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QUASI αgrw-OPEN MAPS IN TOPOLOGICAL SPACES

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ABSTRACT. We introduce the notions of α-generalized regular weakly open sets, Quasi α-generalized regular weakly open maps and Quasi α-generalized regular weakly closed maps in topological spaces.

1. INTRODUCTION

In 2010, A. Vadivel and K. Vairamanickam [5] introduced Quasi rw-open and Quasi rw-closed functions in topological spaces. In 2013, Varun Joshi et al. [6] introduced gprw-closed and gprw-Quasi closed functions in topological spaces. Recently, as a generalization of closed sets, the notion of αgrw-closed sets, αgrw-continuous maps and αgrw-open maps were introduced and studied in [2, 3]. In this paper we introduce and characterize the concept of quasi αgrw-open maps.

2. PRELIMINARIES

Throughout this paper X, Y and Z denote topological spaces (X, τ), (Y, σ) and (Z, η) respectively on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), cl(A), int(A), $\alpha cl(A)$, $\alpha int(A)$ and

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αgrw-cl(A) denote the closure, interior, α-closure, α-interior and αgrw-closure of A respectively. The complement of a set A of (X, τ) is denoted by A^c or $(X - A)$.

Definition 2.1. [4] A subset A of a topological space (X, τ) is called *regular open* if A = int(cl(A)).

Definition 2.2. [1] A subset A of a topological space (X, τ) is called regular semi-open if there is a regular open set Usuch that $U \subseteq$ A $\subseteq cl(U)$.

Definition 2.3. [2] A subset A of a topological space (X, τ) is called *αgrw-closed* if $\alpha cl(A) \subseteq$ U whenever A$\subseteq$ U and U is regular semi-open.

Definition 2.4. [3] A map $f : (X, \tau) \to (Y, \sigma)$ is called

(1) *αgrw-continuous* if $f^{-1}(V)$ is an αgrw-closed set of (X, τ) for every closed set V of (Y, σ),
(2) *αgrw-open* if $f(U)$ is αgrw-open in (Y, σ) for every open set U of (X, τ).

Definition 2.5. [3] For a subset A of (X, τ), αgrw-cl(A)=$\cap\{F : A \subset F, F$ is αgrw-closed in X$\}$.

3. On α-generalized regular weakly open sets

Definition 3.1. A subset A in (X, τ) is called α-generalized regular weakly open (briefly αgrw-open) if A^c is αgrw-closed.

The family of all αgrw-open sets is denoted by $\alpha grw O(X,\tau)$ or $\alpha grw O(X)$.

Theorem 3.1. *A subset A of (X, τ) is αgrw-open if and only if $F \subseteq \alpha int(A)$ whenever F is regular semi-closed and $F \subseteq A$.*

Proof. Suppose that $F \subseteq \alpha int(A)$, whenever F is regular semi-closed and $F \subseteq A$. Let $A^c \subseteq U$, where U is regular semi-open. Then $U^c \subseteq A$, where U^c is regular semi-closed. By hypothesis $U^c \subseteq \alpha int(A)$, which implies $[\alpha int(A)]^c \subseteq U$. i.e., $\alpha cl(A^c) \subseteq U$. Thus A^c is αgrw-closed. Hence A is αgrw-open.

Conversely, suppose that A is αgrw-open, $F \subseteq A$ and F is regular semi-closed. Then F^c is regular semi-open and $A^c \subseteq F^c$. Therefore $\alpha cl(A^c) \subseteq F^c$ and so $F \subseteq [\alpha cl(A^c)]^c = \alpha int(A)$. Hence $F \subseteq \alpha int(A)$.

Lemma 3.1. *If a subset A of X is αgrw-closed, then $\alpha cl(A) - A$ does not contain any non-empty regular semi-closed set.*

Proof. Suppose that A is αgrw-closed in X. Let U be a regular semi-closed set such that $U \subseteq \alpha cl(A) - A$. Then $A \subseteq U^c$. Since A is αgrw-closed, we have $\alpha cl(A) \subseteq U^c$. Consequently, $U \subseteq [\alpha cl(A)]^c$. Thus $U \subseteq (\alpha cl(A)) \cap (\alpha cl(A))^c$. Hence $U = \emptyset$. Therefore $\alpha cl(A) - A$ does not contain any non-empty regular semi-closed set.

Theorem 3.2. *If a subset A is αgrw-open in (X, τ), then $U = X$ whenever U is regular semi-open and $\alpha int(A) \cup A^c \subseteq U$.*

Proof. Let A be αgrw-open, U be regular semi-open such that $\alpha int(A) \cup A^c \subseteq U$. This gives $U^c \subseteq [\alpha int(A) \cup A^c]^c = \alpha cl(A^c) - A^c$. Since A^c is αgrw-closed and U^c is regular semi-closed by Lemma 3.1, it follows that $U^c = \emptyset$. i.e., $X = U$.

Theorem 3.3. *If A is αgrw-open and $\alpha int(A) \subseteq B \subseteq A$, then B is αgrw-open.*

Proof. Suppose that $\alpha int(A) \subseteq B \subseteq A$ and A is αgrw-open. Then $A^c \subseteq B^c \subseteq \alpha cl(A^c)$ and since A^c is αgrw-closed, we have Theorem 3.26 [2], $B^c$is αgrw-closed i.e., B is αgrw-open.

Definition 3.2. Let (X, τ) be a topological space and $E \subseteq X$. αgrw-int(E) is the union of all αgrw-open sets contained in E.

i.e., αgrw-int(E)= $\cup\{E : E \subseteq A$ and E is αgrw-open$\}$.

Lemma 3.2. *Let A be a subset of a space (X, τ). Then $X - \alpha grw$-int(A)=αgrw-cl$(X - A)$.*

Proof. Let $x \in X - \alpha grw$-int(A). Then $x \notin \alpha grw$-int(A). That is, every αgrw-open set B containing x is such that B is not contained in A. This implies every αgrw-open set B containing x is such that $B \cap (X - A) \neq \emptyset$. By Theorem 4.15 [3], $x \in \alpha grw$-cl$(X - A)$. Hence $(X - \alpha grw$-int(A))$\subseteq \alpha grw$-$cl(X - A)$.

Conversely let $x \in \alpha grw$-cl$(X - A)$ then by Theorem 4.15 [3], every αgrw-open set B containing x is such that $B \cap (X - A) \neq \emptyset$. That is every αgrw-open set B containing x is such that B is not contained in A. This implies, $x \notin \alpha grw$-int(A). Thus $x \in X - \alpha grw$-int(A). Hence αgrw-cl$(X - A) \subseteq X - \alpha grw$-int(A). Hence the proof.

Theorem 3.4. *Let (X, τ) be a topological space. Then the following hold:*

(1) *If $A \subseteq X$ is αgrw-closed, then $\alpha cl(A) - A$ is αgrw-open.*
(2) *If A is αgrw-open and B is αgrw-open then $A \cap B$ is αgrw-open.*
(3) *For any $E \subseteq X$, int(E) $\subseteq \alpha grw$-int(E) $\subseteq$ E.*

Proof.

(1) Let A be αgrw-closed. Let F be regular semi-closed such that $F \subseteq \alpha cl(A) - A$. Then by Lemma 3.1, $F = ;$. This implies $F \subseteq \alpha int(\alpha cl(A) - A)$. By Theorem 3.1, $\alpha cl(A) - A$ is αgrw-open.
(2) Let A^c and B^c be αgrw-closed then $A^c \cup B^c$is αgrw-closed by Theorem 3.19 [2]. This implies $A \cap B$ is αgrw-open.
(3) Since every open set is αgrw-open, the proof follows immediately.

Definition 3.3. Let (X, τ) be a topological space. Let $\tau_{\alpha grw} = \{U \subseteq X : \alpha grw$-$cl(X - U) = X - U\}$.

Theorem 3.5. *Let (X, τ) be a topological space. Then the following hold:*

(1) *Every αgrw-closed set is α-closed if and only if $\tau_{\alpha grw} = \alpha O(X, \tau)$.*

(2) *Every αgrw-closed set is closed if and only if $\tau_{\alpha grw} = \tau$.*

Proof. **1. Necessity.** Let $A \in \tau_{\alpha grw}$. Then $\alpha grw\text{-}cl(X - A) = X - A$. By hypothesis, $\alpha cl(X - A) \subseteq \alpha grw\text{-}cl(X - A) = X - A$. This implies $X - A$ is α-closed and hence $A \in \alpha O(X, \tau)$. Let $A \in \alpha O(X, \tau)$, $X - A$ is α-closed which implies $\alpha cl(X - A) = X - A$. Since every α-closed set is αgrw-closed, $\alpha grw\text{-}cl(X - A) \subseteq \alpha cl(X - A) = X - A$. Hence $A \in \tau_{\alpha grw}$.

Sufficiency. Suppose $\tau_{\alpha grw} = \alpha O(X, \tau)$. Let A be αgrw-closed set. Then αgrw-cl(A)=A. This implies $X - A \in \tau_{\alpha grw} = \alpha O(X, \tau)$. So A is α-closed.

2. Necessity. Let $A \in \tau_{\alpha grw}$. Then αgrw-cl$(X - A) = X - A$. By hypothesis, $cl(X - A) \subseteq \alpha grw\text{-}cl(X - A) = X - A$. This implies $X - A$ is closed and hence $A \in \tau$. Let $A \in \tau$, $X - A$ is closed which implies $cl(X - A) = (X - A)$. Since every closed set is αgrw-closed, $\alpha grw\text{-}cl(X - A) \subseteq cl(X - A) = (X - A)$. Hence $A \in \tau_{\alpha grw}$.

Sufficiency. Suppose $\tau_{\alpha grw} = \tau$. Let A be αgrw-closed set. Then αgrw-cl(A) = A. This implies $X - A \in \tau_{\alpha grw} = \tau$. So A is closed.

Theorem 3.6. *If $\alpha grwO(X, \tau)$ is closed under arbitrary union, then $\tau_{\alpha grw}$ is a topology.*

Proof. 1. Clearly, $\emptyset, X \in \tau_{\alpha grw}$.

2. Let $\{A_i : i \in \Lambda\} \in \tau_{\alpha grw}$. Then $\alpha grw\text{-}cl(X - (\bigcup A_i)) = \alpha grw\text{-}cl(\bigcap (X - A_i)) \subseteq \bigcap \alpha grw\text{-}cl(X - A_i) = \bigcap (X - A_i) = X - (\bigcup A_i)$. Therefore $\bigcup A_i \in \tau_{\alpha grw}$.

3. Let $A, B \in \tau_{\alpha grw}$. Now, $\alpha grw\text{-}cl(X - (A \cap B)) = \alpha grw\text{-}cl(X - A) \cup \alpha grw\text{-}cl(X - B) = (X - A) \cup (X - B) = X - (A \cap B)$. Then $A \cap B \in \tau_{\alpha grw}$.

Hence $\tau_{\alpha grw}$ is a topology.

4. Quasi αgrw-open maps

Definition 4.1. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be quasi αgrw-open if the image of every αgrw-open set in X is open in Y.

Definition 4.2. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be quasi αgrw-closed if the image of every αgrw-closed set in X is closed in Y.

It is evident that the concepts of quasi αgrw-openness (resp. αgrw-closedness) and αgrw-continuity coincide if the map is bijection.

Theorem 4.1. *Every quasi αgrw-open map is open.*

Proof. Let U be an open set in X. Since every open set is αgrw-open, U is αgrw-open in X. Then $f(U)$ is open in Y, since f is quasi αgrw-open map. Hence f is open.

Corollary 4.1. *Every quasi αgrw-open map is αgrw-open.*

Proof. It follows from Theorem 4.1 and Theorem 4.8(i) [3].

Remark 1. The converses of Theorem 4.1 and its corollary need not be true as seen from the following example.

Example 4.1. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, X\}$, $Y = \{p, q, r\}$ and $\sigma = \{\emptyset, \{p\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = f(b) = p$, $f(c) = q$ and $f(d) = r$. Then f is open and αgrw-open but it is not quasi αgrw-open.*

Theorem 4.2. *A map $f : (X, \tau) \to (Y, \sigma)$ is said to be quasi αgrw-open if for every subset U of X, $f(\alpha grw\text{-}int(U)) \subseteq int(f(U))$.*

Proof. Assume that U is αgrw-open set in X. Then $f(U) = f(\alpha grw\text{-}int(U)) \subseteq int(f(U))$ but $int(f(U)) \subseteq f(U)$. Consequently, $f(U) = int(f(U))$. Hence $f(U)$ is open in Y. Therefore f is quasi αgrw-open.

Theorem 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two maps and $g \circ f : (X, \tau) \to (Z, \eta)$ be quasi αgrw-open. If g is continuous injective, then f is quasi αgrw-open.*

Proof. Let U be a αgrw-open set in X. Then, we have $(g \circ f)(U)$ is open in Z, since $(g \circ f)$ is quasi αgrw-open. Again g is an injective continuous map, $g^{-1}[(g \circ f)(U)] = f(U)$ is open in Y. This shows that f is quasi αgrw-open.

Definition 4.3. A map $f : (X, \tau) \to (Y, \sigma)$ is called αgrw^*-closed (resp. αgrw^*-open) map if the image of each αgrw-closed (resp. αgrw-open) subset in X is αgrw-closed (resp. αgrw-open) in Y.

Theorem 4.4. *For a topological space (X, τ), the following hold:*

(1) *Every quasi αgrw-closed map is αgrw^*-closed.*
(2) *Every αgrw^*-closed map is αgrw-closed.*

Proof. Obvious.

Remark 2. The converses of the above theorem need not be true as seen from the following examples.

Example 4.2. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$, $Y = \{p, q, r\}$ and $\sigma = \{\emptyset, \{r\}, \{q, r\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = f(b) = p$, $f(c) = q$ and $f(d) = r$. Then f is αgrw^*-closed but not quasi αgrw-closed.*

Example 4.3. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$, $Y = \{p, q, r, s\}$ and $\sigma = \{\emptyset, \{p\}, \{q\}, \{p, q\}, \{p, q, r\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = s$, $f(b) = q$, $f(c) = p$ and $f(d) = r$. Then f is αgrw-closed but not αgrw^*-closed.*

Theorem 4.5. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two maps on topological spaces.*

(1) *If* f *is* αgrw*-closed and* g *is quasi* αgrw*-closed then* $g \circ f : (X, \tau) \to (Z, \eta)$ *is closed.*

(2) *If* f *is quasi* αgrw*-closed and* g *is* αgrw*-closed then* $g \circ f : (X, \tau) \to (Z, \eta)$ *is* αgrw^**-closed.*

(3) *If* f *is* αgrw^**-closed and* g *is quasi* αgrw*-closed then* $g \circ f : (X, \tau) \to (Z, \eta)$ *is quasi* αgrw*-closed.*

Proof. 1. Let F be a closed set in X. Since f is αgrw-closed, $f(F)$ is αgrw-closed set in Y and also g is a quasi αgrw-closed map therefore $g(f(F))$ is closed in Z. Hence $g \circ f$ is a closed map.

2. Let F be a αgrw-closed set in X. Since f is quasi αgrw-closed, $f(F)$ is closed set in Y and also g is an αgrw-closed map therefore $g(f(F))$ is αgrw-closed in Z. Hence $g \circ f$ is an αgrw^*-closed map.

3. Let F be a αgrw-closed set in X. Since f is αgrw^*-closed, $f(F)$ is αgrw-closed set in Y and also g is a quasi αgrw-closed map therefore $g(f(F))$ is closed in Z. Hence $g \circ f$ is a quasi αgrw-closed map.

Theorem 4.6. *Let* $f : (X, \tau) \to (Y, \sigma)$ *and* $g : (Y, \sigma) \to (Z, \eta)$ *be two maps such that* $g \circ f : (X, \tau) \to (Z, \eta)$ *be a quasi* αgrw*-closed map. If* g *is* αgrw*-continuous and injective map then* f *is* αgrw^**-closed.*

Proof. Suppose that F is any αgrw-closed set in X. Since $g \circ f$ is quasi αgrw-closed therefore $(g \circ f)(F)$ is closed in Z and also g is αgrw-continuous and injective map therefore $g^{-1}[(g \circ f)(F)] = f(F)$, which is αgrw-closed in Y. Hence f is αgrw^*-closed.

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