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TOTAL MEAN CORDIALITY OF UMBRELLA, BUTTERFLY AND DUMBBELL GRAPHS

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ABSTRACT. A Total Mean Cordial labeling of a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ such that for each edge xy assign the label $\left\lceil \frac{f(x)+f(y)}{2} \right\rceil$ where $x, y \in V(G)$ and $|ev_f(i) - ev_f(j)| \leq 1$, $i, j \in \{0, 1, 2\}$ where $ev_f(x)$ denotes the total number of vertices and edges labeled with x $(x = 0, 1, 2)$. If there exists a total mean cordial labeling on a graph G, we will call G is Total Mean Cordial. In this paper, we investigate the Total Mean Cordial labeling behavior of fan, umbrella, dumbbell, and butterfly graphs.

1. INTRODUCTION

For standard terminology and notations in graph theory we refer the reader to Harary [4]. New terms and notations shall, however, be specifically defined whenever necessary. By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The number of vertices in G is denoted by $|V(G)|$ and that of edges we denote $|E(G)|$. Graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, database management, secret sharing schemes and models for constraint programming over finite domains,

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for more details see [4]. Cahit [1] introduced the concept of cordial labeling and behaviors of cordial labeling have been studied by several authors, some of them are Diab, Riskin, Seoud, Abdel Maqsoud, Youssef [2, 9, 10, 11]. Ponraj, Ramasamy and Sathish Narayanan [5] introduced the concept of Total Mean Cordial labeling of graphs and studied about the Total Mean Cordial labeling behavior of path, cycle, wheel, lotus inside a circle, bistar, flower graph, $K_{2,n}$, olive tree, the square of a path P_n^2, the corona of $S(P_n)$ with K_1 that is $S(P_n \odot K_1)$, $S(K_{1,n})$ and some more standard graphs in [7]. In [6, 8], Ponraj and Sathish Narayanan investigate the Total Mean Cordial behavior of one point union of 2 cycles C_n with a common vertex that is $C_n^{(2)}$, ladder L_n, book B_m and they proved that $K_n^c + 2K_2$ is Total Mean Cordial if and only if $n = 1, 2, 4, 6, 8$. In this paper we investigate the Total Mean Cordiality of fan, umbrella, dumbbell, and butterfly graphs. Let x be any real number. Then the symbol $\lceil x \rceil$ stands for the smallest integer greater than or equal to x.

2. Preliminary results

Definition 2.1. A Total Mean Cordial labeling of a graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that for each edge xy assign the label $\left\lceil \frac{f(x)+f(y)}{2} \right\rceil$ where $x, y \in V(G)$ and $|ev_f(i) - ev_f(j)| \leq 1$, $i, j \in \{0, 1, 2\}$ where $ev_f(x)$ denotes the total number of vertices and edges labeled with x ($x = 0, 1, 2$). If there exists a total mean cordial labeling on a graph G, we will call G is Total Mean Cordial.

Definition 2.2. The join of two graphs G_1 and G_2 is denoted by $G_1 + G_2$ and whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Definition 2.3. The graph $C_n^{(t)}$ denotes the one point union of t copies of the cycle $C_n : u_1 u_2 \ldots u_n u_1$.

Definition 2.4. The graph $F_n = P_n + K_1$ is called a fan where $P_n : u_1u_2 \dots u_n$ be a path and $V(K_1) = u$.

Definition 2.5. The Umbrella $U_{n,m}$, $m > 1$ is obtained from a fan F_n by pasting the end vertex of the path $P_m : v_1v_2 \dots v_m$ to the vertex of K_1 of the fan F_n.

Definition 2.6. Two even cycles of the same order, say C_n, sharing a common vertex with m pendent edges attached at the common vertex is called a butterfly graph $By_{m,n}$.

Definition 2.7. The graph obtained by joining two disjoint cycles $u_1u_2 \dots u_nu_1$ and $v_1v_2 \dots v_nv_1$ with an edge u_1v_1 is called dumbbell graph Db_n.

Theorem 2.1. [5] Any path P_m is Total Mean Cordial.

The above Theorem 2.1 is used for the investigation of the umbrella graph. So we recall the structure of the Total Mean Cordial labeling of path.

Let $P_m : v_1v_2 \dots v_m$ be the path. When $m = 3t$. Define a map $g : V(P_m) \to \{0, 1, 2\}$ by

$$\begin{aligned} g(v_i) &= 0, \quad 1 \le i \le t \\ g(v_{t+i}) &= 1, \quad 1 \le i \le t \\ g(v_{2t+i}) &= 2, \quad 1 \le i \le t. \end{aligned}$$

When $m = 3t + 1$. The function $g : V(P_m) \to \{0, 1, 2\}$ is given below:

$$\begin{aligned} g(v_i) &= 0, \quad 1 \le i \le t+1 \\ g(v_{t+1+i}) &= 1, \quad 1 \le i \le t \\ g(v_{2t+1+i}) &= 2, \quad 1 \le i \le t. \end{aligned}$$

When $m = 3t + 2$. The map $g : V(P_m) \to \{0, 1, 2\}$ is given below:

$$\begin{aligned} g(v_i) &= 0, \quad 1 \le i \le t+1 \\ g(v_{t+1+i}) &= 1, \quad 1 \le i \le t \\ g(v_{2t+1+i}) &= 2, \quad 1 \le i \le t \\ g(v_{3t+2}) &= 1. \end{aligned}$$

Theorem 2.2. [5] The cycle C_n is Total Mean Cordial if and only if $n \neq 3$.

The investigation of dumbbell graph is based on Theorem 2.2. So we recall the proof technique of Theorem 2.2. Let $C_n : u_1u_2 \ldots u_nu_1$ be the cycle. If $n = 3$, then we have $ev_f(0) = ev_f(1) = ev_f(2) = 2$. But this is an impossible one. Assume $n > 3$. When $n \equiv 0$ $(mod\ 3)$, let $n = 3t$, $t \in \mathbb{Z}^+$. It is easy to see that C_6 is Total Mean Cordial. Take $t \ge 3$. Define $f : V(C_n) \to \{0, 1, 2\}$ by

$$\begin{aligned} f(u_i) &= 0 \quad 1 \le i \le t \\ f(u_{t+i}) &= 2 \quad 1 \le i \le t \\ f(u_{2t+i}) &= 1 \quad 1 \le i \le t-2. \end{aligned}$$

$f(u_{3t-1}) = 0$ and $f(u_{3t}) = 1$. When $n \equiv 1, 2$ $(mod\ 3)$, The labeling f defined in case 2 of theorem 2.1 satisfy Total Mean Cordial condition of C_n.

Theorem 2.3. [8] The graph $C_n^{(2)}$ is Total Mean Cordial.

The investigation of butterfly graph is depends on the Total Mean Cordial labeling of $C_n^{(2)}$. So we once again recall the Total Mean Cordiality of $C_n^{(2)}$. Let the two copies of the cycles be $u_1u_2 \ldots u_nu_1$ and $v_1v_2 \ldots v_nv_1$. Join u_1 and v_1. It is easy to verify that C_n^2 $(3 \le n \le 8)$ is Total Mean Cordial. When $n > 8$. Assign the labels to the vertices of the two copies of the cycle C_n as in Theorem 2.2. If $n \equiv 1 \pmod 3$ then relabel the vertices u_2, u_{t+2} and u_{t+4} by 2, 0, 0 respectively. If $n \equiv 2 \pmod 3$ then relabel the vertex u_{t+2} by 0.

3. Main results

Now we look into the Fan graph.

Theorem 3.1. The fan F_n is Total Mean Cordial if and only if $n \neq 2$, $n \neq 4$.

Proof. Clearly $|V(F_n)| + |E(F_n)| = 3n$.

Case 1. $n \equiv 0 \pmod 6$.

Let $n = 6k$ where $k \in \mathbb{Z}^+$. Define a function $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \leq i \leq 2k \\ f(u_{2k+i}) &= 2, \quad 1 \leq i \leq 3k \\ f(u_{5k+i}) &= 1, \quad 1 \leq i \leq k. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = ev_f(2) = 6k$.

Case 2. $n \equiv 1 \pmod 6$.

If we label u_1 and u by 0 and 2, respectively, then F_1 is Total Mean Cordial. When $n = 7$, Figure 1 shows that F_7 is Total Mean Cordial.

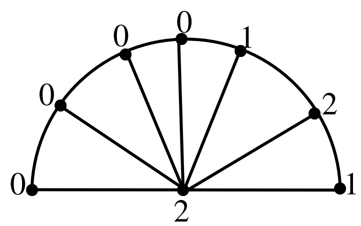

Figure 1

Now, let $n = 6k + 1$ where $k \in \mathbb{Z}^+$ and $k \neq 0$. Define a map $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = f(u_{3k+2}) = 1$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \leq i \leq 3k+1 \\ f(u_{3k+2+i}) &= 2, \quad 1 \leq i \leq 2k \\ f(u_{5k+2+i}) &= 1, \quad 1 \leq i \leq k-1. \end{aligned}$$

Here $ev_f(0) = ev_f(1) = ev_f(2) = 6k + 1$.

Case 3. $n \equiv 2 \pmod 6$.

Subcase 3.1. $n \equiv 2 \pmod{12}$. Obviously, F_2 is not Total Mean Cordial. Assume $n > 2$. Let $n = 12k + 2$ where $k \in \mathbb{Z}^+$. Define $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = 2$, $f(u_{12k+2}) = 0$, $f(u_{6k+2}) = 1$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le 6k+1 \\ f(u_{6k+2+i}) &= 2, \quad 1 \le i \le 3k \\ f(u_{9k+2+i}) &= 1, \quad 1 \le i \le 3k-1. \end{aligned}$$

Here $ev_f(0) = ev_f(1) = ev_f(2) = 12k + 2$.

Subcase 3.2. $n \equiv 8 \pmod{12}$. Let $n = 12k - 4$ where $k \in \mathbb{Z}^+$. Define $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = 2$, $f(u_{12k-4}) = 0$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le 6k-2 \\ f(u_{6k-2+i}) &= 2, \quad 1 \le i \le 3k-1 \\ f(u_{9k-3+i}) &= 1, \quad 1 \le i \le 3k-2. \end{aligned}$$

Here $ev_f(0) = ev_f(1) = ev_f(2) = 12k - 4$.

Case 4. $n \equiv 3 \pmod 6$.

When $n = 3$, Figure 2 shows that F_3 is Total Mean Cordial.

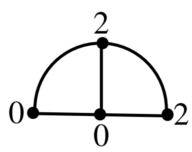

FIGURE 2

Let $n = 6k + 3$ where $k \in \mathbb{Z}^+$. Define a map $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = 0$, $f(u_{6k+3}) = 1$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le 2k+1 \\ f(u_{2k+1+i}) &= 1, \quad 1 \le i \le k \\ f(u_{3k+1+i}) &= 2, \quad 1 \le i \le 3k+1. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = ev_f(2) = 6k + 3$.

Case 5. $n \equiv 4 \pmod 6$.

Subcase 5.1. $n \equiv 4 \pmod{12}$. Then we have the following claim:

Claim. F_4 is not Total Mean Cordial.

Proof of the Claim: Suppose $f(u) = 0$. In this case atleast two zeros should be labeled in the path vertices. This forces $ev_f(0) \geq 5$, a contradiction.

Suppose $f(u) = 1$. In this case, zero should be labeled to u_1, u_4 and u_2 (or u_3). Then $ev_f(2) = 2$, a contradiction.

Suppose $f(u) = 2$. As in above zero should be labeled to u_1, u_4 and u_2 (or u_3). In this case, $ev_f(2) = 2$ or 3, a contradiction. Proof of the claim is complete.

Let $n = 12k + 4$ where $k \in \mathbb{Z}^+$. Define $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = 2$, $f(u_{12k+4}) = 0$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \leq i \leq 6k+2 \\ f(u_{6k+2+i}) &= 2, \quad 1 \leq i \leq 3k+1 \\ f(u_{9k+3+i}) &= 1, \quad 1 \leq i \leq 3k. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = ev_f(2) = 12k + 4$.

Subcase 5.2. $n \equiv 10 \pmod{12}$. Let $n = 12k - 2$ where $k \in \mathbb{Z}^+$. Define a function $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = 2$, $f(u_{6k}) = 1$, $f(u_{12k-2}) = 0$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \leq i \leq 6k-1 \\ f(u_{6k+i}) &= 2, \quad 1 \leq i \leq 3k-1 \\ f(u_{9k-1+i}) &= 1, \quad 1 \leq i \leq 3k-2. \end{aligned}$$

Here $ev_f(0) = ev_f(1) = ev_f(2) = 12k - 2$.

Case 6. $n \equiv 5 \pmod 6$.

Subcase 6.1. $n \equiv 5 \pmod{12}$. Let $n = 12k - 7$ where $k \in \mathbb{Z}^+$. Define $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = 2$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le 6k - 3 \\ f(u_{6k-3+i}) &= 2, \quad 1 \le i \le 3k - 2 \\ f(u_{9k-5+i}) &= 1, \quad 1 \le i \le 3k - 2. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = ev_f(2) = 12k - 7$.

Subcase 6.2. $n \equiv 11 \pmod{12}$. Let $n = 12k - 1$ where $k \in \mathbb{Z}^+$. Define a function $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = 2$, $f(u_{6k+1}) = 1$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le 6k \\ f(u_{6k+1+i}) &= 2, \quad 1 \le i \le 3k - 1 \\ f(u_{9k+i}) &= 1, \quad 1 \le i \le 3k - 1. \end{aligned}$$

Here $ev_f(0) = ev_f(1) = ev_f(2) = 12k - 1$.

Hence F_n is Total Mean Cordial if and only if $n \neq 2$, $n \neq 4$.

□

Next is the Umbrella graph.

Theorem 3.2. The Umbrella $U_{n,m}$, $m > 1$ is Total Mean Cordial.

Proof. Take the vertex set and edge set of F_n as in Theorem 3.1. Further unify the vertices u and v_1. It is clear that $|V(U_{n,m})| + |E(U_{n,m})| = 3n + 2m - 2$. Let f, g respectively be the labellings defined for F_n, P_m in theorems 3.1, 2.1. Define a map $h : V(U_{n,m}) \to \{0, 1, 2\}$ by $h(u) = f(u)$,

$$\begin{aligned} h(u_i) &= f(u_i), \quad 1 \le i \le n \\ h(v_i) &= g(v_i), \quad 2 \le i \le m. \end{aligned}$$

Case 1. $n \equiv 0 \pmod 6$ and $m \equiv 0 \pmod 3$.

Let $n = 6k$ and $m = 3t$ where $k, t \in \mathbb{Z}^+$. When $m = 3$, relabel the vertices v_2, v_3 by 2, 0 respectively. Then $ev_h(0) = ev_h(2) = 6k + 1$, $ev_h(1) = 6k + 2$. For $m = 6$, relabel the vertices v_4, v_5, v_6 by 2, 1, 0 respectively. In this case $ev_h(0) = ev_h(2) = 6k+3$, $ev_h(1) = 6k+4$. Assume $m > 6$. Now, relabel the vertex v_{t+2} by 0. Then $ev_h(0) = ev_h(1) = 6k+2t-1$, $ev_h(2) = 6k+2t$.

Case 2. $n \equiv 0 \pmod 6$ and $m \equiv 1 \pmod 3$.

Let $n = 6k$ and $m = 3t+1$ where $k, t \in \mathbb{Z}^+$. In this case $ev_h(0) = ev_h(1) = ev_h(2) = 6k+2t$.

Case 3. $n \equiv 0 \pmod 6$ and $m \equiv 2 \pmod 3$.

Let $n = 6k$ and $m = 3t+2$ where $k, t \in \mathbb{Z}^+$. In this case $ev_h(0) = 6k+2t$, $ev_h(1) = ev_h(2) = 6k+2t+1$.

Case 4. $n \equiv 1 \pmod 6$ and $m \equiv 0 \pmod 3$.

Let $n = 6k+1$ and $m = 3t$ where $k, t \in \mathbb{Z}^+$.

Subcase 4a. $n = 1$. In this case $U_{n,m} \cong P_{m+1}$. Then by theorem 2.1, P_{m+1} is Total Mean Cordial.

Subcase 4b. $n = 7$. Here relable $h(v_{m-i+1}) = g(v_i), \quad 1 \le i \le m$. Then $ev_h(0) = ev_h(2) = 2t+6$, $ev_h(1) = 2t+7$.

Subcase 4c. $n \neq 1$, $n \neq 7$. For $m = 3$, relabel the vertices v_2, v_3 by 2, 0 respectively. Then $ev_h(0) = ev_h(1) = 6k+2$ and $ev_h(2) = 6k+3$. When $m = 6$, relabel the vertices v_2, v_3, v_4, v_5, v_6 by 1, 2, 2, 0, 0 respectively. Then $ev_h(0) = ev_h(1) = 6k+4$, $ev_h(2) = 6k+5$. When $m > 6$, relabel the vertex v_{t+2} by 0. Then $ev_h(0) = ev_h(1) = 6k+2t$, $ev_h(2) = 6k+2t+1$.

Case 5. $n \equiv 1 \pmod 6$ and $m \equiv 1 \pmod 3$.

Let $n = 6k+1$ and $m = 3t+1$ where $k, t \in \mathbb{Z}^+$.

Subcase 5a. $n = 1$. Similar to subcase 4a.

Subcase 5b. $n = 7$. For $m = 4$, relabel the vertices u_5, u_7, v_2, v_4 by 0, 2, 2, 1 respectively. Here $ev_h(0) = ev_h(1) = ev_h(2) = 9$. When $m = 7$, relabel u_5, u_6, v_2, v_3, v_4, v_6 by 0, 0, 2, 2, 2, 1 respectively. In this case $ev_h(0) = ev_h(1) = ev_h(2) = 11$. When $m > 7$, relabel the vertex v_{t+3} by 0. Then $ev_h(0) = ev_h(1) = ev_h(2) = 2t+7$.

Subcase 5c. $n \neq 1$, $n \neq 7$. For $m = 4$, relabel the vertices u_{3k+2}, v_2, v_3, v_4 by 0, 2, 1, 1, respectively. Then $ev_h(0) = ev_h(1) = ev_h(2) = 6k + 3$. For $m = 7$, relabel the vertices v_2, v_3, v_4, v_5, v_6, v_7 by 2, 2, 0, 1, 0, 0, respectively. Here, $ev_h(0) = ev_h(1) = ev_h(2) = 6k + 5$. When $m > 7$, relabel v_{t+3} by 0. Then $ev_h(0) = ev_h(1) = ev_h(2) = 6k + 2t + 1$.

Case 6. $n \equiv 1 \pmod 6$ and $m \equiv 2 \pmod 3$.

Let $n = 6k + 1$ and $m = 3t + 2$ where $k, t \in \mathbb{Z}^+$.

Subcase 6a. $n = 1$. Similar to subcase 4a.

Subcase 6b. $n = 7$. For $m = 2$, relabel the vertex v_2 by 0. Then $ev_h(0) = ev_h(1) = 8$, $ev_h(2) = 7$. For $m = 5$, relabel v_3, v_5 by 2, 0 respectively. Here $ev_h(0) = 9$, $ev_h(1) = ev_h(2) = 10$. If $m > 5$, relabel $h(v_{m-i+1}) = g(v_i)$, $1 \leq i \leq m$. Then $ev_h(0) = ev_h(1) = 2t + 8$, $ev_h(2) = 2t + 7$.

Subcase 6c. $n \neq 1$, $n \neq 7$. Relabel the vertex u_{3k+2} by 0. Here, $ev_h(0) = ev_h(1) = 6k + 2t + 2$, $ev_h(2) = 6k + 2t + 1$.

Case 7. $n \equiv 2 \pmod 6$ and $m \equiv 0 \pmod 3$.

Let $m = 3t$ where $t \in \mathbb{Z}^+$.

Subcase 7a. $n \equiv 2 \pmod{12}$. Let $n = 12k+2$ where $k \in \mathbb{Z}^+$. Suppose $n = 2$. Then assign the labels 0, 2 to the vertices u_1, u_2 respectively. Then $ev_h(0) = ev_h(2) = 2t+1$, $ev_h(1) = 2t + 2$. Assume $n > 2$. Here relable $h(v_{m-i+1}) = g(v_i)$, $1 \leq i \leq m$. Then $ev_h(0) = ev_h(2) = 12k + 2t + 1$, $ev_h(1) = 12k + 2t + 2$.

Subcase 7b. $n \equiv 8 \pmod{12}$. Let $n = 12k - 4$ where $k \in \mathbb{Z}^+$. Here also relable $h(v_{m-i+1}) = g(v_i)$, $1 \leq i \leq m$. Then $ev_h(0) = ev_h(2) = 12k + 2t - 5$, $ev_h(1) = 12k + 2t - 4$.

Case 8. $n \equiv 2 \pmod 6$ and $m \equiv 1 \pmod 3$.

Let $m = 3t + 1$ where $t \in \mathbb{Z}^+$.

Subcase 8a. $n \equiv 2 \pmod{12}$. Let $n = 12k + 2$ where $k \in \mathbb{Z}^+$. Suppose $n = 2$. Then relabel $h(v_{m-i+1}) = g(v_i)$, $1 \leq i \leq m$ and assign the labels 0, 2 to u_1, u_2

respectively. Then $ev_h(0) = ev_h(1) = ev_h(2) = 2t + 2$. Suppose $n > 2$ and $m = 4$. Let $n = 12k + 2$. Define a function $f : V(U_{n,4}) \to \{0, 1, 2\}$ by $f(v_1) = f(v_4) = 0$, $f(v_2) = f(v_3) = 1$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le 4k + 1 \\ f(u_{4k+1+i}) &= 2, \quad 1 \le i \le 6k + 2 \\ f(u_{10k+3+i}) &= 1, \quad 1 \le i \le 2k - 1. \end{aligned}$$

In this case, $ev_f(0) = ev_f(1) = ev_f(2) = 12k + 4$. When $m = 7$, relabel the vertices v_4, v_5 by 2, 0 respectively. Here $ev_h(0) = ev_h(1) = ev_h(2) = 12k + 6$. For $m > 7$, relabel the vertex v_{t+3} by 0. Then $ev_h(0) = ev_h(1) = ev_h(2) = 12k + 2t + 2$.

Subcase 8b. $n \equiv 8 \pmod{12}$. Let $n = 12k - 4$ where $k \in \mathbb{Z}^+$. For $m = 4$, let $n = 12k - 4$ and $k > 0$. Define a map $f : V(U_{n,4}) \to \{0, 1, 2\}$ by $f(v_1) = f(v_4) = 0$, $f(v_2) = 2$, $f(v_3) = 1$,

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le 4k - 1 \\ f(u_{4k-1+i}) &= 2, \quad 1 \le i \le 6k - 2 \\ f(u_{10k-3+i}) &= 1, \quad 1 \le i \le 2k - 1. \end{aligned}$$

In this case, $ev_f(0) = ev_f(1) = ev_f(2) = 12k - 2$. For $m \ge 7$, assign the labels to the vertices of $U_{n,m}$ as in subcase 8a. It is easy to check that $U_{n,m}$ is Total Mean Cordial.

Case 9. $n \equiv 2 \pmod 6$ and $m \equiv 2 \pmod 3$.

Let $m = 3t + 2$ where $t \in \mathbb{Z}^+$.

Subcase 9a. $n \equiv 2 \pmod{12}$. Let $n = 12k + 2$ where $k \in \mathbb{Z}^+$. Suppose $n = 2$, $m = 2$. Then assign the labels 0, 2 to u_1, u_2 respectively and 0, 2 to v_1, v_2 respectively. Then $ev_f(0) = ev_f(1) = 3$, $ev_f(2) = 2$. For $m > 2$, assign the labels 0, 2 to u_1, u_2 then $ev_h(0) = ev_h(1) = 2t + 3$, $ev_h(2) = 2t + 2$. Assume $n > 2$. When $m = 2$, assign the label 0 to v_2. In this case $ev_h(0) = ev_h(1) = 12k + 3$, $ev_h(2) = 12k + 2$. For $m > 2$, relabel $h(v_{m-i+1}) = g(v_i)$, $2 \le i \le m$. Then $ev_h(0) = ev_h(1) = 12k + 2t + 3$, $ev_h(2) = 12k + 2t + 2$.

Subcase 9b. $n \equiv 8 \pmod{12}$. Let $n = 12k - 4$ where $k \in \mathbb{Z}^+$. For $m = 2$, assign the label 0 to v_2. Then $ev_h(0) = ev_h(1) = 12k - 3$, $ev_h(2) = 12k - 4$. When $m > 2$, relabel $h(v_{m-i+1}) = g(v_i)$, $2 \leq i \leq m$. Then $ev_h(0) = ev_h(1) = 12k + 2t - 3$, $ev_h(2) = 12k + 2t - 4$.

Case 10. $n \equiv 3 \pmod 6$ and $m \equiv 0 \pmod 3$.

Let $n = 6k + 3$ and $m = 3t$ where $k, t \in \mathbb{Z}^+$. For $m = 3$, relabel v_2 by 0. Then $ev_h(0) = 6k + 5$, $ev_h(1) = ev_h(2) = 6k + 4$. For $m = 6$, relabel v_3, v_6 by 0, 1 respectively. In this case $ev_h(0) = 6k + 7$, $ev_h(1) = ev_h(2) = 6k + 6$. When $m > 6$, relabel v_{t+2} by 0. In this case $ev_h(0) = ev_h(1) = 6k + 2t + 2$, $ev_h(2) = 6k + 2t + 3$.

Case 11. $n \equiv 3 \pmod 6$ and $m \equiv 1 \pmod 3$.

Let $n = 6k + 3$ and $m = 3t + 1$ where $k, t \in \mathbb{Z}^+$. In this case $ev_h(0) = ev_h(1) = ev_h(2) = 6k + 2t + 3$.

Case 12. $n \equiv 3 \pmod 6$ and $m \equiv 2 \pmod 3$.

Let $n = 6k + 3$ and $m = 3t + 2$ where $k, t \in \mathbb{Z}^+$. Here $ev_h(0) = 6k + 2t + 3$, $ev_h(1) = ev_h(2) = 6k + 2t + 4$.

Case 13. $n \equiv 4 \pmod 6$ and $m \equiv 0 \pmod 3$.

Let $m = 3t$ where $t \in \mathbb{Z}^+$.

Subcase 13a. $n \equiv 4 \pmod{12}$. Let $n = 12k + 4$ where $k \in \mathbb{Z}^+$. For $n = 4$, assign the labels 0, 2, 2, 0 to u_1, u_2, u_3, u_4 respectively. Then $ev_h(0) = ev_h(2) = 2t + 3$, $ev_h(1) = 2t + 4$. If $n > 4$ then relabel $h(v_{m-i+1}) = g(v_i)$, $2 \leq i \leq m$. Here $ev_h(0) = ev_h(2) = 12k + 2t + 3$, $ev_h(1) = 12k + 2t + 4$.

Subcase 13b. $n \equiv 10 \pmod{12}$. Let $n = 12k - 2$ where $k \in \mathbb{Z}^+$. Here also relabel $h(v_{m-i+1}) = g(v_i)$, $2 \leq i \leq m$. Then $ev_h(0) = ev_h(2) = 12k + 2t - 3$, $ev_h(1) = 12k + 2t - 2$.

Case 14. $n \equiv 4 \pmod 6$ and $m \equiv 1 \pmod 3$.

Let $m = 3t + 1$ where $t \in \mathbb{Z}^+$.

Subcase 14a. $n \equiv 4 \pmod{12}$. Let $n = 12k+4$ where $k \in \mathbb{Z}^+$. For $n = 4$, relabel $h(v_{m-i+1}) = g(v_i)$, $1 \le i \le m$. Also assign the labels 0, 0, 2, 1 to u_1, u_2, u_3, u_4 respectively. Then $ev_h(0) = ev_h(1) = ev_h(2) = 2t+4$. For $n > 4$ and $m = 4$, define a function $\phi : V(U_{n,4}) \to \{0,1,2\}$ by $\phi(v_1) = 0$, $\phi(v_2) = \phi(v_3) = \phi(v_4) = 1$,

$$\begin{aligned} \phi(u_i) &= 0, \quad 1 \le i \le 4k+2 \\ \phi(u_{4k+2+i}) &= 2, \quad 1 \le i \le 6k+3 \\ \phi(u_{10k+5+i}) &= 1, \quad 1 \le i \le 2k-1. \end{aligned}$$

In this case, $ev_\phi(0) = ev_\phi(1) = ev_\phi(2) = 12k+6$. When $n > 4$ and $m = 7$, relabel the vertices v_2, v_4, v_6, v_7 by 2, 2, 0, 0 respectively. Then $ev_h(0) = ev_h(1) = ev_h(2) = 12k+8$. For $n > 7$ and $m > 7$, relabel the vertex v_{t+3} by 0. In this case $ev_h(0) = ev_h(1) = ev_h(2) = 12k+2t+4$.

Subcase 14b. $n \equiv 10 \pmod{12}$. Let $n = 12k-2$ where $k \in \mathbb{Z}^+$. For $m = 4$, define a map $\phi : V(U_{n,4}) \to \{0,1,2\}$ by $\phi(v_1) = 0$, $\phi(v_2) = 2$, $\phi(v_3) = \phi(v_4) = 1$,

$$\begin{aligned} \phi(u_i) &= 0, \quad 1 \le i \le 4k \\ \phi(u_{4k+i}) &= 2, \quad 1 \le i \le 6k-1 \\ \phi(u_{10k-1+i}) &= 1, \quad 1 \le i \le 2k-1. \end{aligned}$$

In this case, $ev_\phi(0) = ev_\phi(1) = ev_\phi(2) = 12k$. When $m = 7$, relabel the vertices u_5, u_7 by 2, 0 respectively. Here, $ev_h(0) = ev_h(1) = ev_h(2) = 12k+2$. For $m > 7$, relabel the vertex v_{t+3} by 0. In this case $ev_h(0) = ev_h(1) = ev_h(2) = 12k+2t-2$.

Case 15. $n \equiv 4 \pmod 6$ and $m \equiv 2 \pmod 3$.

Let $m = 3t+2$ where $t \in \mathbb{Z}^+$.

Subcase 15a. $n \equiv 4 \pmod{12}$. Let $n = 12k+4$ where $k \in \mathbb{Z}^+$. When $n = 4$, assign the labels 0, 2, 2, 0 to u_1, u_2, u_3, u_4, respectively. Then $ev_h(0) = ev_h(1) = 2t+5$, $ev_h(2) = 2t+4$. Assume $n > 4$. For $m = 2$, assign the label 0 to v_2. Then $ev_h(0) = ev_h(1) = 12k+5$, $ev_h(2) = 12k+4$. For $m > 2$, relabel $h(v_{m-i+1}) = g(v_i)$, $2 \le i \le m$. Then $ev_h(0) = ev_h(1) = 12k+2t+5$, $ev_h(2) = 12k+2t+4$.

Subcase 15b. $n \equiv 10 \pmod{12}$. Let $n = 12k - 2$ where $k \in \mathbb{Z}^+$. When $m = 2$, assign the label 0 to v_2. Here $ev_h(0) = ev_h(1) = 12k - 1$, $ev_h(2) = 12k - 2$. For $m > 2$, relabel $h(v_{m-i+1}) = g(v_i)$, $2 \le i \le m$. Then $ev_h(0) = ev_h(1) = 12k + 2t - 1$, $ev_h(2) = 12k + 2t - 2$.

Case 16. $n \equiv 5 \pmod 6$ and $m \equiv 0 \pmod 3$.

Let $m = 3t$ where $t \in \mathbb{Z}^+$.

Subcase 16a. $n \equiv 5 \pmod{12}$. Let $n = 12k-7$ where $k \in \mathbb{Z}^+$. Relabel $h(v_{m-i+1}) = g(v_i)$, $2 \le i \le m$. In this case $ev_h(0) = ev_h(2) = 12k+2t-8$, $ev_h(1) = 12k+2t-7$.

Subcase 16b. $n \equiv 11 \pmod{12}$. Let $n = 12k-1$ where $k \in \mathbb{Z}^+$. Similar to subcase 16a. In this case, $ev_h(0) = ev_h(2) = 12k + 2t - 2$, $ev_h(1) = 12k + 2t - 1$.

Case 17. $n \equiv 5 \pmod 6$ and $m \equiv 1 \pmod 3$.

Let $m = 3t + 1$ where $t \in \mathbb{Z}^+$.

Subcase 17a. $n \equiv 5 \pmod{12}$. Let $n = 12k - 7$ where $k \in \mathbb{Z}^+$. Suppose $m = 4$. Let $n = 12k + 5$ and $k \ge 0$. Define $\phi : V(U_{n,4}) \to \{0, 1, 2\}$ by $\phi(v_1) = \phi(v_3) = 0$, $\phi(v_2) = 2$, $\phi(v_4) = 1$,

$$\begin{aligned} \phi(u_i) &= 0, \quad 1 \le i \le 4k+2 \\ \phi(u_{4k+2+i}) &= 2, \quad 1 \le i \le 6k+3 \\ \phi(u_{10k+5+i}) &= 1, \quad 1 \le i \le 2k. \end{aligned}$$

In this case, $ev_\phi(0) = ev_\phi(1) = ev_\phi(2) = 12k + 7$. For $m = 7$, relabel the vertices v_4, v_5 by 2, 0 respectively. Then $ev_h(0) = ev_h(1) = ev_h(2) = 12k - 3$. When $m > 7$, relabel the vertex v_{t+3} by 0. In this case $ev_h(0) = ev_h(1) = ev_h(2) = 12k + 2t - 7$.

Subcase 17b. $n \equiv 11 \pmod{12}$. Let $n = 12k - 1$ where $k \in \mathbb{Z}^+$. When $m = 4$, let $n = 12k - 1$, $k > 0$. Define a map $\phi : V(U_{n,4}) \to \{0, 1, 2\}$ by $\phi(v_1) = \phi(v_3) = 0$, $\phi(v_2) = 2$, $\phi(v_4) = 1$,

$$\begin{aligned}
\phi(u_i) &= 0, \quad 1 \leq i \leq 4k \\
\phi(u_{4k+i}) &= 2, \quad 1 \leq i \leq 6k \\
\phi(u_{10k+i}) &= 1, \quad 1 \leq i \leq 2k-1.
\end{aligned}$$

In this case, $ev_\phi(0) = ev_\phi(1) = ev_\phi(2) = 12k+1$. For $m = 7$, relabel the vertices v_4, v_5 by 2, 0 respectively. Here $ev_h(0) = ev_h(1) = ev_h(2) = 12k+3$. For $m > 7$, relabel the vertex v_{t+3} by 0. In this case $ev_h(0) = ev_h(1) = ev_h(2) = 12k+2t-1$.

Case 18. $n \equiv 5 \pmod 6$ and $m \equiv 2 \pmod 3$.

Let $m = 3t+2$ where $t \in \mathbb{Z}^+$.

Subcase 18a. $n \equiv 5 \pmod{12}$. Let $n = 12k-7$ where $k \in \mathbb{Z}^+$. For $m = 2$, relabel the vertex v_2 by 0. Then $ev_h(0) = ev_h(1) = 12k-6$, $ev_h(2) = 12k-7$. Assume $m > 2$. Then relabel $h(v_{m-i+1}) = g(v_i), \;\; 2 \leq i \leq m$. Here $ev_h(0) = ev_h(1) = 12k+2t-6$, $ev_h(2) = 12k+2t-7$.

Subcase 18b. $n \equiv 11 \pmod{12}$. Similar to subcase 18a.

Therefore, $U_{n,m}$, $m > 1$ is Total Mean Cordial. $\square$

Example 3.1. A Total Mean Cordial labeling of $U_{9,7}$ is given in Figure 3.

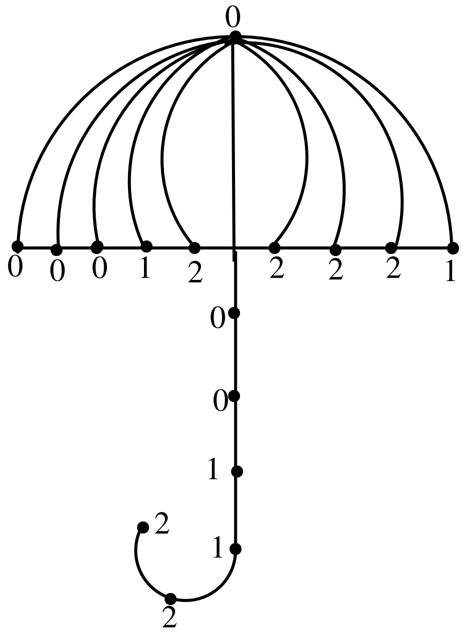

FIGURE 3

We now investigate the butterfly graph.

Theorem 3.3. $By_{m,n}$ is Total Mean Cordial.

Proof. Assign the labels to the vertices of $C_n^{(2)}$ as in theorem 2.3.

Case 1. $n \equiv 0 \pmod 3$ and $m \equiv 0 \pmod 3$.

Let $m = 3k$ and $n = 3t$ where $t, k \in \mathbb{Z}^+$. Assign the label 0 to the first k vertices and 2 to the next $2k$ vertices. Here, $ev_h(0) = 4t + 2k - 1$, $ev_h(1) = ev_h(2) = 4t + 2k$.

Case 2. $n \equiv 0 \pmod 3$ and $m \equiv 1 \pmod 3$.

Let $m = 3k + 1$ and and $n = 3t$ where $t, k \in \mathbb{Z}^+$. Assign the label 0 to the $k + 1$ vertices and 2 to the next $2k$ vertices. In this case, $ev_h(0) = 4t + 2k + 1$, $ev_h(1) = ev_h(2) = 4t + 2k$.

Case 3. $n \equiv 0 \pmod 3$ and $m \equiv 2 \pmod 3$.

Let $m = 3k + 2$ and $n = 3t$ where $t, k \in \mathbb{Z}^+$. Assign the label 0 to the $k + 1$ vertices and 2 to the next $2k + 1$ vertices. Here, $ev_h(0) = ev_h(1) = ev_h(2) = 4t + 2k + 1$.

Case 4. $n \equiv 1 \pmod 3$ and $m \equiv 0 \pmod 3$.

Let $m = 3k$ and $n = 3t + 1$ where $t, k \in \mathbb{Z}^+$. Similar to Case 1. Here, $ev_h(0) = ev_h(1) = ev_h(2) = 4t + 2k + 1$.

Case 5. $n \equiv 1 \pmod 3$ and $m \equiv 1 \pmod 3$.

Let $m = 3k+1$ and $n = 3t+1$ where $t, k \in \mathbb{Z}^+$. Assign the label 0 to the k vertices and 2 to the next $2k+1$ vertices. Here, $ev_h(0) = 4t+2k+1$, $ev_h(1) = ev_h(2) = 4t+2k+2$.

Case 6. $n \equiv 1 \pmod 3$ and $m \equiv 2 \pmod 3$.

Let $m = 3k + 2$ and $n = 3t + 1$ where $t, k \in \mathbb{Z}^+$. Similar to Case 5. Here, $ev_h(0) = 4t + 2k + 3$, $ev_h(1) = ev_h(2) = 4t + 2k + 2$.

Case 7. $n \equiv 2 \pmod 3$ and $m \equiv 0 \pmod 3$.

Let $m = 3k$ and $n = 3t + 2$ where $t, k \in \mathbb{Z}^+$. Similar to Case 1. Here, $ev_h(0) = 4t + 2k + 3$, $ev_h(1) = ev_h(2) = 4t + 2k + 2$.

Case 8. $n \equiv 2 \pmod 3$ and $m \equiv 1 \pmod 3$.

Let $m = 3k + 1$ and $n = 3t + 2$ where $t, k \in \mathbb{Z}^+$. Similar to Case 5. Here, $ev_h(0) = ev_h(1) = ev_h(2) = 4t + 2k + 3$.

Case 9. $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$.

Let $m = 3k+2$ and $n = 3t+2$ where $t, k \in \mathbb{Z}^+$. Assign 0 to k vertices and 2 to $2k+2$ vertices. Then, $ev_h(0) = 4t + 2k + 3$, $ev_h(1) = ev_h(2) = 4t + 2k + 4$. □

Example 3.2. A Total Mean Cordial labeling of $By_{6,6}$ is given in figure 4.

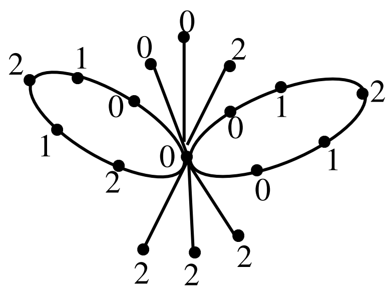

FIGURE 4

The final investigation is about dumbbell graph.

Theorem 3.4. The dumbbell graph Db_n is Total Mean Cordial.

Proof. It is clear that $|V(Db_n)| + |E(Db_n)| = 4n + 1$.

Case 1. $n \equiv 0 \pmod 3$.

For $n = 3$, the figure 5 establish that Db_3 is Total Mean Cordial.

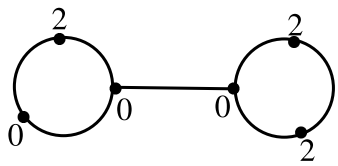

FIGURE 5

Let $n = 3t$ where $t \in \mathbb{Z}^+$ and $t > 1$. Then assign the labels to the vertices of the two cycles as in theorem 2.2. Here, $ev_f(0) = 4t + 1$, $ev_f(1) = ev_f(2) = 4t$.

Case 2. $n \equiv 1 \pmod 3$.

Let $n = 3t + 1$ where $t \in \mathbb{Z}^+$. Then assign the labels to the vertices of the two cycles as in theorem 2.2. Here, $ev_f(0) = ev_f(1) = 4t + 2$, $ev_f(2) = 4t + 1$.

Case 3. $n \equiv 2 \pmod 3$.

Let $n = 3t + 1$ where $t \in \mathbb{Z}^+$. Without loss of generality join u_1 and v_1. Define a function $f : V(Db_n) \to \{0, 1, 2\}$ by

$$\begin{aligned} f(u_i) &= 0, \quad 1 \leq i \leq 2t + 2 \\ f(u_{2t+2+i}) &= 1, \quad 1 \leq i \leq t \\ f(v_i) &= 2, \quad 1 \leq i \leq 2t + 1 \\ f(u_{2t+1+i}) &= 1, \quad 1 \leq i \leq t + 1. \end{aligned}$$

In this case, $ev_f(0) = ev_f(1) = ev_f(2) = 4t + 3$.

Hence Dumbbell graph is Total Mean Cordial. □

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