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COMMON FIXED POINT THEOREMS IN ORDERED MENGER SPACES

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Abstract. The purpose of this paper is to prove common fixed point results for a wide class of contractive mappings in ordered probabilistic metric space. Our results are extensions of the results of Nieto and Rodríguez-Lopez, as well as Ran and Reurings on fixed points of mappings in ordered metric spaces. On the other hand, results of Fang, Mishra, Singh and Jain, as well as Razani and Shirdaryazdi are generalized using partial order and corresponding conditions for the given mappings. Results of several other authors are also partially generalized.

1. Introduction

The study of fixed point theorems in probabilistic metric spaces is an active area of research. The theory of probabilistic metric spaces was introduced by Menger [9] in 1942 and since then the theory of probabilistic metric spaces has developed in many directions, especially in nonlinear analysis and applications. In 1966, Sehgal [19] initiated the study of contraction mapping theorems in probabilistic metric spaces. Since then several generalizations of fixed point theorems in probabilistic metric spaces

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have been obtained by several authors including Sehgal and Bharucha-Reid [20], and Istratescu and Roventa [7]. The study of fixed point theorems in probabilistic metric spaces is useful in the study of existence of solutions of operator equations in probabilistic metric spaces and probabilistic functional analysis. The development of fixed point theory in probabilistic metric spaces was due to Schweizer and Sklar [17], see also [2, 4, 6, 16].

The notion of compatible maps in Menger spaces has been introduced by Mishra [10]. Singh and Jain [21] introduced the concept of weakly commuting mappings in probabilistic metric spaces and proved a common fixed point theorem in PM spaces. Recently, Razani and Shirdaryazdi [15] proved a common fixed point theorem for any even number of self maps in Menger space, which is a generalization of Singh and Jain [21] common fixed point theorem.

Nieto and Rodríguez-Lopez [12], Ran and Reurings [14], as well as Petrusel and Rus [13] presented some new results for contractions in partially ordered metric spaces. The main idea in [11, 12, 14] involve combining the ideas of iterative technique in the contraction mapping principle with those in the monotone technique.

The purpose of this paper is to prove common fixed point results for a wide class of contractive mappings in ordered probabilistic metric space. Our results are extensions of the results of Nieto and Rodríguez-Lopez [11, 12], as well as Ran and Reurings [14] on fixed points of mappings in ordered metric spaces. On the other hand, results of Fang [4], Mishra [10], Singh and Jain [21], as well as Razani and Shirdaryazdi [15] are generalized using partial order and corresponding conditions for the given mappings. Results of several other authors are also partially generalized.

2. Basic notions and auxiliary results

First, we recall some definitions and known results in Menger spaces.

We shall denote by $\mathcal{D}$ the set of all distribution functions, i.e., mappings $F : \mathbb{R} \to \mathbb{R}^+$ which are non-decreasing left-continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$. An example is the function H defined by

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0. \end{cases}$$

For $F_1, F_2 \in \mathcal{D}$, their algebraic sum $F_1 \oplus F_2$ is defined by

$$(F_1 \oplus F_2)(t) = \sup_{t_1+t_2=t} \min\{F_1(t_1), F_2(t_2)\}, \quad \forall t \in \mathbb{R}.$$

It is clear that

$$(F_1 \oplus F_2)(2t) \geq \min\{F_1(t), F_2(t)\}, \quad \forall t \geq 0.$$

A mapping $\triangle : [0,1] \times [0,1] \to [0,1]$ is called an ordered triangular norm (a t-norm, for short) [18] if the following hold:

(i) $\triangle(a, 1) = a$,

(ii) $\triangle(a, b) = \triangle(b, a)$,

(iii) $a \geq b, c \geq d \implies \triangle(a, c) \geq \triangle(b, d)$,

(iv) $\triangle(a, \triangle(b, c)) = \triangle(\triangle(a, b), c)$.

Definition 2.1. A quadruple $(X, \mathcal{F}, \preceq, \triangle)$ is called an ordered Menger probabilistic metric space (an ordered Menger space, for short) if $(X, \preceq)$ is a non-empty ordered set, $\triangle$ is an ordered t-norm and $\mathcal{F}$ is a mapping from $X \times X$ into $\mathcal{D}$ satisfying the following conditions (for $x, y \in X$, we denote $\mathcal{F}(x, y)$ by $F_{x,y}$):

(MS-1) $F_{x,y}(t) = H(t)$ for all $t \in \mathbb{R}$ if and only if $x = y$;

(MS-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$;

(MS-3) $F_{x,y}(s + t) \geq \triangle(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Remark 1. [6, 18] If the ordered t-norm $\triangle$ of an ordered Menger space $(X, \mathcal{F}, \preceq, \triangle)$ satisfies the condition $\sup_{0<t<1} \triangle(t, t) = 1$, then $(X, \mathcal{F}, \preceq, \triangle)$ becomes an ordered Hausdorff topological space in the so-called (ε, λ)-topology $\mathcal{T}$. A basis of $\mathcal{T}$-neighbourhoods of the point x is formed by the sets

$$\{U_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1]\} \ \ (x \in X),$$

where

$$U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

A sequence $\{x_n\}$ in X is $\mathcal{T}$-convergent to $x \in X$ (denoted as $x_n \to^{\mathcal{T}} x$) [18] if $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is a $\mathcal{T}$-Cauchy sequence in X if for any $\varepsilon > 0$ and $\lambda \in (0, 1]$, there exists a positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1-\lambda$, whenever $m, n \geq N$; $(X, \mathcal{F}, \preceq, \triangle)$ is said to be $\mathcal{T}$-complete if each $\mathcal{T}$-Cauchy sequence in X is $\mathcal{T}$-convergent to some point in X.

In what follows, we always assume that $(X, \mathcal{F}, \preceq, \triangle)$ is an ordered Menger space equipped with the (ε, λ)-topology.

Lemma 2.1. [20] *Let $(X, d, \preceq)$ be an ordered metric space and let $\mathcal{F} : X \times X \to \mathcal{D}$ be defined by*

$$\mathcal{F}(x, y)(t) = F_{x,y}(t) = H(t - d(x, y)) \ \ \textit{for} \ \ x, y \in X \ \ \textit{and} \ \ t > 0. \tag{2.1}$$

Then $(X, \mathcal{F}, \preceq, \min)$ is an ordered Menger space, which is complete if $(X, d, \preceq)$ is complete.

The space $(X, \mathcal{F}, \preceq, \min)$ is called the ordered Menger space associated with $(X, d, \preceq)$.

Definition 2.2. [6] A t-norm $\triangle$ is said to be of H-type if the family of functions $\{\triangle^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\triangle^1(t) = \triangle(t, t), \ \ \triangle^m(t) = \triangle(t, \triangle^{m-1}(t)), \ \ m = 1, 2, \dots, \ t \in [0, 1].$$

The ordered t-norm $\triangle_M = \min$ is an obvious, but not a sole example of ordered t-norm of H-type.

Definition 2.3. For a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ let $\phi^n(t)$ denote the n^{th} iteration of $\phi(t)$. Consider the following conditions :

(ϕ_1) ϕ is non-decreasing,

(ϕ_1') ϕ is strictly increasing,

(ϕ_2) ϕ is upper semi-continuous from the right,

(ϕ_3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$.

We will consider the following three classes of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$:

(1) Φ_0 is the class of all functions satisfying conditions (ϕ_1) and (ϕ_3).
(2) Φ_1 is the class of all functions satisfying conditions (ϕ_1') and (ϕ_3).
(3) Φ is the class of all functions satisfying conditions (ϕ_1), (ϕ_2) and (ϕ_3).

Remark 2. If $\phi \in \Phi_0$ then $\phi(t) < t$ for all $t > 0$. Obviously, $\Phi_0 \subset \Phi$ and $\Phi_1 \subset \Phi$.

Lemma 2.2. [3] *Let $\phi \in \Phi_1$ and let the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by*

$$\psi(t) = \phi(t+0) = \lim_{\tau \to 0^+} \phi(t+\tau), \quad t \in \mathbb{R}^+.$$

Then $\psi \in \Phi$.

Lemma 2.3. [5] *Let $(X, \mathcal{F}, \preceq, \triangle)$ be an ordered Menger space with a continuous ordered t-norm $\triangle$, and let $\{x_n\}$ and $\{y_n\}$ be two sequences in X, such that $x_n \to x$, $y_n \to y$ and $x \neq y$. Then*

(1) $\liminf_{n\to\infty} F_{x_n,y_n}(t) > F_{x,y}(t), \quad \forall t > 0;$
(2) $F_{x,y}(t+0) > \limsup_{n\to\infty} F_{x_n,y_n}(t), \quad \forall t > 0.$

Definition 2.4. [10] Let S and T be two self mappings of an ordered Menger space $(X, \mathcal{F}, \preceq, \triangle)$. S and T are said to be compatible if $F_{STx_n,TSx_n}(t) \to 1$ for all $t > 0$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$ for some $u \in X$.

Definition 2.5. [8] Let S and T be two self mappings of an non-empty set X. S and T are said to be weakly compatible if $TSu = STu$ whenever $Tu = Su$ for some $u \in X$.

Remark 3. Two compatible self maps are weakly compatible, but the converse is not true [21].

Recently Abbas et al. [1] introduced the following two new concepts in partial ordered sets as follows:

Definition 2.6. [1] Let $(X, \preceq)$ be a partially ordered set and $f, g : X \to X$.

(1) f is said to be a weak annihilator of g if $fgx \preceq x$ for all $x \in X$.
(2) f is said to be dominating if $x \preceq fx$ for all $x \in X$.

3. Main results

In order to prove our main results, we will first note that the following two lemmas can be proved in the same way as Lemma 3.1 and Lemma 3.2 from [4] (since they do not depend on the partial order $\preceq$ in X).

Lemma 3.1. *Let $\{y_n\}$ be a sequence in an ordered Menger space $(X, \mathcal{F}, \preceq, \triangle)$, where $\triangle$ is a t-norm of H-type. If there exists a function $\phi \in \Phi$ such that*

$$F_{y_n,y_{n+1}}(\phi(t)) \geq \min\{F_{y_{n-1},y_n}(t), F_{y_n,y_{n+1}}(t)\}, \quad \forall t > 0 \ \ and \ \ n \in \mathbb{N},$$

then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 3.2. *Let $(X, \mathcal{F}, \preceq, \triangle)$ be an ordered Menger space and $x, y \in X$. If there exists $\phi \in \Phi$ such that*

$$F_{x,y}(\phi(t) + 0) \geq F_{x,y}(t), \quad \forall t > 0,$$

then $x = y$.

Theorem 3.1. *Let* P, Q, L *and* M *be self mappings of a complete ordered Menger space* $(X, \mathcal{F}, \preceq, \triangle)$ *with a continuous ordered t-norm* $\triangle$ *of H-type. Let the following conditions be satisfied:*

(i) $L(X) \subseteq Q(X)$, $M(X) \subseteq P(X)$,

(ii) L *and* M *are dominating maps,*

(iii) L *is a weak annihilator of* Q *and* M *is a weak annihilator of* P,

(iv) *there exists* $\phi \in \Phi$ *or* $\phi \in \Phi_1$ *such that*

$$F_{Lx,My}(\phi(t)) \geq \min \left\{ \begin{array}{c} F_{Px,Lx}(t), F_{Qy,My}(t), F_{Px,Qy}(t), \\ F_{Qy,Lx}(t), [F_{Px,\xi} \oplus F_{\xi,My}]((2-\beta)t) \end{array} \right\} \tag{3.1}$$

for all $x, y, \xi \in X$ *with* x *and* y *comparable w.r.t.* $\preceq$, $\beta \in (0, 2)$ *and* $t > 0$.

(v) *for each non-decreasing sequence* $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n, $y_n \to u$ *implies that* $x_n \preceq u$.

Assume either

(a) $\{L, P\}$ *is compatible,* L *or* P *is continuous and* $\{Q, M\}$ *is weakly compatible or*

(b) $\{Q, M\}$ *is compatible,* Q *or* M *is continuous and* $\{L, P\}$ *is weakly compatible.*

Then L, Q, M *and* P *have a common fixed point. Moreover, the set of common fixed points of* L, Q, M *and* P *is totally ordered if and only if* L, Q, M *and* P *have one and only one common fixed point.*

Proof. Let $x_0 \in X$. It follows from condition (i) that there exist $x_1, x_2 \in X$ such that $Lx_0 = Qx_1 = y_0$ and $Mx_1 = Px_2 = y_1$. We can construct inductively two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Lx_{2n} = Qx_{2n+1} = y_{2n} \text{ and } Mx_{2n+1} = Px_{2n+2} = y_{2n+1}$$

for $n = 0, 1, 2, \dots$. Using conditions (ii) and (iii) we get that

$$x_{2n} \preceq Lx_{2n} = y_{2n} = Qx_{2n+1} \preceq LQx_{2n+1} \preceq x_{2n+1} \tag{3.2}$$

and

$$x_{2n+1} \preceq Mx_{2n+1} = y_{2n+1} = Px_{2n+2} \preceq MPx_{2n+2} \preceq x_{2n+2} \tag{3.3}$$

for $n = 0, 1, 2, \dots$.

Assume that there exists $\phi \in \Phi$ such that (3.1) holds. Putting $x = x_{2n}$, $y = x_{2n+1}$ and $\xi = y_{2n}$ in (3.1), we get

$$\begin{aligned}
F_{y_{2n},y_{2n+1}}(\phi(t)) &\geq F_{Lx_{2n},Mx_{2n+1}}(\phi(t)) \\
&\geq \min\left\{\begin{array}{c} F_{Px_{2n},Lx_{2n}}(t), F_{Qx_{2n+1},Mx_{2n+1}}(t), F_{Px_{2n},Qx_{2n+1}}(t), \\ F_{Qx_{2n+1},Lx_{2n}}(t), [F_{Px_{2n},y_{2n}} \oplus F_{y_{2n},Mx_{2n+1}}]((2-\beta)t) \end{array}\right\} \\
&\geq \min\left\{\begin{array}{l} F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n+1}}(t), \\ F_{y_{2n-1},y_{2n}}((2-\beta)t/2), F_{y_{2n},y_{2n+1}}((2-\beta)t/2) \end{array}\right\}.
\end{aligned}$$

Letting $\beta \to 0$, we get

$$F_{y_{2n},y_{2n+1}}(\phi(t)) \geq \min\{F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n+1}}(t)\}. \tag{3.4}$$

Similarly, we can prove that

$$F_{y_{2n+1},y_{2n+2}}(\phi(t)) \geq \min\{F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n+2}}(t)\}. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$F_{y_n,y_{n+1}}(\phi(t)) \geq \min\{F_{y_{n-1},y_n}(t), F_{y_n,y_{n+1}}(t)\}, \quad n = 1, 2, \dots.$$

Hence, the conditions of Lemma 3.1 are satisfied, so that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, we conclude that $y_n \to z$ for some $z \in X$. Thus,

$$\lim_{n\to\infty} Lx_{2n} = \lim_{n\to\infty} Px_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Mx_{2n+1} = z. \tag{3.6}$$

Using (3.2), (3.3) and condition (v), we get that $x_n \preceq z$ for each $n \in \mathbb{N}$.

Case-I. Suppose that (a) holds. Since P is continuous and $\{L, P\}$ is compatible, we have

$$\lim_{n\to\infty} LPx_{2n} = \lim_{n\to\infty} PLx_{2n} = Pz.$$

Also, L is a dominating mapping, so

$$x_{2n} \preceq Lx_{2n} = Qx_{2n+1}.$$

Now taking $x = Px_{2n} = y_{2n-1}$, $y = x_{2n+1}$, $\xi = LPx_{2n}$ and $\beta = 1$ in (3.1), we have

$$\begin{aligned} F_{LPx_{2n},Mx_{2n+1}}(\phi(t)) &\geq \min\left\{\begin{array}{l} F_{PPx_{2n},LPx_{2n}}(t), F_{Qx_{2n+1},Mx_{2n+1}}(t), F_{PPx_{2n},Qx_{2n+1}}(t), \\ F_{Qx_{2n+1},LPx_{2n}}(t), [F_{PPx_{2n},LPx_{2n}} \oplus F_{LPx_{2n},Mx_{2n+1}}](t) \end{array}\right\} \\ &\geq \min\left\{\begin{array}{l} F_{PPx_{2n},LPx_{2n}}(t), F_{Qx_{2n+1},Mx_{2n+1}}(t), F_{PPx_{2n},Qx_{2n+1}}(t), \\ F_{Qx_{2n+1},LPx_{2n}}(t), F_{PPx_{2n},LPx_{2n}}(\varepsilon), F_{LPx_{2n},Mx_{2n+1}}(t-\varepsilon) \end{array}\right\}, \end{aligned}$$

where $\varepsilon \in (0, t)$. Letting $n \to \infty$, by Lemma 2.3 we get

$$F_{Pz,z}(\phi(t)) \geq \min\{1, 1, F_{Pz,z}(t), F_{z,Pz}(t), 1, F_{Pz,z}(t-\varepsilon)\}.$$

Now, letting $\varepsilon \to 0$, we get

$$F_{Pz,z}(\phi(t) + 0) \geq F_{Pz,z}(t), \quad \forall t > 0,$$

which implies that $Pz = z$ by Lemma 3.2.

Putting $x = \xi = z$, $y = x_{2n+1}$ (which is possible by condition (v)) and $\beta = 1$ in (3.1), we get

$$F_{Lz,Mx_{2n+1}}(\phi(t)) \geq \min\left\{\begin{array}{l} F_{Pz,Lz}(t), F_{Qx_{2n+1},Mx_{2n+1}}(t), F_{Pz,Qx_{2n+1}}(t), \\ F_{Qx_{2n+1},Lz}(t), [F_{Pz,z} \oplus F_{z,Mx_{2n+1}}](t) \end{array}\right\}.$$

Letting $n \to \infty$ in the above inequality, we have

$$F_{Lz,z}(\phi(t)+0) \geq \min\{F_{z,Lz}(t), 1, 1, F_{z,Lz}(t), 1, 1\},$$

$$F_{Lz,z}(\phi(t)+0) \geq F_{z,Lz}(t),$$

for all $t > 0$. By Lemma 3.2, we conclude that $Lz = z$. Thus, z is a common fixed point of L and P.

Moreover, from $Lz = z$ and (3.6), it can be proved that z is also a common fixed point of M and Q, i.e., $Mz = Qz = z$. Indeed, since $L(X) \subset Q(X)$, there exists $v \in X$ such that $z = Lz = Qv$. Putting $x = x_{2n}$, $y = v$, $\xi = z$ and $\beta = 1$ in condition (3.1), we get

$$F_{Lx_{2n},Mv}(\phi(t)) \geq \min \left\{ \begin{array}{l} F_{Px_{2n},Lx_{2n}}(t), F_{Qv,Mv}(t), F_{Px_{2n},Qv}(t), \\ F_{Qv,Lx_{2n}}(t), [F_{Px_{2n},z} \oplus F_{z,Mv}](t) \end{array} \right\}.$$

As $n \to \infty$ in the above inequality, we get

$$F_{z,Mv}(\phi(t)) \geq \min\{1, F_{z,Mv}(t), 1, 1, 1, F_{z,Mv}(t-\varepsilon)\}$$

that is, for any $\varepsilon \in (0,t)$,

$$F_{z,Mv}(\phi(t)) \geq F_{z,Mv}(t-\varepsilon), \quad \forall t > 0.$$

Letting $\varepsilon \to 0$, we have

$$F_{z,Mv}(\phi(t)+0) \geq F_{z,Mv}(t),$$

for all $t > 0$, which implies that $Mv = z$ by Lemma 3.2. So, we have $Qv = z = Mv$, i.e., v is a coincidence point of Q and M. Since $\{M, Q\}$ is weakly compatible, we have $MQv = QMv$, and so $Mz = Qz = z$. Therefore, z is a common fixed point of L, M, P and Q.

Case-II. Suppose that L is continuous. Since $Lx_{2n} \to z$ and $Px_{2n} \to z$, we have $LLx_{2n} \to Lz$ and $LPx_{2n} \to Lz$. As $\{L, P\}$ is compatible, we have $F_{PLx_{2n},LPx_{2n}}(t) \to 1$

for all $t > 0$. It is then easy to prove that $PLx_{2n} \to Lz$. Again using condition (3.1), on taking $x = Lx_{2n} = y_{2n}$, $y = x_{2n+1}$, $\xi = Lz$ and $\beta = 1$, we obtain that

$$\begin{aligned} F_{LLx_{2n},Mx_{2n+1}}(\phi(t)) &\geq \min\left\{\begin{array}{l} F_{PLx_{2n},LLx_{2n}}(t), F_{Qx_{2n+1},Mx_{2n+1}}(t), F_{PLx_{2n},Qx_{2n+1}}(t), \\ F_{Qy,LLx_{2n}}(t), [F_{PLx_{2n},Lz} \oplus F_{Lz,Mx_{2n+1}}]((2-\beta)t) \end{array}\right\} \\ &\geq \min\left\{\begin{array}{l} F_{PLx_{2n},LLx_{2n}}(t), F_{Qx_{2n+1},Mx_{2n+1}}(t), F_{PLx_{2n},Qx_{2n+1}}(t), \\ F_{Qy,LLx_{2n}}(t), F_{PLx_{2n},Lz}(\varepsilon), F_{Lz,Mx_{2n+1}}(t-\varepsilon) \end{array}\right\} \end{aligned}$$

where $\varepsilon \in (0, t)$. Letting $n \to \infty$, by Lemma 2.3 we get

$$\begin{aligned} F_{Lz,z}(\phi(t)+0) &\geq \min\{F_{Lz,Lz}(t), F_{z,z}(t), F_{Lz,z}(t), F_{z,Lz}(t), F_{Lz,Lz}(\varepsilon), F_{Lz,z}(t-\varepsilon)\} \\ &\geq F_{Lz,z}(t-\varepsilon). \end{aligned}$$

Letting now $\varepsilon \to 0$, it follows that $F_{Lz,z}(\phi(t)+0) \geq F_{Lz,z}(t)$, for all $t > 0$, which implies that $Lz = z$.

In the same way as in Case-I, from $Lz = z$ and (3.6), it is not difficult to prove that $Mz = Qz = z$. In what follows, we need only to show that $Pz = z$.

Since $M(X) \subset P(X)$, there exists $w \in X$ such that $z = Mz = Pw$. By condition (3.1), taking $x = w, y = x_{2n+1}, \xi = z$ and $\beta = 1$, we get

$$\begin{aligned} F_{Lw,Mx_{2n+1}}(\phi(t)) &\geq \min\left\{\begin{array}{l} F_{Pw,Lw}(t), F_{Qx_{2n+1},Mx_{2n+1}}(t), F_{Pw,Qx_{2n+1}}(t), \\ F_{Qx_{2n+1},Lw}(t), [F_{Pw,z} \oplus F_{z,Mx_{2n+1}}](t) \end{array}\right\} \\ &\geq \min\left\{\begin{array}{l} F_{Pw,Lw}(t), F_{Qx_{2n+1},Mx_{2n+1}}(t), F_{Pw,Qx_{2n+1}}(t), \\ F_{Qx_{2n+1},Lw}(t), F_{Pw,z}(t), F_{z,Mx_{2n+1}}(t) \end{array}\right\}. \end{aligned}$$

When $n \to \infty$ in above inequality, by Lemma 2.3,

$$F_{Lw,z}(\phi(t)+0) \geq \min\{F_{z,Lw}(t), 1, 1, F_{z,Lw}(t), 1\}$$

$$F_{Lw,z}(\phi(t)+0) \geq F_{z,Lw}(t)$$

for all $t > 0$, which implies that $Lw = z = Pw$. Note that $\{L, P\}$ is compatible, so it is also weakly compatible. Hence $Pz = PLw = LPw = Lz = z$. This shows that z is a common fixed point of L, M, P and Q.

Finally, we show that the common fixed point is unique if the set of common fixed points is totally ordered. Let u be another common fixed point of L, M, P and Q (which is comparable with z). Then $Lu = Pu = Mu = Qu = u$. Thus, putting $x = \xi = z, y = u$ and $\beta = 1$ in condition (3.1), we get

$$\begin{aligned} F_{Lz,Mz}(\phi(t)) &\geq \min\{F_{Pz,Lz}(t), F_{Qu,Mu}(t), F_{Pz,Qu}(t), F_{Qu,Lz}(t), [F_{Pz,z} \oplus F_{z,Mu}](t)\} \\ &\geq \min\{1, 1, F_{z,u}(t), F_{u,z}(t), F_{u,z}(t)\} \\ &\geq F_{u,z}(t) \end{aligned}$$

for $t > 0$. This implies that $z = u$. Therefore, z is a unique common fixed point of L, M, P and Q.

If there exists $\phi \in \Phi_1$ such that (3.1) holds, we define a new function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as in Lemma 2.2, so that $\phi \in \Phi$. Moreover, by the definition of ϕ and (3.1), we have

$$\begin{aligned} F_{Lx,My}(\psi(t)) &= F_{Lx,My}(\phi(t+0)) \geq F_{Lx,My}(\phi(t)) \\ &\geq \min\left\{\begin{array}{l} F_{Px,Lx}(t), F_{Qy,My}(t), F_{Px,Qy}(t), F_{Qy,Lx}(t), \\ F_{Px,\xi} \oplus F_{\xi,My}]((2-\beta)t) \end{array}\right\} \end{aligned}$$

for all $x, y, \xi \in X$, with x, y comparable, $\beta \in (0, 2)$ and $t > 0$. This shows that the function ψ also satisfies (3.1). Therefore the conclusion follows as above. $\square$

Taking $\phi(t) = kt$ in Theorem 3.1, we obtain the following

Corollary 3.1. *Let P, Q, L and M be self mappings of a complete ordered Menger space $(X, \mathcal{F}, \preceq, \triangle)$ with a continuous ordered t-norm $\triangle$ of H-type. Let the following conditions be satisfied:*

(i) $L(X) \subseteq Q(X)$, $M(X) \subseteq P(X)$,

(ii) *L and M are dominating maps,*

(iii) *L is a weak annihilator of Q and M is a weak annihilator of P,*

(iv) *for some $k \in (0,1)$ and all $x, y, \xi \in X$, with x, y comparable, $\beta \in (0,2)$ and $t > 0$,*

$$F_{Lx,My}(kt) \geq \min \left\{ \begin{array}{l} F_{Px,Lx}(t), F_{Qy,My}(t), F_{Px,Qy}(t), F_{Qy,Lx}(t), \\ [F_{Px,\xi} \oplus F_{\xi,My}]((2-\beta)t) \end{array} \right\} \tag{3.7}$$

(v) *for each non-decreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n, $y_n \to u$ implies that $x_n \preceq u$.*

Assume either

(a) *$\{L, P\}$ is compatible, L or P is continuous and $\{Q, M\}$ is weakly compatible or*

(b) *$\{Q, M\}$ is compatible, Q or M is continuous and $\{L, P\}$ is weakly compatible.*

Then L, Q, M and P have a common fixed point. Moreover, the set of common fixed points of L, Q, M and P is totally ordered if and only if L, Q, M and P have one and only one common fixed point.

Corollary 3.2. *Let P, Q, L and M be self mappings of a complete ordered Menger space $(X, \mathcal{F}, \preceq, \triangle)$ with $\triangle(t,t) \geq t$ for all $t \in [0,1]$. Let the following conditions be satisfied:*

(i) *$L(X) \subseteq Q(X)$, $M(X) \subseteq P(X)$,*

(ii) *L and M are dominating maps,*

(iii) *L is a weak annihilator of Q and M is a weak annihilator of P,*

(iv) *for some $k \in (0,1)$ and all $x, y, \xi \in X$, with x, y comparable, $\beta \in (0,2)$ and $s > 0$*

$$F_{Lx,My}(ks) \geq \min \left\{ \begin{array}{l} F_{Px,Lx}(s), F_{Qy,My}(s), F_{Px,Qy}(s), F_{Qy,Lx}(s), \\ F_{Px,\xi}(\beta s), F_{\xi,My}((2-\beta)s) \end{array} \right\} \tag{3.8}$$

(v) *for each non-decreasing sequence* $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n, $y_n \to u$ *implies that* $x_n \preceq u$.

Assume either

(a) $\{L, P\}$ *is compatible,* L *or* P *is continuous and* $\{Q, M\}$ *is weakly compatible or*

(b) $\{Q, M\}$ *is compatible,* Q *or* M *is continuous and* $\{L, P\}$ *is weakly compatible.*

Then L, Q, M *and* P *have a common fixed point. Moreover, the set of common fixed points of* L, Q, M *and* P *is totally ordered if and only if* L, Q, M *and* P *have one and only one common fixed point.*

Proof. If $\triangle = \min$, then by (MS-3) we have

$$F_{Px,My}(t) \geq \sup_{0<s<t} \min\{F_{Px,\xi}(s), F_{\xi,My}(t-s)\}$$

for all $x, y, \xi \in X$ and $t > 0$. Moreover, note that the ordered t-norm $\triangle$ satisfies $\triangle(t,t) \geq t$ for all $t \in [0,1]$ if and only if $\triangle = \min$. Hence, it is clear that (3.8) implies (3.7). Thus, the conclusion follows from Corollary 3.1. $\square$

If $P = Q$ in Theorem 3.1, we have the following consequence.

Corollary 3.3. *Let* P, L *and* M *be self mappings of a complete ordered Menger space* $(X, \mathcal{F}, \preceq, \triangle)$ *with a continuous ordered t-norm* $\triangle$ *of H-type. Let the following conditions be satisfied:*

(i) $L(X) \subseteq P(X)$, $M(X) \subseteq P(X)$,

(ii) L *and* M *are dominating maps,*

(iii) L *and* M *are weak annihilators of* P,

(iv) *there exists* $\phi \in \Phi$ *or* $\phi \in \Phi_1$ *such that*

$$F_{Lx,My}(\phi(t)) \geq \min \left\{ \begin{array}{l} F_{Px,Lx}(t), F_{Py,My}(t), F_{Px,Py}(t), F_{Py,Lx}(t), \\ {[F_{Px,\xi} \oplus F_{\xi,My}]((2-\beta)t)} \end{array} \right\}$$

for all $x, y, \xi \in X$, with x, y comparable, $\beta \in (0,2)$ and $t > 0$,

(v) *for each non-decreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n, $y_n \to u$ implies that $x_n \preceq u$.*

Assume either

(a) *$\{L, P\}$ is compatible, L or P is continuous and $\{M, P\}$ is weakly compatible or*

(b) *$\{M, P\}$ is compatible, P or M is continuous and $\{L, P\}$ is weakly compatible.*

Then L, M and P have a common fixed point. Moreover, the set of common fixed points of L, M and P is totally ordered if and only if L, M and P have one and only one common fixed point.

If $L = M$ and $P = Q$ in Theorem 3.1, then we have the following consequence.

Corollary 3.4. *Let P, L be self mappings of a complete ordered Menger space $(X, \mathcal{F}, \preceq, \triangle)$ with a continuous ordered t-norm $\triangle$ of H-type. Let the following conditions be satisfied:*

(i) *$L(X) \subseteq P(X)$,*

(ii) *L is a dominating mapping,*

(iii) *L is a weak annihilator of P,*

(iv) *there exists $\phi \in \Phi$ or $\phi \in \Phi_1$ such that*

$$F_{Lx,Ly}(\phi(t)) \geq \min \left\{ \begin{array}{l} F_{Px,Lx}(t), F_{Py,Ly}(t), F_{Px,Py}(t), F_{Py,Lx}(t), \\ {[F_{Px,\xi} \oplus F_{\xi,Ly}]((2-\beta)t)} \end{array} \right\}$$

for all $x, y, \xi \in X$, with x, y comparable, $\beta \in (0,2)$ and $t > 0$,

(v) *for each non-decreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n, $y_n \to u$ implies that $x_n \preceq u$,*

(vi) *$\{L, P\}$ is compatible, L or P is continuous.*

Then L and P have a common fixed point. Moreover, the set of common fixed points of L and P is totally ordered if and only if L and P have one and only one common fixed point.

Example 3.1. *Let $X = [0, 1]$ with the metric d defined by $d(x, y) = |x-y|$ and define $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X, t > 0$. Clearly $(X, \mathcal{F}, \preceq, \triangle)$ is a complete ordered Menger space with a continuous ordered t-norm $\triangle$ of H-type. Let P, Q, L and M be maps from X into itself defined as*

$$Px = \frac{x}{2},\ Qx = \frac{x}{5},\ Lx = 0,\ Mx = \frac{x}{6}$$

for all $x \in X$. Then

$$L(X) = \{0\} \subset \left[0, \frac{1}{5}\right] = Q(X)$$

and

$$M(X) = \left[0, \frac{1}{6}\right] \subset \left[0, \frac{1}{2}\right] = P(X).$$

Clearly all conditions of Theorem 3.1 are satisfied and common fixed point is 0.

We conclude by stating some more results in ordered Menger spaces which can be proved similarly as the corresponding results in [4].

The following common fixed point theorem for six self-maps on X, is an extension of [4, Theorem 3.2] and thus a generalization of the one from Singh and Jain [21].

Theorem 3.2. *Let A, B, S, T, L and M be self mappings of a complete ordered Menger space $(X, \mathcal{F}, \preceq, \triangle)$ with a continuous ordered t-norm $\triangle$ of H-type. Let the following conditions be satisfied:*

(i) *$L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$,*

(ii) *$AB = BA$, $ST = TS$, $LB = BL$, $MT = TM$,*

(iii) *L and M are dominating maps,*

(iv) *L is a weak annihilator of ST and M is a weak annihilator of AB,*

(v) *there exists* $\phi \in \Phi$ *or* $\phi \in \Phi_1$ *such that*

$$F_{Lx,My}(\phi(t)) \geq \min \begin{Bmatrix} F_{ABx,Lx}(t), F_{STy,My}(t), F_{ABx,STy}(t), F_{STy,Lx}(t), \\ [F_{ABx,\xi} \oplus F_{\xi,My}]((2-\beta)t) \end{Bmatrix} \tag{3.9}$$

for all $x, y, \xi \in X$*, with* x, y *comparable,* $\beta \in (0, 2)$ *and* $t > 0$*,*

(vi) *for each non-decreasing sequence* $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n*,* $y_n \to u$ *implies that* $x_n \preceq u$*.*

Assume either

(a) $\{L, AB\}$ *is compatible,* L *or* AB *is continuous and* $\{ST, M\}$ *is weakly compatible or*

(b) $\{ST, M\}$ *is compatible,* ST *or* M *is continuous and* $\{L, AB\}$ *is weakly compatible.*

Then A, B, S, T, L *and* M *have a common fixed point. Moreover, the set of common fixed points of* A, B, S, T, L *and* M *is totally ordered if and only if* A, B, S, T, L *and* M *have one and only one common fixed point.*

Corollary 3.5. *Let* A, B, S, T, L *and* M *be self mappings of a complete ordered Menger space* $(X, \mathcal{F}, \preceq, \triangle)$ *with a continuous ordered t-norm* $\triangle$ *of H-type. Let the following conditions be satisfied:*

(i) $L(X) \subseteq ST(X)$*,* $M(X) \subseteq AB(X)$*,*

(ii) $AB = BA$*,* $ST = TS$*,* $LB = BL$*,* $MT = TM$*,*

(iii) L *and* M *are dominating maps,*

(iv) L *is a weak annihilator of* ST *and* M *is a weak annihilator of* AB*,*

(v) *for some* $k \in (0, 1)$ *and all* $x, y, \xi \in X$*, with* x, y *comparable,* $\beta \in (0, 2)$ *and* $t > 0$*,*

$$F_{Lx,My}(kt) \geq \min \begin{Bmatrix} F_{ABx,Lx}(t), F_{STy,My}(t), F_{ABx,STy}(t), \\ F_{STy,Lx}(t), [F_{ABx,\xi} \oplus F_{\xi,My}]((2-\beta)t) \end{Bmatrix},$$

(vi) *for each non-decreasing sequence* $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n, $y_n \to u$ *implies that* $x_n \preceq u$.

Assume either

(a) $\{L, AB\}$ *is compatible,* L *or* AB *is continuous and* $\{ST, M\}$ *is weakly compatible or*

(b) $\{ST, M\}$ *is compatible,* ST *or* M *is continuous and* $\{L, AB\}$ *is weakly compatible.*

Then A, B, S, T, L *and* M *have a common fixed point. Moreover, the set of common fixed points of* A, B, S, T, L *and* M *is totally ordered if and only if* A, B, S, T, L *and* M *have one and only one common fixed point.*

Theorem 3.3. *Let* $P_1, P_2, P_3, \ldots, P_{2n}, Q_0$ *and* Q_1 *be self mappings of a complete ordered Menger space* $(X, \mathcal{F}, \preceq, \triangle)$ *with a continuous ordered t-norm* $\triangle$ *of H-type. Let the following conditions be satisfied:*

(i) $Q_0(X) \subseteq P_1P_3P_5 \ldots P_{2n-1}(X)$, $Q_1(X) \subseteq P_2P_4P_6 \ldots P_{2n}(X)$,

(ii) $P_2(P_4P_6 \ldots P_{2n}) = (P_4P_6 \ldots P_{2n})P_2$,

$P_2P_4(P_6 \ldots P_{2n}) = (P_6 \ldots P_{2n})P_2P_4$,

$\vdots$

$P_2P_4P_6 \ldots P_{2n-2}(P_{2n}) = (P_{2n})P_2P_4P_6 \ldots P_{2n-2}$,

$Q_0(P_4P_6 \ldots P_{2n}) = (P_4P_6 \ldots P_{2n})Q_0$,

$Q_0(P_6 \ldots P_{2n}) = (P_6 \ldots P_{2n})Q_0$,

$\vdots$

$Q_0P_{2n} = P_{2n}Q_0$

$P_1(P_3P_5 \ldots P_{2n-1}) = (P_3P_5 \ldots P_{2n-1})P_1$,

$P_1P_3(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})P_1P_3$,

$\vdots$

$P_1P_3P_5 \ldots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1P_3P_5 \ldots P_{2n-3}$,

$$Q_1(P_3P_5\dots P_{2n-1}) = (P_3P_5\dots P_{2n-1})Q_1,$$
$$Q_1(P_5\dots P_{2n-1}) = (P_5\dots P_{2n-1})Q_1,$$
$$\vdots$$
$$Q_1P_{2n-1} = P_{2n-1}Q_1;$$

(iii) *Q_0 and Q_1 are dominating maps,*

(iv) *Q_0 is a weak annihilator of $P_1P_3P_5\dots P_{2n-1}$ and Q_1 is a weak annihilator of $P_2P_4P_6\dots P_{2n}$,*

(v) *there exists $\phi \in \Phi$ or $\phi \in \Phi_1$ such that*

$$F_{Q_0x,Q_1y}(\phi(t)) \geq \min\left\{\begin{array}{l} F_{P_2P_4P_6\dots P_{2n}x,Q_0x}(t), F_{P_1P_3P_5\dots P_{2n-1}y,Q_1y}(t), \\ F_{P_2P_4P_6\dots P_{2n}x,P_1P_3P_5\dots P_{2n-1}y}(t), F_{P_1P_3P_5\dots P_{2n-1}y,Q_0x}(t), \\ [F_{P_2P_4P_6\dots P_{2n}x,\xi} \oplus F_{\xi,Q_1y}]((2-\beta)t) \end{array}\right\}$$

for all $x, y, \xi \in X$, with x, y comparable, $\beta \in (0, 2)$ and $t > 0$,

(vi) *for each non-decreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n, $y_n \to u$ implies that $x_n \preceq u$.*

Assume either

(a) *$\{Q_0, P_2P_4P_6\dots P_{2n}\}$ is compatible, Q_0 or $P_2P_4P_6\dots P_{2n}$ is continuous and $\{P_1P_3P_5\dots P_{2n-1}, Q_1\}$ is weakly compatible or*

(b) *$\{P_1P_3P_5\dots P_{2n-1}, Q_1\}$ is compatible, $P_1P_3P_5\dots P_{2n-1}$ or Q_1 is continuous and $\{Q_0, P_2P_4P_6\dots P_{2n}\}$ is weakly compatible.*

Then $P_1, P_2, P_3, \dots, P_{2n}, Q_0$ and Q_1 have a common fixed point. Moreover, the set of common fixed points of $P_1, P_2, P_3, \dots, P_{2n}, Q_0$ and Q_1 is totally ordered if and only if $P_1, P_2, P_3, \dots, P_{2n}, Q_0$ and Q_1 have one and only one common fixed point.

Proof. The proof is similar to that of Theorem 3.2 (i.e., [4, Theorem 3.3]). □

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