14], *Beaumont and Zuckerman* [8]*) A torsion-free ring of rank one is either a zero ring or isomorphic to a subring of the rational number field of the form*

$$m\mathbb{Z}(q_j^{-1}; j \in J) \text{ with } (m, q_j) = 1,$$

where $\{q_j\}_{j \in J}$ is the set of primes at which $q_j A = A$. A torsion-free group of rank one is not a nil group if and only if its type is idempotent.

Proof. Let $t = t(A)$. From Propositions 3.1 and 3.2 we infer that for $Mult(A) \cong Hom(A \otimes A, A) \neq 0$, it is necessary and sufficient that $t^2 = t$, or equivalently, t be idempotent. Thus, A is a nil group if and only if t is not idempotent. Now, all the rings on A can be listed easily, under the hypothesis that t is idempotent. Choose $a \neq 0$ in A such that $\chi(a)$ consists of $0s$ and ∞s. If R is a non-zero-ring on A, then $a^2 = ma$

for some rational number $m \neq 0$. Without loss of generality, it can be supposed that m is a positive integer not divisible by any prime q at which $\chi(a)$ is ∞; otherwise a could be replaced by a suitable rational multiple of a with the same characteristic for which $m > 0$ is an integer of the stated kind. If $\{q_j\}_{j\in J}$ is the set of primes at which $\chi(a)$ is infinity [$i.e., q_jA = A$], and if $\mathbb{Z}(q_j^{-1}; j \in J)$ denotes the subring of $\mathbb{Q}$, generated by all the q_j^{-1}, then there is a ring-isomorphism $R \cong m\mathbb{Z}(q_j^{-1}; j \in J)$. In fact, it is readily seen that the map $ra \longmapsto mr$ for $r \in \mathbb{Z}(q_j^{-1}; j \in J)$ is bijective and preserves both addition and multiplication. □

4. Non-Homogeneous Rank Two Groups

In this section, we consider the non-homogeneous groups. According to Remark 2, the type set of the group may have two or three elements.

Proposition 4.1. *(Aghdam [1]) Let A be a torsion-free group of rank two, $T(A) = \{t_1, t_2\}$ and $t_1 < t_2$. Let $x, y \in A$ be such that $t(x) = t_1, t(y) = t_2$. Assume U, U_0, V, V_0 are the rank one groups belonging to $\{x, y\}$. If $t(U_0) = t(U)$ then $< y >^*$ is a direct summand of A. In particular, if $kU \leq U_0$ or $kV \leq V_0$ for some integer $k \neq 0$, then A is decomposable.*

Proof. We have $A/ < y >^* \cong U$, hence $t(A/ < y >^*) = t(U)$. Let a be in A but not in $< y >^*$; then $t(a) = t_1$. By assumption we have $t(U) = t(U_0) = t_1$, therefore the type of all elements in A but not in $< y >^*$ are equal $t(U) = t(A/ < y >^*)$. By Proposition 2.2, $< y >^*$ is a direct summand of A. In particular, if $kU \leq U_0$ or $kV \leq V_0$ for some integer $k \neq 0$, then because of $U/U_0 \cong V/V_0$ we have that $t(U) = t(U_0)$, and hence A is decomposable. □

Theorem 4.1. *(Najafizadeh, Aghdam and Karimi [12]) Let A be a torsion-free group of rank two and $T(A) = \{t_1, t_2\}$ such that $t_1 < t_2$. Let $x, y \in A$ be such that $t(x) = t_1, t(y) = t_2$. Then*

(1) $xy = cy,\ yx = dy,\ y^2 = ey,$ *for some* $c, d, e \in \mathbb{Q}$.

(2) *If* $t_1^2 \neq t_1$ *then* $x^2 = by$ *for some* $b \in \mathbb{Q}$.

(3) *If* $t_2^2 \neq t_2$ *then* $y^2 = 0$.

(4) *If* $t_1 t_2 > t_2$ *then* $xy = yx = 0$.

Proof. 1) The hypothesis $t_1 < t_2$ implies that $t(xy) \geq t(y) = t_2$, hence xy and y belong to $A(t_2)$ which is a rank one subgroup of A. Therefore xy and y are dependent elements. That is $xy = cy$ for some $c \in \mathbb{Q}$. By the same reasoning we deduce that $xy = dy$ and $y^2 = ey$ for some $b, c \in \mathbb{Q}$.
2) Clearly $t(x^2) > t(x) = t_1$ since $t_1^2 \neq t_1$. But $T(A) = \{t_1, t_2\}$, hence $t(x^2) = t_2$. This implies that $x^2 \in A(t_2)$. Thus $x^2 = by$ for some $b \in \mathbb{Q}$.
3) If t_2 is not idempotent then $t(y^2) > t(y) = t_2$. Therefore $t(y^2) \notin T(A)$ which implies $y^2 = 0$.
4) We have $t(xy) \geq t(x)t(y) = t_1 t_2 > t_2$, thus $t(xy) \notin T(A)$, that is $xy = 0$. By the similar way $yx = 0$. □

Theorem 4.2. *(Aghdam [1]) Let A be a torsion-free indecomposable Abelian group of rank two. Let $T(A) = \{t_1, t_2\}$ such that $t_1 < t_2$. If $\{x, y\}$ is an independent set such that $t(x) = t_1, t(y) = t_2$, then all non-trivial rings on A satisfy the following multiplication table:*

$$x^2 = by,\ xy = yx = y^2 = 0,\ b \text{ is a rational number.}$$

Proof. Let $(A, *)$ be a non-trivial ring over A. Since $t_1 < t_2$, in general we have

$$x^2 = ax + by,\ \ xy = cy,\ \ yx = dy,\ \ y^2 = ey.$$

We are going to prove that $a = c = d = e = 0$.

Let U, U_0, V, V_0 be the rank one groups belonging to $\{x, y\}$. We claim $xy = yx$. If not, then $c \neq d$, and for an arbitrary element $g = ux + vy$ of A,

$$gx = ux^2 + vyx, \;\; xg = ux^2 + vxy, \;\; gx - xg = v(d - c)y,$$

implying that $(d - c)v \in V_0$ for all $v \in V$. Hence there is an integer $k \neq 0$ such that $kV \leq V_0$. Now by Proposition 4.1, $< y >^*$ is a direct summand of A, which is a contradiction. Hence $c = d$ and $xy = yx = cy$.

We claim that $a = 0$. If not, take two arbitrary elements $g_1 = ux+vy, \;\; g_2 = rx+sy$ of A. Then

$$g_1g_2 = urx^2 + (su + rv)xy + vsy^2 = aurx + (urb + suc + rvc + vse)y.$$

This implies that $aU^2 \leq U \leq U^2$, whence $t(U) = t(U^2)$. Consequently,

$$\text{if } a \neq 0 \text{ then } t(U) \text{ is idempotent.} \tag{4.1}$$

A is not homogeneous, hence by Proposition 2.3, A should have two non-zero elements $X = rx + sy$ and $Y = \alpha x + \beta y$ such that $XY = 0$, i.e.

$$XY = (rx + sy)(\alpha x + \beta y) = a\alpha rx + (\alpha rb + s\alpha c + r\beta c + \beta se)y = 0.$$

Since x, y are independent elements, $a\alpha r = 0$. By assumption $a \neq 0$, hence we should have one of the following cases:

(1) $\alpha = 0, \;\; r = 0,$

(2) $\alpha = 0, \;\; r \neq 0,$

(3) $\alpha \neq 0, \;\; r = 0.$

In case (1), s and β must be non-zero, as $X \neq 0, Y \neq 0$. Hence, $0 = XY = s\beta y^2 = s\beta ey$, which implies that $e = 0$.

In case (2), $\{X = rx + sy, y\}$ is an independent set of A, and

$$0 = XY = (rx + sy)(\beta y) = \beta(rx + sy)y,$$

since $\alpha = 0$. However, $Y \neq 0$, therefore $\beta \neq 0$ so that

$$Xy = (rx + sy)y = 0. \tag{4.2}$$

Let H, H_0, F, F_0 be the rank one groups belonging to $\{X, y\}$, and let $g = hX + fy$ be an arbitrary element of A where $h \in H, f \in F$. By (4.2) and by the assumption that $y^2 = ey$ we have

$$gy = hXy + fy^2 = efy,$$

so we conclude that ef belongs to F_0 for all $f \in F$. If $e \neq 0$ then there is an integer $k \neq 0$ such that $kF \leq F_0$, so by Proposition 4.1, $< y >^*$ is a direct summand of A, contradicting the indecomposability of A. Hence $e = 0$.
Similarly, in case (3) we also conclude that $e = 0$. Therefore,

$$\text{if } \ a \neq 0 \ \text{ then } \ e = 0. \tag{4.3}$$

Let $g = ux + vy$ be an arbitrary element of A with $u \in U, v \in V$. By (4.3) we have $gy = uxy + vy^2 = cuy$, so if c is not zero then $cU \leq V_0$, hence

$$t(U) \leq t(V_0). \tag{4.4}$$

Now, using (4.1) and (4.4) we prove that $t(U) = t(U_0)$. By (4.1), $t(U)$ is idempotent, therefore $h_p^U(1) = 0$ or ∞ except for finitely many prime numbers. $U_0 \leq U$ implies that $t(U_0) \leq t(U)$, so that $h_p^U(1) = 0$ implies $h_p^{U_0}(1) = 0$ and $h_p^U(1) < \infty$ implies $h_p^{U_0}(1) < \infty$. It remains to prove that $h_p^{U_0}(1) = \infty$ if $h_p^U(1) = \infty$. Let $1/p^n \in U$ and $h_p^U(1) = \infty$. Then by the definition of U there is $K/m \in V$ such that $g = (1/p^n)x + (K/m)y \in A$. Let $m = m'p^i$ where $(m', p) = 1$. Then

$$g = (1/p^n)x + (K/m'p^i)y, \;\; m'g = (m'/p^n)x + (K/p^i)y, \;\; (m'g - K(y/p^i)) = (m'/p^n)x.$$

By (4.4), $1/p^i \in V_0$, so that $1/p^n \in U_0$. This is correct for all $n < \infty$, hence $h_p^{U_0}(1) = \infty$, so we conclude that $t(U_0) \leq t(U)$. But $t(U_0) \leq t(U)$, therefore $t(U) = t(U_0)$. By

Proposition 4.1, $< y >^*$ will be a direct summand of A which is in contradiction with indecomposability. Consequently $c = 0$.

By assuming $a \neq 0$ we got $c = 0$ and $e = 0$, that is $x^2 = ax + by,\ \ xy = yx = y^2 = 0$. Thus $\{z = ax + by, y\}$ is an independent set of A, and $z^2 = a^2x^2 + b^2y^2 + 2abxy = a^2x^2 = a^2z,\ \ zy = yz = y^2 = 0$. Let W, W_0, T, T_0 be the rank one groups belonging to $\{z, y\}$. Let $g = wz + ty$ be an arbitrary element of A and $w \in W, t \in T$. Then $gz = wz^2 = a^2wz$.

Since we supposed $a \neq 0$, we have $a^2W \leq W_0 \leq W$, hence $t(W_0) \leq t(W)$. Again by Proposition 4.1, $< y >^*$ is a direct summand of A which is a contradiction. All contradictions are due to the assumption $a \neq 0$. Consequently $a = 0$.

So far we proved that

$$x^2 = by,\ \ xy = yx = cy,\ \ y^2 = ey.$$

Let $g = ux + vy$ be an arbitrary element of A. Then

$$gx = uby + vcy = (ub + cv)y,\ \ gy = cuy + evy = (cu + ev)y$$

hence

$$ub + cv = v_0,\ \ cu + ev = v_0' \text{ for some } v_0,\ \ v_0' \text{ in } V_0.$$

This implies that $(c^2 - be)v = v_0''$ for some v_0'' in V_0. If $c^2 - be \neq 0$ then there is an integer $k \neq 0$ such that $kU \leq U_0$, which implies by Proposition 4.1 that $< y >^*$ is a direct summand of A. This is a contradiction. Therefore

$$c^2 - be = 0. \tag{4.5}$$

If $b = 0$ then $gy = uxy + vy^2 = evy$. Again this is a contradiction, hence

$$b \neq 0. \tag{4.6}$$

By (4.5) and (4.6)

$$e = 0 \text{ if and only if } c = 0. \tag{4.7}$$

If $e \neq 0$ and $c \neq 0$ then $\{z_1 = -cx + by, y\}$ is an independent set of A. We get

$$z_1^2 = (-cx + by)^2 = c^2x^2 + b^2y^2 - 2cbxy = c^2by + eb^2y - 2c^2by = b(eb - c^2)y = 0,$$

$$z_1y = yz_1 = -cxy + by^2 = -c^2y + eby = (-c^2 + eb)y = 0 \text{ (by (4.5))},$$

$$y^2 = ey.$$

Let M, M_0, N, N_0 be the rank one groups belonging to $\{z_1, y\}$, and let $g = mz_1 + ny$ be an arbitrary element of A where $m \in M$ and $n \in N$. Then $gy = ny^2 = eny$, hence $eN \leq N_0$. It follows now that there is an integer $k \neq 0$ such that $kN \leq N_0$, so by Proposition 4.1, $< y >^*$ is a direct summand of A, contradicting the indecomposability of A. Therefore $c = 0$ or $e = 0$, whence by (4.7) $c = 0$ and $e = 0$. □

Theorem 4.3. *(Aghdam and Najafizadeh* [3]*) Let A be a torsion-free group of rank two and $T(A) = \{t_0, t_1, t_2\}$ such that $t_0 < t_1$ and $t_0 < t_2$. Let $x, y \in A$ such that $t(x) = t_1$ and $t(y) = t_2$. If t_1, t_2 are incomparable, then any ring on A satisfies $x^2 = ax$, $y^2 = by$, $xy = yx = 0$ for some $a, b \in \mathbb{Q}$.*

Proof. Let $z \in A$ such that $t(z) = t_0$. Then $z \notin A(t_1)$. But since $A(t_1)$ is a pure subgroup of A, it is of rank one. Now since $t(x^2) \geq t(x) = t_1$, both x^2 and x belong to $A(t_1)$ so they are dependent, that is, $x^2 = ax$ for some $a \in \mathbb{Q}$. Similarly, $y^2 = by$ for some $b \in \mathbb{Q}$.

On the other hand, $t(yx) \geq t(x)$, so yx and x belong to $A(t_1)$, therefore $yx = ex$ for some $e \in \mathbb{Q}$ and similarly $yx = fy$ for some $f \in \mathbb{Q}$. Now if $yx \neq 0$ then $t(x) = t(xy) = t(y)$, contrary to our hypothesis, therefore $yx = 0$. By the same reasoning, $xy = 0$. □

Theorem 4.4. *(Aghdam and Najafizadeh [3]) Let A be a torsion-free group of rank two and $T(A) = \{t_0, t_1, t_2\}$ such that $t_0 < t_1$ and $t_0 < t_2$. Let $x, y \in A$ such that $t(x) = t_1$ and $t(y) = t_2$. If $t_1^2 = t_1$ and $t_2^2 \neq t_2$, then any ring on A satisfies $x^2 = ax$, $y^2 = xy = yx = 0$ for some $a \in \mathbb{Q}$.*

Proof. The hypotheses ensure that $t_1 \neq t_2$. Moreover, in view of Proposition 2.1, t_1 and t_2 are incomparable. Now let R be an arbitrary non-trivial ring on A. Then by Theorem 4.3,

$$x^2 = ex, \; xy = yx = 0, \; y^2 = by,$$

for some $a, b \in \mathbb{Q}$. If $b \neq 0$, then $t(y) = t(y^2) \geq t^2(y)$, which implies that $t(y)$ is idempotent, a contradiction to our hypothesis, so $y^2 = 0$. Furthermore, since R is non-trivial, a is non-zero. □

Theorem 4.5. *(Aghdam and Najafizadeh [3]) Let A be a torsion-free group of rank two and $T(A) = \{t_0, t_1, t_2\}$ such that $t_0 < t_1$, $t_0 < t_2$ and t_1, t_2 incomparable. Let $x, y \in A$ such that $t(x) = t_1$ and $t(y) = t_2$. If $t_1^2 = t_1$ and $t_2^2 = t_2$, then any ring on A satisfies $x^2 = ax$, $y^2 = by$, $xy = yx = 0$ for some rational numbers a and b which are not both zero.*

Proof. Follows from Theorem 4.3. □

Proposition 4.2. *(Aghdam, Karimi and Najafizadeh [4]) Let $A = A_1 \oplus A_2$ be a completely decomposable non-homogeneous group of rank two with $t(A_1) = t_1$ and $t(A_2) = t_2$. Let x and y be non-zero elements of A_1 and A_2 respectively. If A is non-nil, then any ring on A satisfies one of the following cases:*

(1) *$T(A) = \{t_0, t_1, t_2\}$ with $t_0 < t_1$ and $t_0 < t_2$.*
   (a) *t_1 and t_2 are incomparable and in general $x^2 = ax$, $y^2 = by$, $xy = yx = 0$ for some $a \in U_0$ and $b \in V_0$.*
   (b) *If $t_1^2 = t_1$ and $t_2^2 \neq t_2$, then $x^2 = ax$, $y^2 = xy = yx = 0$ for some $a \in U_0$.*

(c) *If* $t_1^2 = t_1$ *and* $t_2^2 = t_2$, *then* $x^2 = ax$, $y^2 = by$, $xy = yx = 0$ *for some* $a \in U_0$ *and* $b \in V_0$.

(2) $T(A) = \{t_1, t_2\}$ *with* $t_1 < t_2$.

(d) *If* $t_1^2 \neq t_1$ *and* $t_2^2 = t_2$, *then* $x^2 = ay$, $y^2 = by$, $xy = cy$, $yx = dy$ *for some* $a, b, c, d, f \in V_0$.

(e) *If* $t_1^2 \neq t_1$ *and* $t_2^2 \neq t_2$, *then* $x^2 = ay$, $y^2 = 0$, $xy = cy$, $yx = dy$, *for some* $a, c, d \in V_0$.

(f) *If* $t_1^2 = t_1$ *and* $t_2^2 = t_2$, *then* $x^2 = a'x + by$, $y^2 = cy$, $xy = dy$, $yx = fy$, *for some* $a' \in U_0$ *and* $b, c, d, f \in V_0$, *in which if* $b \neq 0$, *then* $a' \neq 0$.

(g) *If* $t_1^2 = t_1$ *and* $t_2^2 \neq t_2$, *then* $x^2 = a'x + by$, $y^2 = 0$, $xy = dy$, $yx = fy$, *for some* $a' \in U_0$ *and* $a, b, c, d, f \in V_0$, *in which if* $b \neq 0$, *then* $a' \neq 0$.

Proof. 1) See [3, Proposition 2.7, Lemma 3.1 and Lemma 3.3].
2) (d) Clearly, $t(x^2) \geqq t(x)^2 = t_1^2 > t_1$. Now the hypothesis that $T(A)$ contains two elements, implies that $t(x^2) = t_2$. This yields $x^2 = ay$ for some $a \in V_0$. By the same reasoning, the other parts are obtained. □

5. Homogeneous Rank Two Groups

In this section, we investigate the rings over rank two homogeneous groups. The works are due to Aghdam and Najafizadeh [3]. These results give some properties of the rings over homogeneous rank two groups; and are not about the rings structures over such groups. At first, we recall some notions related to this case. An element x in a ring R is called a semi-identity element if there exist an integer $m \geq 1$ such that $xy = yx = my$ for all $y \in R$. A generalized number n is a formal infinite product of powers of primes. The exponent of a prime p in a generalized number n is denoted by $h_p(n)$ which may be a natural number, 0 or ∞. Generalized numbers are multiplied by adding corresponding exponents with the convention that $\infty + a = \infty$ for all a. A natural number is identified with the generalized number by its prime decomposition. Two generalized numbers n and m are said to be equivalent if there exist natural numbers a and b with $an = bm$.

Let A be a torsion-free Abelian group. Following [7], we define the nucleus $N = N(A)$ of A by

$$N = \{\alpha \in \mathbb{Q} \ : \ \alpha.x \in A, \ \forall x \in A\}.$$

Clearly, N is a subgroup of $\mathbb{Q}$. Moreover, N is a subring of $\mathbb{Q}$, which implies that the type of N is idempotent. It is clear from the definition that A may be treated as an $N-$module. A consequence of this observation is that every element of A has the type greater than or equal to the type of N. If A is homogeneous and the type of A is idempotent, then type of N is equal with the type of A. Indeed, N may be embedded in the endomorphism ring $End_{\mathbb{Z}}(A)$ as the pure subring of $End_{\mathbb{Z}}(A)$ generated by the identity I_A of $End_{\mathbb{Z}}(A)$.

Let A be an indecomposable homogeneous group with idempotent type and R a subring of rational numbers containing 1 and having the same type as $t(A) = t(N)$. Let $\{x, y\}$ be an independent set of A. Then, $A/(Rx + Ry)$ is a torsion group. In particular,

$$A/(Rx + Ry) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}_{p_i^{\alpha_i}}.$$

We call the generalized number $n = \prod_{i=1}^{\infty} p_i^{\alpha_i}$ the cocharacteristic of (x, y) in A. The cotype of A is the set of cocharacteristics of independent pairs in A and is denoted by $cot(A)$. A generalized number n is idempotent if and only if $h_p(n)$ is 0 or ∞ for all primes p, but a finite number of primes. The cotype of A is idempotent if there exists an idempotent cocharacteristic in A.

Proposition 5.1. *Let n be a cocharacteristics in A. Then m is also a cocharacteristic in A if and only if m is equivalent to n.*

Proof. See [13, Proposition 2.7]. □

Proposition 5.2. *The cotype of A is an equivalence class of generalized numbers.*

Proof. See [13, Proposition 2.8]. □

Let A and B be torsion-free Abelian groups of rank two. Let n and m be cocharacteristics of A and B respectively. Moreover, suppose that

$$\chi(n) = (\alpha_1, \alpha_2, \cdots, \alpha_i, \cdots), \ \ \chi(m) = (\beta_1, \beta_2, \cdots, \beta_i, \cdots)$$

where $\alpha_i = h_{p_i}(n)$ and $\beta_i = h_{p_i}(m)$. Then, we say that $cot(A) \leq cot(B)$ exactly if

(1) $\alpha_i = \beta_i$ if $\alpha_i = \infty$,

(2) $\alpha_i \leq \beta_i$ for all but a finite number of i.

By Proposition 5.1, this definition of the inequality is independent of the cotype representatives chosen. We recall that given two torsion-free groups A and B, we say that they are quasi-isomorphic if there exist subgroups $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cong B'$ and A', B' have finite index in A and B respectively. In this situation, we write $A \sim B$.

Proposition 5.3. *(Aghdam and Najafizadeh* [2]*) Let A and B be homogeneous torsion-free Abelian groups of rank two such that $A \subseteq B$. If $t(A) = t(B)$, then $cot(A) \leq cot(B)$.*

Proof. Straightforward. □

Theorem 5.1. *(Aghdam and Najafizadeh* [2]*) Let A be a torsion-free Abelian group of rank two. If there exists an associative ring without zero-divisors over A, then $cot(A)$ is idempotent.*

Proof. By Theorem 2.3, A contains independent elements x and y such that U, U_0, V and V_0, the groups of rank one belonging to $\{x, y\}$, are idempotent. Now we define

$$\phi : A/ < x >^* \oplus < y >^* \longrightarrow U/U_0$$

$$\phi(\bar{a}) = \bar{u}$$

where $a = ux + vy$. Clearly, ϕ is onto. Moreover, If $\phi(\bar{a}) = 0$ for some $a = ux + vy$, then $u \in U_0$, that is $(a - ux) = vy \in < y >^*$. Hence $a \in < x >^* \oplus < y >^*$. We deduce that $Ker(\phi) = 0$. Consequently,

$$A/ < x >^* \oplus < y >^* \cong U/U_0 \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}_{p_i^{\alpha_i}}$$

with only finitely many $\alpha_i \in (0, \infty)$. By Proposition 2.3, A is homogeneous. Let R be a subring of the rational numbers containing 1 and $t(A) = t(R)$. Take $X = x/m$ and $Y = y/n$ with integers $m, n \geq 1$ and $h_p(X), h_p(Y) \in \{0, \infty\}$ for all primes p. Then $RX =< x >^*$ and $RY =< y >^*$. Moreover, $\{X, Y\}$ is an independent set in A. Thus,

$$A/RX \oplus RY = A/ < x >^* \oplus < y >^* \cong U/U_0 \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}_{p_i^{\alpha_i}}$$

with only finitely many $\alpha_i \in (0, \infty)$. Therefore, $cot(A)$ is idempotent. □

Theorem 5.2. *(Aghdam and Najafizadeh* [2]*) Let A be an indecomposable homogeneous torsion-free group of rank two. Then, $End_{\mathbb{Z}}(A)$ is a homogeneous group with rank less than or equal to two. Moreover, if $t(A)$ is idempotent, then*

1. *$t(A) = t(End_{\mathbb{Z}}(A))$.*
2. *If $End_{\mathbb{Z}}(A)$ is of rank two, then $cot(End_{\mathbb{Z}}(A)) \leq cot(A)$. Furthermore, $End_{\mathbb{Z}}(A)$ has idempotent cotype.*

Proof. Let x_0 be a fixed non-zero element of A. Now we define

$$\psi : End_{\mathbb{Z}}(A) \longrightarrow A$$

$$\psi(\phi) = \phi(x_0),$$

where $\phi \in End_{\mathbb{Z}}(A)$. Let $\psi(\phi) = 0$. Then $\phi(x_0) = 0$. But $x_0 \neq 0$ hence, in view of Proposition 2.4, $\phi = 0$. We conclude that ψ is a monomorphism. Therefore,

$$End_{\mathbb{Z}}(A) \cong Im(\psi) \subseteq A. \tag{5.1}$$

This implies that $End_{\mathbb{Z}}(A)$ has rank less than or equal to the rank of A. Hence, the rank of $End_{\mathbb{Z}}(A)$ is less than or equal to two. Now from Proposition 2.3 and Corollary 2.1, we get that $End_{\mathbb{Z}}(A)$ is a homogeneous group.

For the rest of proof, we suppose that $t(A)$ is idempotent. Then,

(1) We observe that $End_{\mathbb{Z}}(A)$ is homogeneous. Hence

$$t(A) = t(< I_A >^*) = t(End_{\mathbb{Z}}(A)). \tag{5.2}$$

(2) By Corollary 2.1, $End_{\mathbb{Z}}(A)$ is associative and without zero-divisors. Hence, if $End_{\mathbb{Z}}(A)$ has rank two, then by Theorem 5.1, $End_{\mathbb{Z}}(A)$ has idempotent cotype. Now in view of (5.1), (5.2) and Proposition 5.3, we have

$$cot(End_{\mathbb{Z}}(A)) \leq cot(A).$$

□

Theorem 5.3. *(Aghdam and Najafizadeh* [2]*) Let A be an indecomposable homogeneous torsion-free group of rank two. If A is a non-nil group, then $cot(A) = cot(End_{\mathbb{Z}}(A))$. Moreover, $cot(A)$ and $cot(End_{\mathbb{Z}}(A))$ are idempotent.*

Proof. Let $(A, .)$ be a non-trivial ring over A. If $End_{\mathbb{Z}}(A)$ has rank one, then A is a rigid group. Hence, by Remark 1, A is a nil group, a contradiction. Therefore, by Theorem 5.2, $End_{\mathbb{Z}}(A)$ has rank equal to two. Moreover, by part (2) of Theorem 5.2, $cot(End_{\mathbb{Z}}(A))$ is idempotent and

$$cot(End_{\mathbb{Z}}(A)) \leq cot(A). \tag{5.3}$$

Now we define

$$\phi : A \longrightarrow End_{\mathbb{Z}}(A)$$

$$\phi(x) = L_x$$

where $L_x : A \longrightarrow A$ is the left multiplication by x. In view of Proposition 2.4, ϕ is a monomorphism. Hence, $A \cong Im(\phi) \leq End_{\mathbb{Z}}(A)$. By part (2) of Theorem 5.2, $t(A) = t(End_{\mathbb{Z}}(A))$. Now by Proposition 5.3, we get

$$cot(A) \leq cot(End_{\mathbb{Z}}(A)). \tag{5.4}$$

Consequently, by (5.3) and (5.4) we have $cot(A) = cot(End_{\mathbb{Z}}(A))$. Moreover, both $cot(A)$ and $cot(End_{\mathbb{Z}}(A))$ are idempotent. □

Proposition 5.4. *(Aghdam and Najafizadeh* [2]*) Let A and B be torsion-free groups such that $A \sim B$. Then, B is a non-nil group exactly if A is.*

Proof. Let $A \sim B$. Then, there exists an isomorphism $\phi : A \to B$ and an integer $n \geq 1$ such that $nB \subseteq \phi(A)$. If B a non-nil group, then define the multiplication in A as follows:

$$x.y = \phi^{-1}[n\phi(x)\phi(y)].$$

□

Theorem 5.4. *(Aghdam and Najafizadeh* [2]*) Let A be an indecomposable homogeneous torsion-free group of rank two. Then A is a non-nil group if and only if $A \sim End_{\mathbb{Z}}(A)$.*

Proof. $\Longrightarrow$) Let A be a non-nil group. Then by Theorem 5.3,

$$cot(A) = cot(End_{\mathbb{Z}}(A)).$$

Now in view of (5.1) in the proof of the Theorem 5.2, it is clear that the index of $Im(\alpha)$ in $End_{\mathbb{Z}}(A)$ is finite. Consequently $A \sim End_{\mathbb{Z}}(A)$.

$\Longleftarrow$) Suppose that $A \sim End_{\mathbb{Z}}(A)$. Then there exist a ring over the endomorphism group $End_{\mathbb{Z}}(A)$. Hence, by Proposition 5.4, A is a non-nil group. □

For the case of indecomposable homogeneous rank two torsion-free groups, we have the following theorem.

Theorem 5.5. *(Aghdam* [1], *Aghdam and Najafizadeh* [2]*) Let A be an indecomposable homogeneous torsion-free group of rank two. Then any non-trivial ring over A is without zero divisors. Moreover, such a ring is associative, commutative and contains a semi-identity.*

Proof. Let $(A, .)$ be a ring over A and let $xy = 0$ for some $x, y \in A$, $x \neq 0$, $y \neq 0$. By Proposition 2.4, any non-trivial element of $End_{\mathbb{Z}}(A)$ is monic. For the left multiplication L_x we have $L_x(y) = xy = 0$, which implies that $L_x = 0$, so

$$x^2 = L_x(x) = 0. \tag{5.5}$$

Let $\{x, z\}$ be an independent set of A. Then we have

$$xz = L_x(z) = 0. \tag{5.6}$$

Furthermore, since the right multiplication R_z is 0 or monic, and $R_z(x) = xz = 0$, therefore $R_z = 0$. Hence,

$$z^2 = R_z(z) = 0. \tag{5.7}$$

Taking now the left multiplication L_z, by (5.7) we get that L_z is 0, so

$$zx = L_z(x) = 0. \tag{5.8}$$

By assumption $\{x, z\}$ is an independent set of A, consequently by (5.5), (5.6), (5.7) and (5.8), $(A, .)$ is a trivial ring, a contradiction. This shows that any non-trivial ring over A is without zero-divisors.

Clearly, $\phi_1 : A \longrightarrow End_{\mathbb{Z}}(A)$ and $\phi_2 : A \longrightarrow End_{\mathbb{Z}}(A)$ which are defined as $\phi_1(x) = L_x$ and $\phi_2(x) = R_x$ respectively, for any $x \in A$, are group homomorphisms. By Proposition 2.4, ϕ_1 and ϕ_2 are monic, hence

$$A \cong Im(\phi_1) = \{L_x : x \in A\}, \;\; A \cong Im(\phi_2) = \{R_x : x \in A\}.$$

By Lemma 5.4, $A \sim End_{\mathbb{Z}}(A)$. Hence,

$$Im(\phi_1) \sim End_{\mathbb{Z}}(A), \;\; Im(\phi_2) \sim End_{\mathbb{Z}}(A).$$

This shows that there exist integer numbers $m, n > 1$ such that

$$mEnd_{\mathbb{Z}}(A) \subseteq Im(\phi_1), \;\; nEnd_{\mathbb{Z}}(A) \subseteq Im(\phi_2).$$

We deduce that there exist x, w in A such that $mI_A = L_x$ and $nI_A = R_w$. Thereby, for all $y \in A$ we have

$$my = xy \;\text{ and }\; ny = yw. \tag{5.9}$$

In particular,

$$mw = xw \;\text{ and }\; nx = xw.$$

The latest equation implies that $mw = nx$, hence $w = nx/m$. Moreover, (5.9) implies that $ny = y(nx/m)$. Hence $my = yx$. Now in view of (5.9), we have $xy = yx = my$ for every $y \in A$. Therefore, x is a semi-identity for the ring $(A, .)$.

Now let $\{x, y\}$ be an independent subset of A where x is a semi-identity. Then $xy = yx$ implies that the ring is commutative.

Now we prove that this ring is associative. To do this, we only need to prove the followings:

(1) $x^2x = xx^2$,

(2) $x^2y = x(xy)$,

(3) $(xy)x = x(yx)$,

(4) $(xy)y = xy^2$,

(5) $y^2x = y(yx)$,

(6) $y^2y = yy^2$.

We observe that (1) and (6) hold by the commutativity of the ring. For the others, we use:

(1) $x^2 = mx$,

(2) $xy = yx = my$,

(3) $y^2x = xy^2 = my^2$.

□

For the case of decomposable homogeneous rank two torsion-free groups, we have the following proposition.

Proposition 5.5. *Let $A = A_1 \oplus A_2$ be a completely decomposable homogeneous torsion-free group of rank two. Let x and y be non-zero elements of A_1 and A_2, respectively. If A is non-nil, then*

$$x^2 = a_1x + b_1y,\ y^2 = a_2x + b_2y,\ xy = a_3x + b_3y,\ yx = a_4x + b_4y,$$

for some $a_i, b_i \in \mathbb{Q},\ i = 1, 2, 3, 4.$

Proof. Obvious. □

6. Commutative and Non-Commutative Rank Two Rings

In this section, a necessary and sufficient condition for the existence of a non-commutative ring over an Abelian group A is given. Moreover, all such rings are determined. The works are due to R. A. Beaumont and R. J. Wisner [7].

Let A be a torsion-free Abelian group, let $x_1, x_2, \cdots, x_n$ be elements of A, and let $a_1/b_1, a_2/b_2, \cdots, a_n/b_n$ be rational numbers. Denote the least common multiple of the integers b_i $(i = 1, 2, \cdots, n)$ by $[b_i]$. If the equation

$$[b_i]x = \sum_{j=1}^{n}[b_i](a_j/b_j)x_j$$

has a solution $x \in A$, then this solution is unique and we write

$$x = \sum_{j=1}^{n}(a_j/b_j)x_j.$$

Let $x = \sum_{j=1}^{n} r_j x_j$ and $y = \sum_{j=1}^{n} s_j x_j$, where r_i and s_i $(i = 1, 2, \cdots, n)$ are rational numbers, be elements of A as described above. Then it is routine to check that

$$x \pm y = \sum_{j=1}^{n}(r_j \pm s_j)x_j.$$

Further, if R is a ring with A as its additive group, the distributive laws in R yield

$$x.y = \sum_{i,j=1}^{n}(r_j.s_j)(x_i.x_j). \tag{6.1}$$

Now let x ba an element of a torsion-free Abelian group A, and R_x be the set of rational numbers r such that $rx \in A$. We recall that the nucleus D of A is $D = \bigcap_{x \in A} R_x$.

Lemma 6.1. *(Beaumont and Wisner* [7]*) Let R be a ring over a torsion-free group A of rank two. If R is non-commutative, then the elements z and z^2 are dependent elements of A for every $z \in R$. If R is commutative and if R contains an element $x \in R$ such that $x^2 \neq 0$, then there exist an element $z \in R$ such that z and z^2 are independent elements of A.*

Proof. Suppose first that R is non-commutative and let $z \in R$. If z and z^2 were independent, then every element of A could be written $rz + sz^2$ for rationals r, s, and this implies that R is commutative by (6.1).

Now suppose that R is commutative and that R contains an element x such that $x^2 \neq 0$. Assume that z and z^2 are dependent for every $z \in R$. Let x and y be independent elements of A. Then we have

$$x^2 = rx,\ \ y^2 = sy,\ \ xy = yx = tx + uy,$$

for rationals r, s, t, u, where we may assume that $r \neq 0$. We obtain $x^2y = rxy$ and $x^2y = tx^2 + uxy = rtx + uxy$. Hence $(r - u)xy = rtx$, and we consider two cases:

Case I: $r = u$. Here $t = 0$ and $xy = yx = uy = ry$. We have $(x + y)^2 = rx + ry + ry + sy$. By hypothesis, $(x + y)^2 = a(x + y)$ for some rational number a. Hence $(r - a)x + (2r + s - a)y = 0$, and this implies $r = a$, $r + s = 0$. Similarly $(x - y)^2 = rx - ry - ry + sy$ and $(x - y)^2 = b(x - y)$ yields $r = b,\ \ r = s$. Hence $r + s = 2r = 0$, which is a contradiction.

Case II: $r \neq u$. Here $(r - u)xy = rtx$ combined with

$$(r - u)xy = (r - u)tx + (r - u)uy$$

yields $u = 0$. From $y^2x = syx$ and $y^2x = tyx + uy^2 = tyx$, we obtain either $s = t$ in which case $xy = yx = sx$, or $xy = yx = 0$. As in case I, the computation of $(x + y)^2$ and $(x - y)^2$ yields $r = 0$, whichever of the alternatives holds.

Hence in each case, the assumption that z and z^2 are dependent for every $z \in R$ leads to a contradiction, and this completes the proof of the lemma. □

Lemma 6.2. *(Beaumont and Wisner* [7]*) Let R be a non-commutative ring over a torsion-free group A of rank two. Then there exist independent elements $x,\ y \in A$ and a rational number $a \neq 0$ in the nucleus D of A such that x and y satisfy one of the following multiplication tables:*

$$x^2 = ax,\ \ xy = ay,\ \ yx = 0,\ \ y^2 = 0; \tag{6.2}$$

$$x^2 = ax,\ \ xy = 0,\ \ yx = ay,\ \ y^2 = 0. \tag{6.3}$$

Proof. By Lemma 6.1, for every $z \in A$, the elements z and z^2 are dependent elements of A. Let $A =< x, y >^*$ where x and y are the independent elements of A, let $x^2 = ax,\ y^2 = by,\ xy = cx + dy,\ yx = ex + fy$, where a, b, c, d, e, f are rational multipliers. Let $K, L \in R$, where $K = mx + ny$ and $L = sx + ty$. By (6.1),

$$KL = msx^2 + mtxy + nsyx + nty^2.$$

If $a = b = 0$, then $KL = mtxy + nsyx$, and R would have a trivial commutative multiplication if $xy = yx = 0$. Hence not both $xy = 0$ and $yx = 0$, say $xy \neq 0$. Then $0 = xy^2 = cxy + dy^2 = cxy$ implies $c = 0$, and $0 = x^2y = dxy$ implies $d = 0$. But then $xy = cx + dy = 0$, which is a contradiction. Similarly $yx \neq 0$ leads to a contradiction.

Hence we may assume not both $a = 0$ and $b = 0$ and we consider the case $a \neq 0,\ b = 0$. By calculating xy^2, y^2x, x^2y, yx^2 we find $c = e = f = 0,\ a = d$. Thus we have

$$x^2 = ax,\ y^2 = 0,\ xy = ay,\ yx = 0.$$

If $a = 0,\ b \neq 0$, we similarly obtain

$$x^2 = 0,\ y^2 = by,\ xy = 0,\ yx = bx.$$

Suppose now that $a \neq 0, b \neq 0$. If $xy = 0$ then $0 = (xy)x = x(yx) = ex^2 + fxy = ex^2$ implies $e = 0$, and $0 = y(xy) = (yx)y = exy + fy^2 = fy^2$ implies $f = 0$, so that $yx = ex + fy = 0$. But with $xy = yx = 0$, R would be commutative. Hence either $c \neq 0$ or $d \neq 0$. Since $axy = x^2y = x(xy) = cx^2 + dxy = acx + dxy$, we have $(a - d)xy = acx$. Now $a = d$ implies $c = 0$, and $a \neq d$ implies $d = 0$ since otherwise there would be a dependency between x and y. Thus we have two cases to consider.

Case I: $c = 0,\ d \neq 0$. Then $a = d$ and $xy = ay$.

Case II: $c \neq 0,\ d = 0$. Then $bxy = xy^2 = (xy)y = cxy$ implies $b = c$. So that $xy = bx$.

By an analysis similar to the above, we can show that either $yx = bx$ or $yx = ay$. Thus there are two apparent cases where R is not commutative.

$$x^2 = ax, \; xy = ay, \; yx = bx, \; y^2 = by; \tag{6.4}$$

$$x^2 = ax, \; xy = bx, \; yx = ay, \; y^2 = by. \tag{6.5}$$

Now let $z = xy - yx$. If (6.4) holds, then $z = ay - bx$ and x and z are independent elements of A such that

$$x^2 = ax, \; xz = az, \; zx = 0, \; z^2 = 0, \; a \neq 0. \tag{6.6}$$

If (6.5) holds, $z = bx - ay$ and we have

$$x^2 = ax, \; xz = 0, \; zx = az, \; z^2 = 0. \tag{6.7}$$

We complete the proof of the lemma by showing that $a \in D$. Let $mx + nz$ be an arbitrary element of A. Then by (6.6) $x(mx + nz) = mx^2 + nxz = max + naz = a(mx + nz) \in A$. Hence $a \in D$. Similarly if (6.7) holds, $(mx + nz)x = max + naz = a(mx + nz) \in A$. □

Lemma 6.3. *(Beaumont and Wisner [7]) Let R be a ring over a torsion-free group A of rank two such that there exist independent elements $x, y \in A$ which satisfy (6.2) or (6.3) of Lemma 6.2, and let U be the group of rank one belonging to x. Then $aU \subseteq D$.*

Proof. Let $r \in U$. Then there exists $z \in A$ such that $z = rx + sy$. Let $w = ex + fy$ be an arbitrary element of A. Then by (6.2), $zw = rfax + rfay = arw \in A$. Hence $ar \in D$ for all $r \in U$. Similarly by (6.3), $wz = arw \in A$. □

Theorem 6.1. *(Beaumont and Wisner [7]) Let A be a torsion-free group of rank two. Then R is a non-commutative ring over A if and only if multiplication in A is defined by $xy = \xi(x)y$ or $xy = \xi(y)x$, for $x, y \in A$, where ξ is a non-trivial homomorphism of A into the nucleus D of A.*

Proof. If R is a non-commutative ring over A, then by Lemma 6.2, $A =< x_1, x_2 >^*$ where x_1, x_2 satisfy (6.2) or (6.3). Suppose (6.2) is satisfied. For $rx_1 + sx_2$, $mx_1 + nx_2 \in A$, we have

$$(rx_1 + sx_2)(mx_1 + nx_2) = rmax_1 + rnax_2 = ra(mx_1 + nx_2).$$

It follows from Lemma 6.3 that the mapping $\xi : A \longrightarrow D$ defined $\xi(rx_1 + sx_2) = ra$ is a non-trivial homomorphism of A into D. Similarly, if (6.3) is satisfied, $(rx_1 + sx_2)(mx_1 + nx_2) = ma(rx_1 + sx_2) = \xi(mx_1 + nx_2)(rx_1 + sx_2)$.

Conversely, if ξ is a non-trivial homomorphism of A into D, then multiplication defined by $xy = \xi(x)y$ for $x, y \in A$ is associative and distributive with respect to addition. Since ξ is non-trivial, there exists $K \in A$ such that $\xi(K) \neq 0$. Since A has rank two, there exists $L \in A$ such that K and L are independent. If $KL = LK$, then $\xi(K)L = \xi(L)K$ which implies $\xi(K) = \xi(L) = 0$, which is a contradiction. Hence the multiplication yields a non-commutative ring R over A. An analogous discussion can be given for $xy = \xi(y)x$. □

Corollary 6.1. *(Beaumont and Wisner* [7]*) If A is a torsion-free Abelian group of arbitrary rank and if ξ is a non-trivial homomorphism of A into the nucleus D of A, then multiplication defined by $xy = \xi(x)y$ or $xy = \xi(y)x$, for $x, y \in A$ yields a (non-zero) ring R over A. R is non-commutative if and only if the rank of A is greater than one.*

Proof. The fact that the given multiplication is well-defined, associative, and distributive with respect to addition does not depend on the rank of A, and hence the result follows as in Theorem 6.1.

If R is non-commutative, then A cannot have rank 1. In fact, by Theorem 3.1, the only (non-zero) rings over torsion-free groups of rank one are isomorphic to subrings of the field of rational numbers. Conversely, if the rank of A is at least two, the fact that R is non-commutative follows from the proof of Theorem 6.1. □

Corollary 6.2. *(Beaumont and Wisner [7]) A non-commutative ring R over a torsion-free group of rank two contains an ideal I such that*

(1) $I^2 = 0$,

(2) *the additive group I^+ of I has rank one.*

In particular, R contains proper divisors of zero.

Proof. We note that $< y >^*$, the pure subgroup of A generated by y in (6.2) and (6.3), Lemma 6.2, is an ideal in R with the stated properties.

The non-commutative rings R over A occur in anti-isomorphic pairs, defined by $xy = \xi(x)y$ and $xy = \xi(y)x$ in Theorem 6.1. Thus, to determine the essentially different rings over A, we need only consider those defined by $xy = \xi(x)y$. □

Now we consider the rings over a torsion-free group of rank two which contain no proper divisors of zero. By Corollary 6.2, such a ring is necessarily commutative. The next theorem characterizes those rings over A which contain no proper divisors of zero.

Theorem 6.2. *(Beaumont and Wisner [7]) Let A be a torsion-free group of rank two. Then R is a ring over A without proper divisors of zero if and only if R is isomorphic to a subring of a quadratic extension $R(\alpha)$ of R.*

Proof. Let R be a ring over A without divisors of zero. Then, as remarked above, R is commutative. Hence, by Lemma 6.1, there exists $x \in A$ such that x and x^2 are independent. Then $x^3 = rx + sx^2$, where $r = r_1/r_2$ and $s = s_1/s_2$. Hence $[r_2, s_2]x$ and $([r_2, s_2]x)^2$ are independent elements of A where $([r_2, s_2]x)^3 = ([r_2, s_2])^2 r([r_2, s_2]x) + [r_2, s_2]s([r_2, s_2]x)^2$, so we may assume that x and x^2 are independent elements of A, where $x^3 = ax + bx^2$ with a and b integers.

Consider the polynomial $X^2 - bX - a$, and let α and β be its zeros. Then α and β are not rational. For suppose the contrary. Then $x^4 - bx^3 - ax^2 = 0$, and since $\alpha + \beta = b$ and $\alpha.\beta = -a$, we have

$$x^4 - bx^3 - ax^2 = (x^2 - \alpha x)(x^2 - \beta x) = 0.$$

Since α and β are integers, and since x and x^2 are independent, $x^2 - \alpha x$ and $x^2 - \beta x$ are non-zero elements of R. But this contradicts the hypothesis that R has no divisors of zero.

Since α is a zero of $X^2 - bX - a$, we have $\alpha^3 = \alpha a + b\alpha^2$. Hence the correspondence $\varphi : R \to R(\alpha)$ defined by $mx + nx^2 \to m\alpha + n\alpha^2$ is a ring homomorphism. Moreover, φ is an isomorphism for $m\alpha + n\alpha^2 = 0$ implies $(m + nb)\alpha + na = 0$, and since α is not rational, this yields $m + nb = 0$ and $na = 0$. But again since α is not rational, $a \neq 0$. Hence $m = n = 0$.

Conversely, any subring of a quadratic extension $R(\alpha)$ of R is an integral domain. $\square$

In order to derive a necessary condition for the existence of a ring R without zero-divisors over the group A, Beaumont and Wisner find additional necessary and sufficient conditions that a ring R over A has no zero-divisors.

Theorem 6.3. *(Beaumont and Wisner* [7]*) Let A be a torsion-free group of rank two. Then R is a ring over A without zero divisors if and only if there exists an element $y \in R$ such that*

(1) *y and y^2 are independent.*

(2) *$y^3 = cy$, where c is a non-square integer.*

Proof. In the proof of Theorem 6.2, x can be chosen so that $x^3 = ax + bx^2$ where $b = 2q$ is even (since the element x in the proof of Theorem 6.2 can be replaced by

$2x$). Now $\alpha^2 - 2q\alpha - a = 0$ so that we have

$$(\alpha - q)^2 = q^2 + a; (a\alpha - aq)^2 = a^2(q^2 + a);$$

$$a\alpha - aq = a\alpha - q(\alpha - q)^2 + q^3 = a\alpha - q(\alpha^2 - 2q\alpha) = (a + 2q)\alpha - q\alpha^2; \tag{6.8}$$

$$[(a + 2q)\alpha - q\alpha^2]^2 = (a\alpha - aq)^2 = a^2(q^2 + a); \tag{6.9}$$

$$[(a + 2q)\alpha - q\alpha^2]^3 = a^2(q^2 + a)[(a + 2q)\alpha - q\alpha^2]. \tag{6.10}$$

Now let $y = (a+2q)x - qx^2$. Then $\varphi(y) = (a+2q)\alpha - q\alpha^2$, where φ is the isomorphism of R into $R(\alpha)$ given in Theorem 6.2. We note first that y and y^2 are independent. For if $ey + fy^2 = 0$ for integers e and f, we would have $e\varphi(y) + f\varphi(y^2) = 0$, which by (6.8) and (6.9) would yield

$$e(a\alpha - aq) + fa^2(q^2 + a) = 0.$$

But this implies that α is rational unless $e = f = 0$.

It is immediate from (6.10) that $y^3 = a^2(q^2 + a)y$ where $a^2(q^2 + a)$ is a non-square integer.

Conversely, suppose that there exists an element $y \in R$ which satisfies (1) and (2). Assume that R has proper divisors of zero, so that there exist non-zero elements $ry + sy^2$ and $uy + vy^2$ such that $(ry + sy^2)(uy + vy^2) = 0$. Using (2), this yields

$$c(rv + su)y + (ru + svc)y^2 = 0.$$

Since y and y^2 are independent, we have

$$c(rv + su) = 0 \text{ and } ru + svc = 0.$$

Since $c \neq 0$ by (2), and not both r and s are zero, we obtain

$$0 = uv - uvc = uv(1 - c)$$

which implies (since $c \neq 1$ by (2)) that $u = 0$ or $v = 0$. Suppose $u = 0$. Then $crv = svc = 0$, which implies $r = s = 0$, since not both u and v are zero. But this is a contradiction. Similarly, a contradiction is obtained if $v = 0$. □

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