

## EXTENSION OF CERONE'S GENERALIZATIONS OF STEFFENSEN'S INEQUALITY

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ABSTRACT. The aim of this paper is to extend Cerone's generalization of Steffensen's inequality to positive finite measures and to give weaker conditions for obtained extension. Further, our intension is to obtain extensions of known generalizations of Steffensen's inequality in order to allow bounds that involve any two subintervals using measure theoretic aspects.

### 1. INTRODUCTION

Firstly, let us recall Steffensen's inequality (see [13]):

**Theorem 1.1.** *Suppose that  $f$  is nonincreasing and  $g$  is integrable on  $[a, b]$  with  $0 \leq g \leq 1$  and  $\lambda = \int_a^b g(t)dt$ . Then we have*

$$(1.1) \quad \int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt.$$

Steffensen's inequality has been generalized and refined by many mathematicians since it plays an important role not only in the theory of inequalities but also in statistics, functional equations, time scales, special functions, etc. A comprehensive

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survey on Steffensen's inequality, its generalizations and applications can be found in [11].

It is known that Steffensen inequality can be used in an improvement of Jensen's inequality for convex functions, the well known Jensen-Steffensen inequality. For some other related results see [12] and [16].

Starting point for this paper is Cerone's generalization of Steffensen's inequality given in [2]. This generalization allows bounds that involve any two subintervals instead of requiring that they include the end points.

**Theorem 1.2.** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be integrable functions on  $[a, b]$  and let  $f$  be nonincreasing. Further, let  $0 \leq g \leq 1$  and*

$$\lambda = \int_a^b g(t)dt = d_i - c_i,$$

where  $[c_i, d_i] \subseteq [a, b]$  for  $i = 1, 2$  and  $d_1 \leq d_2$ . Then

$$\int_{c_2}^{d_2} f(t)dt - r(c_2, d_2) \leq \int_a^b f(t)g(t)dt \leq \int_{c_1}^{d_1} f(t)dt + R(c_1, d_1)$$

holds, where

$$r(c_2, d_2) = \int_{d_2}^b (f(c_2) - f(t))g(t)dt \geq 0$$

and

$$R(c_1, d_1) = \int_a^{c_1} (f(t) - f(d_1))g(t)dt \geq 0.$$

Milovanović and Pečarić in [6] obtained weaker conditions on function  $g$ . In [14] Vasić and Pečarić showed that these weaker conditions are necessary and sufficient. Hence, we have the following theorem.

**Theorem 1.3.** *Let  $f$  and  $g$  be integrable functions on  $[a, b]$  and let  $\lambda = \int_a^b g(t)dt$ .*

- (a) *The second inequality in (1.1) holds for every nonincreasing function  $f$  if and only if*

$$\int_a^x g(t)dt \leq x - a \quad \text{and} \quad \int_x^b g(t)dt \geq 0, \quad \text{for every } x \in [a, b].$$

- (b) *The first inequality in (1.1) holds for every nonincreasing function  $f$  if and only if*

$$\int_x^b g(t)dt \leq b - x \quad \text{and} \quad \int_a^x g(t)dt \geq 0, \quad \text{for every } x \in [a, b].$$

Motivated by Cerone's result given in Theorem 1.2, Pečarić, Perušić and Smoljak in [10] extended some generalizations of Steffensen's inequality from [9] to allow bounds that involve any two subintervals. The aim of this paper is to further extend these generalizations to positive finite measures on Borel  $\sigma$ -algebra.

Throughout the paper by  $\mathcal{B}([a, b])$  we denote Borel  $\sigma$ -algebra on  $[a, b]$ .

## 2. MAIN RESULTS

We begin with an extension of Cerone's result given in Theorem 1.2 to positive finite measures.

**Theorem 2.1.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f$  be nonincreasing and  $g$  be measurable function on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $[c, d] \subseteq [a, b]$  and*

$$(2.1) \quad \mu([c, d]) = \int_{[a, b]} g(t)d\mu(t).$$

*Then*

$$(2.2) \quad \int_{[a, b]} f(t)g(t)d\mu(t) \leq \int_{[c, d]} f(t)d\mu(t) + R_\mu(c, d)$$

*holds, where*

$$R_\mu(c, d) = \int_{[a, c)} (f(t) - f(d))g(t)d\mu(t) \geq 0.$$

*Proof.* Let us consider a corresponding difference:

$$\begin{aligned}
 (2.3) \quad & \int_{[c,d]} f(t) d\mu(t) + R_\mu(c, d) - \int_{[a,b]} f(t) g(t) d\mu(t) \\
 &= \int_{[c,d]} f(t) d\mu(t) + \int_{[a,c]} (f(t) - f(d)) g(t) d\mu(t) - \int_{[a,b]} f(t) g(t) d\mu(t) \\
 &= \int_{[c,d]} f(t) d\mu(t) - \int_{[c,b]} f(t) g(t) d\mu(t) - f(d) \int_{[a,b]} g(t) d\mu(t) + f(d) \int_{[c,b]} g(t) d\mu(t) \\
 &= \int_{[c,d]} f(t) d\mu(t) - \int_{[c,b]} f(t) g(t) d\mu(t) - f(d) \int_{[c,d]} d\mu(t) + f(d) \int_{[c,b]} g(t) d\mu(t) \\
 &= \int_{[c,d]} (f(t) - f(d))(1 - g(t)) d\mu(t) + \int_{(d,b]} (f(d) - f(t)) g(t) d\mu(t)
 \end{aligned}$$

where we used (2.1).

Since  $0 \leq g \leq 1$ ,  $f$  is nonincreasing and  $\mu$  is positive, terms under the integral sign are nonnegative, hence the first sum in this chain is nonnegative, i.e.

$$\int_{[c,d]} f(t) d\mu(t) + R_\mu(c, d) \geq \int_{[a,b]} f(t) g(t) d\mu(t).$$

□

**Theorem 2.2.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f$  be nonincreasing and  $g$  be measurable function on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $[c, d] \subseteq [a, b]$  and*

$$(2.4) \quad \mu((c, d]) = \int_{[a,b]} g(t) d\mu(t).$$

*Then*

$$(2.5) \quad \int_{(c,d]} f(t) d\mu(t) - r_\mu(c, d) \leq \int_{[a,b]} f(t) g(t) d\mu(t)$$

*holds, where*

$$r_\mu(c, d) = \int_{(d,b]} (f(c) - f(t)) g(t) d\mu(t) \geq 0.$$

*Proof.* Similar to the proof of Theorem 2.1 we obtain the corresponding difference

$$\begin{aligned} & \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c,d]} f(t)d\mu(t) + r_\mu(c, d) \\ &= \int_{(c,d]} (f(c) - f(t))(1 - g(t))d\mu(t) + \int_{[a,c]} (f(t) - f(c))g(t)d\mu(t) \geq 0. \end{aligned}$$

□

Motivated by Pachpatte's result from [7] under the additional assumption on the function  $f$  we can replace conditions (2.1) and (2.4) by weaker conditions given in the following theorems.

**Theorem 2.3.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f$  be nonincreasing and nonnegative, and  $g$  be measurable function on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $[c, d] \subseteq [a, b]$  and*

$$(2.6) \quad \mu([c, d]) \geq \int_{[a,b]} g(t)d\mu(t).$$

*Then (2.2) holds.*

*Proof.* Since  $f(d) \geq 0$  using (2.6) in (2.3) we obtain the claim of this theorem. □

**Theorem 2.4.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f$  be nonincreasing and nonnegative, and  $g$  be measurable function on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $[c, d] \subseteq [a, b]$  and*

$$\mu((c, d]) \leq \int_{[a,b]} g(t)d\mu(t).$$

*Then (2.5) holds.*

*Proof.* Similar to the proof of Theorem 2.3. □

**Remark 1.** Taking  $c = a$  and  $d = a + \lambda$  in Theorems 2.1 and 2.3 or taking  $c = b - \lambda$  and  $d = b$  in Theorems 2.2 and 2.4 we obtain results given in [3]. Further, if we take  $c = a$ ,  $d = a + \lambda$  and consider the Lebesgue measure in Theorem 2.1 or take  $c = b - \lambda$ ,

$d = b$  and consider the Lebesgue measure in Theorem 2.2 we obtain Cerone's result given in Theorem 1.2.

In the sequel we need the following lemmas to generalize the above results for the function  $f/k$ .

**Lemma 2.1.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $k$  be measurable functions on  $[a, b]$  such that  $k$  is positive. Further, let  $[c, d] \subseteq [a, b]$  with  $\int_{[c, d]} h(t)k(t)d\mu(t) = \int_{[a, b]} g(t)k(t)d\mu(t)$ . Then the following identity holds:*

$$(2.7) \quad \begin{aligned} & \int_{[c, d]} f(t)h(t)d\mu(t) - \int_{[a, b]} f(t)g(t)d\mu(t) = \int_{[a, c]} \left( \frac{f(d)}{k(d)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) \\ & + \int_{[c, d]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) k(t)[h(t) - g(t)]d\mu(t) + \int_{(d, b]} \left( \frac{f(d)}{k(d)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t). \end{aligned}$$

*Proof.* We have

$$(2.8) \quad \begin{aligned} & \int_{[c, d]} f(t)h(t)d\mu(t) - \int_{[a, b]} f(t)g(t)d\mu(t) = \int_{[c, d]} \frac{f(t)}{k(t)} k(t)[h(t) - g(t)]d\mu(t) \\ & - \int_{[a, c]} \frac{f(t)}{k(t)} g(t)k(t)d\mu(t) - \int_{(d, b]} \frac{f(t)}{k(t)} g(t)k(t)d\mu(t) \\ & = \int_{[a, c]} \left( \frac{f(d)}{k(d)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) + \int_{(d, b]} \left( \frac{f(d)}{k(d)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) \\ & + \int_{[c, d]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) k(t)[h(t) - g(t)]d\mu(t) + \frac{f(d)}{k(d)} \left[ \int_{[c, d]} k(t)h(t)d\mu(t) \right. \\ & \left. - \int_{[a, c]} g(t)k(t)d\mu(t) - \int_{[c, d]} k(t)g(t)d\mu(t) - \int_{(d, b]} g(t)k(t)d\mu(t) \right]. \end{aligned}$$

Since

$$\int_{[c, d]} k(t)h(t)d\mu(t) = \int_{[a, b]} k(t)g(t)d\mu(t),$$

we have

$$\int_{[c, d]} k(t)h(t)d\mu(t) - \int_{[a, c]} g(t)k(t)d\mu(t) - \int_{[c, d]} k(t)g(t)d\mu(t) - \int_{(d, b]} g(t)k(t)d\mu(t) = 0.$$

Hence, (2.7) follows from (2.8).  $\square$

**Lemma 2.2.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $k$  be measurable functions on  $[a, b]$  such that  $k$  is positive. Further, let  $[c, d] \subseteq [a, b]$  with  $\int_{(c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$ . Then the following identity holds:*

$$(2.9) \quad \begin{aligned} & \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c,d]} f(t)h(t)d\mu(t) = \int_{[a,c]} \left( \frac{f(t)}{k(t)} - \frac{f(c)}{k(c)} \right) g(t)k(t)d\mu(t) \\ & + \int_{(c,d]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) k(t)[h(t) - g(t)]d\mu(t) + \int_{(d,b]} \left( \frac{f(t)}{k(t)} - \frac{f(c)}{k(c)} \right) g(t)k(t)d\mu(t). \end{aligned}$$

*Proof.* Similar to the proof of Lemma 2.1.  $\square$

**Theorem 2.5.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $k$  be measurable functions on  $[a, b]$  such that  $k$  is positive,  $0 \leq g \leq h$  and  $f/k$  is nonincreasing. Further, let  $[c, d] \subseteq [a, b]$  with*

$$(2.10) \quad \int_{[c,d]} h(t)k(t)dt = \int_{[a,b]} g(t)k(t)dt.$$

*Then*

$$(2.11) \quad \int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[c,d]} f(t)h(t)d\mu(t) + \mathfrak{R}_\mu(c, d)$$

*holds, where*

$$(2.12) \quad \mathfrak{R}_\mu(c, d) = \int_{[a,c]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) g(t)k(t)d\mu(t) \geq 0.$$

*Proof.* Since  $f/k$  is nonincreasing,  $k$  and  $\mu$  are positive and  $0 \leq g \leq h$  we have

$$(2.13) \quad \int_{[c,d]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) k(t)[h(t) - g(t)]d\mu(t) \geq 0,$$

$$(2.14) \quad \int_{(d,b]} \left( \frac{f(d)}{k(d)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) \geq 0$$

and  $\Re_\mu(c, d) \geq 0$ . Now, from (2.7), (2.13) and (2.14) we have

$$\begin{aligned}
 (2.15) \quad & \int_{[c,d]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) + \int_{[a,c]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) g(t)k(t)d\mu(t) \\
 &= \int_{[c,d]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) k(t)[h(t) - g(t)]d\mu(t) + \int_{(d,b]} \left( \frac{f(d)}{k(d)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) \geq 0.
 \end{aligned}$$

Hence, (2.11) holds.  $\square$

**Theorem 2.6.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $k$  be measurable functions on  $[a, b]$  such that  $k$  is positive,  $0 \leq g \leq h$  and  $f/k$  is nonincreasing. Further, let  $[c, d] \subseteq [a, b]$  with*

$$(2.16) \quad \int_{(c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t).$$

Then

$$(2.17) \quad \int_{(c,d]} f(t)h(t)d\mu(t) - \mathfrak{r}_\mu(c, d) \leq \int_{[a,b]} f(t)g(t)d\mu(t)$$

holds, where

$$(2.18) \quad \mathfrak{r}_\mu(c, d) = \int_{(d,b]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) \geq 0.$$

*Proof.* Since  $f/k$  is nonincreasing,  $k$  and  $\mu$  are positive and  $0 \leq g \leq h$  we have

$$(2.19) \quad \int_{[a,c]} \left( \frac{f(t)}{k(t)} - \frac{f(c)}{k(c)} \right) k(t)g(t)d\mu(t) \geq 0,$$

$$(2.20) \quad \int_{(c,d]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) k(t)[h(t) - g(t)]d\mu(t) \geq 0$$

and  $\mathfrak{r}_\mu(c, d) \geq 0$ . Now, from (2.9), (2.19) and (2.20) we have

$$\begin{aligned}
 (2.21) \quad & \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c,d]} f(t)h(t)d\mu(t) + \int_{(d,b]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) \\
 &= \int_{[a,c]} \left( \frac{f(t)}{k(t)} - \frac{f(c)}{k(c)} \right) g(t)k(t)d\mu(t) + \int_{(c,d]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) k(t)[h(t) - g(t)]d\mu(t) \geq 0.
 \end{aligned}$$



Hence, (2.17) holds.  $\square$

Remark 2. If we additionally assume that the function  $f$  is nonnegative, conditions (2.10) and (2.16) can be replaced by weaker conditions

$$\int_{[c,d]} h(t)k(t)dt \geq \int_{[a,b]} g(t)k(t)dt \quad \text{and} \quad \int_{(c,d]} h(t)k(t)d\mu(t) \leq \int_{[a,b]} g(t)k(t)d\mu(t).$$

Remark 3. If we take  $c = a$ ,  $d = a + \lambda$  and consider the Lebesgue measure in Theorem 2.5 we obtain Mercer's generalization of the right-hand Steffensen's inequality (see Theorem 3 in [5]). If we take  $c = b - \lambda$ ,  $d = b$  and consider the Lebesgue measure in Theorem 2.6 we obtain a similar generalization of the left-hand Steffensen's inequality which is obtained in [9] from a generalization given by Pečarić in [8].

For  $h \equiv 1$  and  $k \equiv 1$  Theorems 2.5 and 2.6 reduce to Theorems 2.1 and 2.2.

Inequalities (2.11) and (2.17) can be refined using similar reasoning as in [9] and [15]. These refinements are given in the following theorems.

**Theorem 2.7.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $k$  be measurable functions on  $[a, b]$  such that  $k$  is positive,  $0 \leq g \leq h$  and  $f/k$  is nonincreasing. Further, let  $[c, d] \subseteq [a, b]$  with  $\int_{[c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$ . Then*

$$\begin{aligned} & \int_{[a,b]} f(t)g(t)d\mu(t) \\ & \leq \int_{[c,d]} f(t)h(t)d\mu(t) - \int_{[c,d]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) k(t)[h(t) - g(t)]d\mu(t) + \mathfrak{R}_\mu(c, d) \\ & \leq \int_{[c,d]} f(t)h(t)d\mu(t) + \mathfrak{R}_\mu(c, d) \end{aligned}$$

holds, where  $\mathfrak{R}_\mu(c, d)$  is defined by (2.12).

*Proof.* Similar to the proof of Theorem 2.5.  $\square$

**Theorem 2.8.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $k$  be measurable functions on  $[a, b]$  such that  $k$  is positive,  $0 \leq g \leq h$  and  $f/k$  is nonincreasing. Further, let  $[c, d] \subseteq [a, b]$  with  $\int_{(c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$ . Then*

$$\begin{aligned} & \int_{(c,d]} f(t)h(t)d\mu(t) - \mathfrak{r}_\mu(c, d) \\ & \leq \int_{(c,d]} f(t)h(t)d\mu(t) + \int_{(c,d]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) k(t)[h(t) - g(t)]d\mu(t) - \mathfrak{r}_\mu(c, d) \\ & \leq \int_{[a,b]} f(t)g(t)d\mu(t) \end{aligned}$$

holds, where  $\mathfrak{r}_\mu(c, d)$  is defined by (2.18).

*Proof.* Similar to the proof of Theorem 2.6. □

Remark 4. If we take  $c = a$ ,  $d = a + \lambda$  and consider the Lebesgue measure in Theorem 2.7, or  $c = b - \lambda$ ,  $d = b$  and consider the Lebesgue measure in Theorem 2.8, we obtain generalizations of Wu and Srivastava refinement of Steffensen's inequality given in [9]. Additionally taking  $k \equiv 1$  we obtain Wu and Srivastava refinement given in [15].

Furthermore, from Theorems 2.7 and 2.8 (taking  $h \equiv 1$  and  $k \equiv 1$ ) we can obtain a refinement of Theorems 2.1 and 2.2.

### 3. WEAKER CONDITIONS

The aim of this section is to replace conditions on the function  $g$  in previous section with weaker conditions as in Theorem 1.3.

**Theorem 3.1.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $k$  be  $\mu$ -integrable functions on  $[a, b]$  such that  $k$  is positive,  $h$  is nonnegative and  $f/k$  is nonincreasing and right-continuous. Further, let  $[c, d] \subseteq [a, b]$  with  $\int_{[c,d]} h(t)k(t)d\mu(t) =$*

$\int_{[a,b]} g(t)k(t)d\mu(t)$ . If

$$(3.1) \quad \int_{[c,x]} k(t)g(t)d\mu(t) \leq \int_{[c,x]} k(t)h(t)d\mu(t), \quad c \leq x \leq d$$

and

$$(3.2) \quad \int_{[x,b]} k(t)g(t)d\mu(t) \geq 0, \quad d < x \leq b,$$

then

$$(3.3) \quad \int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[c,d]} f(t)h(t)d\mu(t) + \int_{[a,c]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) g(t)k(t)d\mu(t).$$

*Proof.* We use identity (2.15) and define a new measure  $\nu$  on  $\sigma$ -algebra  $\mathcal{B}((a, b])$  such that, on an algebra of finite disjoint unions of half open intervals, we set

$$\nu((x, y]) = \frac{f(x)}{k(x)} - \frac{f(y)}{k(y)}, \quad \text{for } a < x < y \leq b,$$

and then we pass to  $\mathcal{B}((a, b])$  in a unique way (for details see [1, p. 21]). Hence, using Fubini, we have

$$\begin{aligned} & \int_{[c,d]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) + \int_{[a,c]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) g(t)k(t)d\mu(t) \\ &= \int_{[c,d]} \left( \int_{(t,d]} d\nu(x) \right) k(t)[h(t) - g(t)]d\mu(t) + \int_{(d,b]} \left( \int_{(d,t]} d\nu(x) \right) g(t)k(t)d\mu(t) \\ &= \int_{[c,d]} \left( \int_{[c,x]} k(t)[h(t) - g(t)]d\mu(t) \right) d\nu(x) + \int_{(d,b]} \left( \int_{[x,b]} g(t)k(t)d\mu(t) \right) d\nu(x). \end{aligned}$$

So we have (3.3) when (3.1) and (3.2) hold.  $\square$

**Theorem 3.2.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $k$  be  $\mu$ -integrable functions on  $[a, b]$  such that  $k$  is positive,  $h$  is nonnegative and  $f/k$  is non-increasing and right-continuous. Further, let  $[c, d] \subseteq [a, b]$  with  $\int_{[c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$ . If*

$$(3.4) \quad \int_{[x,d]} k(t)g(t)d\mu(t) \leq \int_{[x,d]} k(t)h(t)d\mu(t), \quad c < x \leq d$$

and

$$(3.5) \quad \int_{[a,x)} k(t)g(t)d\mu(t) \geq 0, \quad a \leq x \leq c,$$

then

$$(3.6) \quad \int_{(c,d]} f(t)h(t)d\mu(t) - \int_{(d,b]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t).$$

*Proof.* Defining a new measure  $\nu$  as in the proof of Theorem 3.1 and using identity (2.21) and Fubini we obtain

$$\begin{aligned} & \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c,d]} f(t)h(t)d\mu(t) + \int_{(d,b]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) \\ &= \int_{[a,c]} \left( \int_{[a,x)} g(t)k(t)d\mu(t) \right) d\nu(x) + \int_{(c,d]} \left( \int_{[x,d]} k(t)[h(t) - g(t)]d\mu(t) \right) d\nu(x). \end{aligned}$$

So we have (3.6) when (3.4) and (3.5) hold.  $\square$

Taking  $k \equiv 1$  and  $h \equiv 1$  in Theorems 3.1 and 3.2 we obtain weaker conditions for the function  $g$  in an extension of Cerone's result obtained in Theorems 2.1 and 2.2.

**Theorem 3.3.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f$  and  $g$  be  $\mu$ -integrable functions on  $[a, b]$  such that  $f$  is nonincreasing and right-continuous. Further, let  $[c, d] \subseteq [a, b]$  with  $\mu([c, d]) = \int_{[a,b]} g(t)d\mu(t)$ . If*

$$(3.7) \quad \int_{[c,x)} g(t)d\mu(t) \leq \mu([c, x)), \quad c \leq x \leq d \quad \text{and} \quad \int_{[x,b]} g(t)d\mu(t) \geq 0, \quad d < x \leq b,$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[c,d]} f(t)d\mu(t) + \int_{[a,c]} (f(t) - f(d))g(t)d\mu(t).$$

**Theorem 3.4.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f$  and  $g$  be  $\mu$ -integrable functions on  $[a, b]$  such that  $f$  is nonincreasing and right-continuous. Further, let  $[c, d] \subseteq [a, b]$  with  $\mu((c, d]) = \int_{[a, b]} g(t) d\mu(t)$ . If*

$$(3.8) \quad \int_{[x, d]} g(t) d\mu(t) \leq \mu([x, d]), \quad c < x \leq d \quad \text{and} \quad \int_{[a, x)} g(t) d\mu(t) \geq 0, \quad a \leq x \leq c,$$

then

$$\int_{(c, d]} f(t) d\mu(t) - \int_{(d, b]} (f(c) - f(t)) g(t) d\mu(t) \leq \int_{[a, b]} f(t) g(t) d\mu(t).$$

Remark 5. If we take  $c = a$  and  $d = a + \lambda$  conditions (3.7) become

$$\int_{[a, x)} g(t) d\mu(t) \leq \mu([a, x)), \quad a \leq x \leq a + \lambda \quad \text{and} \quad \int_{[x, b]} g(t) d\mu(t) \geq 0, \quad a + \lambda < x \leq b.$$

For  $a + \lambda < x \leq b$  we have

$$\begin{aligned} \int_{[a, x)} g(t) d\mu(t) &= \int_{[a, b]} g(t) d\mu(t) - \int_{[x, b]} g(t) d\mu(t) = \mu([a, a + \lambda]) - \int_{[x, b]} g(t) d\mu(t) \\ &\leq \mu([a, a + \lambda]) \leq \mu([a, x)). \end{aligned}$$

Also, for  $a \leq x \leq a + \lambda$  we have

$$\begin{aligned} \int_{[x, b]} g(t) d\mu(t) &= \int_{[a, b]} g(t) d\mu(t) - \int_{[a, x)} g(t) d\mu(t) = \mu([a, a + \lambda]) - \int_{[a, x)} g(t) d\mu(t) \\ &\geq \mu([a, a + \lambda]) - \mu([a, x)) = \mu([x, a + \lambda]) \geq 0. \end{aligned}$$

Hence, for  $c = a$  and  $d = a + \lambda$  conditions (3.7) are equivalent to

$$\int_{[a, x)} g(t) d\mu(t) \leq \mu([a, x)) \quad \text{and} \quad \int_{[x, b]} g(t) d\mu(t) \geq 0, \quad \text{for every } x \in [a, b].$$

Similarly, if we take  $c = b - \lambda$  and  $d = b$  conditions (3.8) are equivalent to

$$\int_{[x, b]} g(t) d\mu(t) \leq \mu([x, b]) \quad \text{and} \quad \int_{[a, x)} g(t) d\mu(t) \geq 0, \quad \forall x \in [a, b].$$

Therefore, if we take  $c = a$  and  $d = a + \lambda$  in Theorem 3.3, or  $c = b - \lambda$  and  $d = b$  in Theorem 3.4 we obtain sufficient conditions from [4].

In the following theorems we obtain weaker conditions for refinements given in Theorems 2.7 and 2.8.

**Theorem 3.5.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $k$  be  $\mu$ -integrable functions on  $[a, b]$  such that  $k$  is positive,  $h$  is nonnegative and  $f/k$  is non-increasing and right-continuous. Further, let  $[c, d] \subseteq [a, b]$  with  $\int_{[c, d]} h(t)k(t)d\mu(t) = \int_{[a, b]} g(t)k(t)d\mu(t)$ . If (3.1) and (3.2) hold, then*

$$\begin{aligned}
 \int_{[a, b]} f(t)g(t)d\mu(t) &\leq \int_{[c, d]} f(t)h(t)d\mu(t) + \int_{[a, c]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) g(t)k(t)d\mu(t) \\
 (3.9) \quad &\quad - \int_{[c, d]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) k(t)[h(t) - g(t)]d\mu(t) \\
 &\leq \int_{[c, d]} f(t)h(t)d\mu(t) + \int_{[a, c]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) g(t)k(t)d\mu(t).
 \end{aligned}$$

*Proof.* Using identity (2.7), defining a new measure  $\nu$  as in the proof of Theorem 3.1 and using Fubini we have

$$\begin{aligned}
 &\int_{[c, d]} f(t)h(t)d\mu(t) - \int_{[a, b]} f(t)g(t)d\mu(t) + \int_{[a, c]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) g(t)k(t)d\mu(t) \\
 &- \int_{[c, d]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) k(t)[h(t) - g(t)]d\mu(t) = \int_{(d, b]} \left( \frac{f(d)}{k(d)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) \\
 &= \int_{(d, b]} \left( \int_{(d, t]} d\nu(x) \right) g(t)k(t)d\mu(t) = \int_{(d, b]} \left( \int_{[x, b]} g(t)k(t)d\mu(t) \right) d\nu(x) \geq 0
 \end{aligned}$$

when

$$\int_{[x, b]} g(t)k(t)d\mu(t) \geq 0, \quad d < x \leq b.$$

Furthermore,

$$\begin{aligned}
 &\int_{[c, d]} \left( \frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) k(t)[h(t) - g(t)]d\mu(t) \\
 &= \int_{[c, d]} \left( \int_{[c, x]} k(t)[h(t) - g(t)]d\mu(t) \right) d\nu(x) \geq 0
 \end{aligned}$$

when

$$\int_{[c, x]} k(t)g(t)d\mu(t) \leq \int_{[c, x]} k(t)h(t)d\mu(t), \quad c \leq x \leq d.$$

Hence (3.9) holds when (3.1) and (3.2) hold.  $\square$

**Theorem 3.6.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $k$  be  $\mu$ -integrable functions on  $[a, b]$  such that  $k$  is positive,  $h$  is nonnegative and  $f/k$  is non-increasing and right-continuous. Further, let  $[c, d] \subseteq [a, b]$  with  $\int_{(c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$ . If (3.4) and (3.5) hold, then*

$$\begin{aligned} & \int_{(c,d]} f(t)h(t)d\mu(t) - \int_{(d,b]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) \leq \int_{(c,d]} f(t)h(t)d\mu(t) \\ & - \int_{(d,b]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) g(t)k(t)d\mu(t) + \int_{(c,d]} \left( \frac{f(c)}{k(c)} - \frac{f(t)}{k(t)} \right) k(t)[h(t) - g(t)]d\mu(t) \\ & \leq \int_{[a,b]} f(t)g(t)d\mu(t). \end{aligned}$$

*Proof.* Similar to the proof of Theorem 3.5 using identity (2.9).  $\square$

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