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ON SEQUENCE SPACES EQUATIONS OF THE FORM $E_T + F_x = F_b$ FOR SOME TRIANGLE T

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ABSTRACT. Given any sequence $a = (a_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_a for the set of all sequences $y = (y_n)_{n\geq 1}$ such that $y/a = (y_n/a_n)_{n\geq 1} \in E$; in particular, $s_a^{(c)}$ denotes the set of all sequences y such that y/a converges. We denote by w_∞ and w_0 the sets of all sequences y such that $\sup_n \left(n^{-1}\sum_{k=1}^{n}|y_k|\right) < \infty$ and $\lim_{n\to\infty}\left(n^{-1}\sum_{k=1}^{n}|y_k|\right) = 0$. We also use the sets of analytic and entire sequences denoted by $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ and defined by $\sup_n |y_n|^{1/n} < \infty$ and $\lim_{n\to\infty} |y_n|^{1/n} = 0$, respectively. In this paper we explicitly calculate the solutions of (SSE) of the form $E_T + F_x = F_b$ in each of the cases $E = c_0$, c, ℓ_∞, ℓ_p, $(p \geq 1)$, w_0, $\mathbf{\Gamma}$, or $\mathbf{\Lambda}$, $F = c$, or ℓ_∞, and T is either of the triangles Δ, or Σ, where Δ is the operator of the first difference, and Σ is the operator defined by $\Sigma_n y = \sum_{k=1}^{n} y_k$. For instance the solvability of the (SSE) $\Gamma_\Sigma + \mathbf{\Lambda}_x = \mathbf{\Lambda}_b$ consists in determining the set of all positive sequences $x = (x_n)_n$ that satisfy the statement: $\sup_n \left\{(|y_n|/b_n)^{1/n}\right\} < \infty$ if and only if there are u, $v \in \omega$ with $y = u + v$ such that

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n} u_k\right|^{1/n} = 0 \text{ and } \sup_n \left\{\left(\frac{|v_n|}{x_n}\right)^{1/n}\right\} < \infty \text{ for all } y.$$

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1. Introduction

We write ω for the set of all complex sequences $y = (y_n)_{n\geq 1}$, ℓ_∞, c and c_0 for the sets of all bounded, convergent and null sequences, respectively, also $\ell_p = \{y \in \omega : \sum_{k=1}^{\infty} |y_k|^p < \infty\}$ for $1 \leq p < \infty$. We then consider the sets of analytic and entire sequences denoted by $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ and defined by $\sup_n |y_n|^{1/n} < \infty$ and $\lim_{n\to\infty} |y_n|^{1/n} = 0$, respectively. If y, $z \in \omega$, then we write $yz = (y_n z_n)_{n\geq 1}$. Let $U = \{y \in \omega : y_n \neq 0\}$ and $U^+ = \{y \in \omega : y_n > 0\}$. We write $z/u = (z_n/u_n)_{n\geq 1}$ for all $z \in \omega$ and all $u \in U$, in particular $1/u = e/u$, where $e = \mathbf{1}$ is the sequence with $e_n = 1$ for all n. Finally, if $a \in U^+$ and E is any subset of ω, then we put $E_a = (1/a)^{-1} * E = \{y \in \omega : y/a \in E\}$. Let E and F be subsets of ω. Then the set $M(E, F) = \{y \in \omega : yz \in F \text{ for all } z \in E\}$ is called the *multiplier space of* E *and* F. In [2], the sets s_a, s_a^0 and $s_a^{(c)}$ were defined for positive sequences a by $(1/a)^{-1} * E$ and $E = \ell_\infty, c_0, c$, respectively. In [3] the sum $E_a + F_b$ and the product $E_a * F_b$ were defined where E, F are any of the symbols s, s^0, or $s^{(c)}$. Then in [7] were given solvability of sequences spaces equations inclusion $G_b \subset E_a + F_b$ where E, F, $G \in \{s^0, s^{(c)}, s\}$ and some applications to sequence spaces inclusions with operators. As above we define the sets of $a-$*analytic and* $a-$*entire sequences*, by $(1/a)^{-1} * E$ and $E = \mathbf{\Lambda}$, or $\mathbf{\Gamma}$, (see [4]). Recall that the spaces w_∞ and w_0 of strongly bounded and summable sequences are the sets of all y such that $(n^{-1}\sum_{k=1}^{n} |y_k|)_n$ is bounded and tend to zero respectively. These spaces were studied by Maddox [19] and Malkowsky [20]. In [10] were given some properties of well known operators defined by the sets $W_a = (1/a)^{-1} * w_\infty$ and $W_a^0 = (1/a)^{-1} * w_0$.

In this paper we extend some results given in [15, 7, 5, 6, 14, 8]. In [14] for given sequences a and b was determined the set of all positive sequences x for which $y_n/b_n \to l$ if and only if there are sequences u and v for which $y = u + v$ and $u_n/a_n \to 0$, $v_n/x_n \to l'$ $(n \to \infty)$ for all y and for some scalars l and l'. This statement is equivalent to the sequence spaces equation $s_a^0 + s_x^{(c)} = s_b^{(c)}$. In [8] was determined the set of all $x \in U^+$ such that for every sequence y, we have $y_n/b_n \to l$ if and only if there are sequences u and v with $y = u + v$ and $|u_n/a_n|^{1/n} \to 0$ and $v_n/x_n \to l'$ $(n \to \infty)$ for some scalars l and l'. This statement means $\mathbf{\Gamma}_a + s_x^{(c)} = s_b^{(c)}$. So we are led to deal with special *sequence spaces equations (SSE) with operator,* which are determined by an identity, for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $\left(E_{f(x)}\right)_T$ where f maps U^+ to itself, E is a linear space of sequences, x is the unknown and T is a triangle. It can be found in [6] a solvability of the (SSE) $E_a + (c_x)_{B(r,s)} = c_x$ where $E = s$, s^0, or $s^{(c)}$ and x is the unknown. In [14] were determined the sets of all positive sequences x that satisfy each of the systems $s_a^0 + (s_x)_\Delta = s_b$, $s_x \supset s_b$ and $s_a + (c_x)_\Delta = c_b$, $c_x \supset c_b$. Then it can be found a resolution of the (SSE) with operators defined by $(E_a)_{C(\lambda)D_\tau} + (c_x)_{C(\mu)D_\tau} = c_b$ with $E = c_0$, or ℓ_∞. Recently in [9] can be found a study on the (SSE) with operator $(E_a)_{C(\lambda)C(\mu)} + (E_x)_{C(\lambda\sigma)C(\mu)} = E_b$, where $b \in \widehat{C}_1$ and E is any of the sets ℓ_∞, or c_0. For $E = c_0$ the resolution of this equation consists in determining the set of all $x \in U^+$ such that for every sequence y the condition $y_n/b_n \to 0$ $(n \to \infty)$ holds if and only if there are u, $v \in \omega$ such that $y = u + v$ and

$$\frac{1}{\lambda_n a_n}\sum_{k=1}^{n}\left(\frac{1}{\mu_k}\sum_{i=1}^{k}u_i\right) \to 0 \text{ and } \frac{1}{\lambda_n\sigma_n x_n}\sum_{k=1}^{n}\left(\frac{1}{\mu_k}\sum_{i=1}^{k}v_i\right) \to 0 \ \ (n \to \infty). \tag{1.1}$$

In this paper we deal with a class of (SSE) with operators of the form $E_T + F_x = F_b$, where T is either Δ or Σ and E is any of the sets c_0, c, ℓ_∞, ℓ_p, $(p \geq 1)$, w_0, $\mathbf{\Gamma}$, or $\mathbf{\Lambda}$ and $F = c$, ℓ_∞ or $\mathbf{\Lambda}$. For instance the solvability of the (SSE) defined by the equation $\Gamma_\Sigma + \mathbf{\Lambda}_x = \mathbf{\Lambda}_b$ consists in determining the set of all positive sequences $x = (x_n)_n$ that satisfy the statement: $\sup_n \left\{(|y_n|/b_n)^{1/n}\right\} < \infty$ if and only if there are u, $v \in \omega$ with

$y = u + v$ such that

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n} u_k\right|^{1/n} = 0 \text{ and } \sup_n\left\{\left(\frac{|v_n|}{x_n}\right)^{1/n}\right\} < \infty \text{ for all } y.$$

This paper is organized as follows. In Section 2 we recall some definitions and results on sequence spaces and matrix transformations. In Section 3 are recalled general results on the multiplier $M(E, F)$ and on the classes (ℓ_∞, c) and (ℓ_p, F) where F is any of the sets c_0, c, or ℓ_∞. In Section 4 we deal with the sets of analytic and entire sequences. In Section 5 we deal with the sets of strongly and summable sequences by the Cesàro method and recall some results of the multiplier $M(E, F)$ where E and F are any of the sets w_0, w_∞, c_0, c, ℓ_∞, or ℓ_1. In Section 6 we recall some results on the solvability of (SSE) of the form $E_a + F_x = F_b$ with $\mathbf{1} \in F$ and we deal we deal with the solvability (SSE) with operator $E_T + F_x = F_b$ in the general case. In Section 7 we apply the previous results to solve (SSE) using the operator of the first difference and that are of the form $E_\Delta + F_x = F_b$, where $E = c_0$, c, ℓ_∞, ℓ_p, $(p \geq 1)$, w_0, $\mathbf{\Gamma}$, or $\mathbf{\Lambda}$ and $F = c$, or ℓ_∞. Then using the operator Σ we solve (SSE) of the form $E_\Sigma + F_x = F_b$, where $E = c_0$, c, ℓ_∞, ℓ_p, $(p \geq 1)$, w_0, $\mathbf{\Gamma}$, or $\mathbf{\Lambda}$ and $F = c$, ℓ_∞, and the (SSE) $\mathbf{\Gamma}_\Sigma + \Lambda_x = \Lambda_b$.

2. Premilinaries and notations

An FK space is a *complete metric space*, for which convergence implies *coordinatewise convergence*. A *BK space* is a Banach space of sequences that is, an *FK space*. A BK space E *is said to have AK if for every sequence* $y = (y_k)_{k\geq 1} \in E$*, then* $y = \lim_{p\to\infty}\sum_{k=1}^{p} y_k e^{(k)}$*, where* $e^{(k)} = (0, ..., 1, ...)$*,* 1 *being in the* $k-th$ *position.*

For a given infinite matrix $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ we define the operators $A_n = (\mathbf{a}_{nk})_{k\geq 1}$ for any integer $n \geq 1$, by $A_n y = \sum_{k=1}^{\infty} \mathbf{a}_{nk} y_k$, where $y = (y_k)_{k\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator A defined by $Ay = (A_n y)_{n\geq 1}$ mapping between sequence spaces. When A *maps* E *into* F*, where*

E and F are subsets of ω, we write that $A \in (E,F)$, (cf. [19, 25]). It is well known that if E has AK, then the set $\mathcal{B}(E)$ of all *bounded linear operators* L mapping in E, with norm $\|L\| = \sup_{y\neq 0}\left(\|L(y)\|_E / \|y\|_E\right)$ satisfies the identity $\mathcal{B}(E) = (E,E)$. We denote by ω, c_0, c, ℓ_∞ the sets of all sequences, the sets of null, convergent and bounded sequences. For any subset F of ω, we write $F_A = \{y \in \omega \ : \ Ay \in F\}$. By Σ we denote the operator defined by $\Sigma_n y = \sum_{k=1}^n y_k$ for all sequences y. Then we write $cs = c_\Sigma$, $bs = (\ell_\infty)_\Sigma$ and $cs_0 = (c_0)_\Sigma$ for the sets of all convergent, bounded and convergent to zero series. More precisely we have $cs = \{y : \sum_{k=1}^\infty y_k \text{ is convergent}\}$, $bs = \left\{y : \left(\sum_{k=1}^n y_k\right)_n \in \ell_\infty\right\}$ and $cs_0 = \left\{y : \left(\sum_{k=1}^n y_k\right)_n \in c_0\right\}$. Let $U^+ \subset \omega$ be the set of all sequences $\mathbf{u} = (u_n)_{n\geq 1}$ with $u_n > 0$ for all n. Then for given sequence $\mathbf{u} = (u_n)_{n\geq 1} \in \omega$ we define the diagonal matrix $D_{\mathbf{u}}$ by $[D_{\mathbf{u}}]_{nn} = u_n$ for all n. It is interesting to rewrite the set $E_{\mathbf{u}}$ using a diagonal matrix. Let E be any subset of ω and $\mathbf{u} \in U^+$ we have

$$E_{\mathbf{u}} = D_{\mathbf{u}}E = \{y = (y_n)_n \in \omega \ : y/\mathbf{u} \in E\}.$$

We will use the sets s_a^0, $s_a^{(c)}$, s_a and ℓ_a^p defined as follows (cf. [2]). For given $a \in U^+$ and $p \geq 1$ we put $D_a c_0 = s_a^0$, $D_a c = s_a^{(c)}$, $D_a \ell_\infty = s_a$, and $D_a \ell^p = \ell_a^p$. We will frequently write c_a instead of $s_a^{(c)}$ to simplify. Each of the spaces $D_a E$, where $E \in \{c_0, c, \ell_\infty\}$ is a *BK space normed* by $\|y\|_{s_a} = \sup_{n\geq 1}\left(|y_n|/a_n\right)$ and s_a^0 *has AK*. If $a = (r^n)_{n\geq 1}$ with $r > 0$, we write s_r, s_r^0 and $s_r^{(c)}$ for the sets s_a, s_a^0 and $s_a^{(c)}$ respectively. When $r = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$. Recall that $S_1 = (s_1, s_1)$ is a Banach algebra and $(c_0, s_1) = (c, \ell_\infty) = (s_1, s_1) = S_1$. We have $A \in S_1$ if and only if

$$\sup_n \left(\sum_{k=1}^\infty |\mathbf{a}_{nk}|\right) < \infty. \tag{2.1}$$

We will also use the characterization of (c_0, c_0). We have $A \in (c_0, c_0)$ if and only if (2.1) holds and $\lim_{n\to\infty} \mathbf{a}_{nk} = 0$ for all k. We will use the well known property, stated as follows. For any given triangle T, the operator T' represented by a triangle belongs to (E_T, F) if and only if $T'T^{-1} \in (E, F)$ for any subsets $E,\ F \subset \omega$.

3. The multipliers of some sets and matrix transformations

3.1. **The multipliers of classical sets.** First we need to recall some well known results. Let y and z be sequences and let E and F be two subsets of ω, we then write $yz = (y_n z_n)_n$ and

$$M(E, F) = \{y \in \omega : \ yz \in F \text{ for all } z \in E\},$$

$M(E, F)$ is called the *multiplier space of* E *and* F. In the following we will use the next well known results.

Lemma 3.1. *Let* E, $\widetilde{E}$, F *and* $\widetilde{F}$ *be arbitrary subsets of* ω. *Then*
(i) $M(E, F) \subset M\left(\widetilde{E}, F\right)$ *for all* $\widetilde{E} \subset E$,
(ii) $M(E, F) \subset M\left(E, \widetilde{F}\right)$ *for all* $F \subset \widetilde{F}$.

Lemma 3.2. *Let* $a,\ b \in U^+$ *and let* E *and* F *be two subsets of* ω. *Then* $D_a E \subset D_b F$ *if and only if* $a/b \in M(E, F)$.

Lemma 3.3. *Let* $a,\ b \in U^+$ *and* $E,\ F \subset \omega$. *Then* $A \in (D_a E, D_b F)$ *if and only if* $D_{1/b} A D_a \in (E, F)$.

Notice that this lemma can be extended to the case when $a \in \omega$ and b is a nonzero sequence.

By [3, Lemma 3.1, p. 648] and [3, Example 1.28, p. 157], we obtain the next result.

Lemma 3.4. *We have*
i) $M(c, c_0) = M(\ell_\infty, c) = M(\ell_\infty, c_0) = c_0$ *and* $M(c, c) = c$;
ii) $M(E, \ell_\infty) = M(c_0, F) = \ell_\infty$ *for* $E,\ F = c_0,\ c,$ *or* ℓ_∞ .

3.2. **The classes (ℓ_∞, c) and (ℓ_p, F) where F is any of the sets c_0, c, or ℓ_∞.** As a direct consequence of the famous Kojima-Shur Theorem we obtain the next lemma.

Lemma 3.5. *Let $A = (\mathbf{a}_{nk})_{nk}$ be an infinite matrix. Then*

i) if $\lim_{n\to\infty} \mathbf{a}_{nk} = 0$ for all k, then $A \in (\ell_\infty, c)$ if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |\mathbf{a}_{nk}| = 0. \tag{3.1}$$

ii) $A \in (\ell_\infty, c_0)$ if and only if (3.1) holds.

For the convenience of the reader we recall the next well-known result, (see for instance [22, Theorem 1.37, pp. 160-161]), which will be frequently used in the following.

Lemma 3.6. *i) Case $1 < p < \infty$. Let $q = p/(p-1)$. Then we have*

a) $A \in (\ell_p, \ell_\infty)$ if and only if condition

$$\sup_n \left(\sum_{k=1}^{\infty} |\mathbf{a}_{nk}|^q \right) < \infty \tag{3.2}$$

holds.

b) $A \in (\ell_p, c_0)$ if and only if condition (3.2) holds and $\lim_{n\to\infty} \mathbf{a}_{nk} = 0$ for all k.

c) $A \in (\ell_p, c)$ if and only if condition (3.2) holds and $\lim_{n\to\infty} \mathbf{a}_{nk} = l_k$ for some $l_k \in \mathbb{C}$ and for all k.

ii) Case $p = 1$. We have

a) $A \in (\ell_1, \ell_\infty)$ if and only if

$$\sup_{n,k} |\mathbf{a}_{nk}| < \infty \tag{3.3}$$

holds.

b) $A \in (\ell_1, c_0)$ if and only if condition (3.3) holds and $\lim_{n\to\infty} \mathbf{a}_{nk} = 0$ for all k.

c) $A \in (\ell_1, c)$ if and only if condition (3.3) holds and $\lim_{n\to\infty} \mathbf{a}_{nk} = l_k$ for some $l_k \in \mathbb{C}$ and for all k.

Remark 1. We deduce from Lemma 3.6 the identity $M(\ell_p, \chi) = \ell_\infty$ for $\chi = c_0$, c, or ℓ_∞ and $(p \geq 1)$.

4. On the sets of analytic and entire sequences

4.1. **Some definitions and properties of $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$.** A sequence $y = (y_n)_{n \geq 1}$ is said to be analytic if $\sup_n |y_n|^{1/n} < \infty$. The linear space of all *analytic sequences* is denoted by Λ. It is well known that Γ is the linear space of all *entire sequences* defined by $\lim_{n \to \infty} |y_n|^{1/n} = 0$. The sets Λ and Γ are *metric spaces* with the metric defined for any sequences y, z, by $d(y, z) = \sup_n |y_n - z_n|^{1/n}$. Then Λ is an *FK space since it is a complete metric space*, and convergence implies *coordinatewise convergence*; it is the same for Γ since it is a closed subset of Λ. For a study of the sets Λ and Γ**,** we refer the reader to [23].

Concerning the multipliers $M(\mathbf{\Gamma}, F)$, $M(\mathbf{\Lambda}, F)$, $M(E, \mathbf{\Lambda})$ and $M(E, \mathbf{\Gamma})$ for E, $F \in \{c_0, c, \ell_\infty, \mathbf{\Gamma}, \mathbf{\Lambda}\}$ recall the following.

Lemma 4.1. *[8, Proposition 4.2] We have*

(i) $M(\mathbf{\Gamma}, F) = \mathbf{\Lambda}$ *for* $F \in \{c_0, c, \ell_\infty, \mathbf{\Gamma}, \mathbf{\Lambda}\}$,

(ii) $M(\mathbf{\Lambda}, F) = \mathbf{\Gamma}$ *for* $F \in \{c_0, c, \ell_\infty, \mathbf{\Gamma}\}$,

(iii) $M(E, \mathbf{\Lambda}) = \mathbf{\Lambda}$ *for* $E \in \{c_0, c, \ell_\infty, \mathbf{\Gamma}, \mathbf{\Lambda}\}$,

(iv) $M(E, \mathbf{\Gamma}) = \mathbf{\Gamma}$ *for* $E \in \{c_0, c, \ell_\infty, \mathbf{\Lambda}\}$.

4.2. **Some properties of the sets $\mathbf{\Gamma}_a$ and $\mathbf{\Lambda}_a$.** For $a \in U^+$ we put $\Lambda_a = D_a \Lambda$. So $y \in \Lambda_a$ if $\sup_n (|y_n| / a_n)^{1/n} < \infty$ and Λ_a is called the set of all $a-$*analytic sequences.* For $a = \mathbf{1}$ we write $\Lambda_1 = \Lambda$. Similarly we put $\Gamma_a = D_a \Gamma$ and $y = (y_n)_{n \geq 1} \in \Gamma_a$ if and only if $\lim_{n \to \infty} (|y_n| / a_n)^{1/n} = 0$, we write $\Gamma_1 = \Gamma$ and Γ_a is the set of all $a-$*entire sequences.*

In the following we use the triangle $C(\lambda)$ defined for any nonzero sequence $\lambda = (\lambda_n)_{n \geq 1}$ by $[C(\lambda)]_{nk} = 1/\lambda_n$ for $k \leq n$. It can be shown that the triangle $\Delta(\lambda)$ whose the nonzero entries are defined by $[\Delta(\lambda)]_{nn} = \lambda_n$, for all n and by $[\Delta(\lambda)]_{n,n-1} = -\lambda_{n-1}$, for all $n \geq 2$, is the inverse of $C(\lambda)$, that is, $C(\lambda)(\Delta(\lambda)y) = \Delta(\lambda)(C(\lambda)y)$ for all $y \in \omega$. It is well known that $\Delta = \Delta(\mathbf{1}) \in (\omega, \omega)$, is the operator of the first difference and we have $\Delta y_n = y_n - y_{n-1}$ for all $n \geq 1$ with $y_0 = 0$. The inverse $\Delta^{-1} = \Sigma$ is defined by $\Sigma_{nk} = 1$ for $k \leq n$, (see for instance [2, 13]). For any given $a \in U^+$ we have $[C(a)a]_n = (a_1 + ... + a_n)/a_n$ for all n. Then we let

$$\widehat{C}_{\mathbf{\Lambda}} = \left\{a \in U^+ : [C(a)a]_n \leq k^n \text{ for all } n \text{ and for some } k > 0\right\}.$$

We obtain the next results which is a consequence of [8, Proposition 3.1, p. 101].

Lemma 4.2. *Let a, $b \in U^+$. Then*

a) We have $\Lambda_a = \Lambda_b$ if and only if $\mathbf{\Gamma}_a = \mathbf{\Gamma}_b$, and the equality $\Lambda_a = \Lambda_b$ is equivalent to the statement $k_1^n \leq a_n/b_n \leq k_2^n$ for all n and for some k_1, $k_2 > 0$.

b) $(\mathbf{\Lambda}_a)_\Delta = \Lambda_b$ if and only if $\Lambda_a = \Lambda_b$ and $a \in \widehat{C}_{\mathbf{\Lambda}}$.

Now we recall some results on the spaces $c_0(p)$ and $\ell_\infty(q)$ that generalize the sets Λ and $\mathbf{\Gamma}$.

4.3. **On the sets $(c_0(p), c_0(q))$ and $(c_0(p), \ell_\infty(q))$.** Let $p = (p_n)_{n\geq 1} \in U^+ \bigcap \ell_\infty$ be a sequence and put

$$\begin{aligned} \ell_\infty(p) &= \left\{y = (y_n)_{n\geq 1} : \sup_n |y_n|^{p_n} < \infty\right\}, \\ c_0(p) &= \left\{y = (y_n)_{n\geq 1} : \lim_{n\to\infty} |y_n|^{p_n} = 0\right\}. \end{aligned}$$

The set $c_0(p)$ is a *complete paranormed space* with $g(y) = \sup_n \left(|y_n|^{p_n/L}\right)$, where $L = \max\{1, \sup_n p_n\}$, ([18, Theorem 1]) and $\ell_\infty(p)$ is a *paranormed space with g* only if $\inf_n p_n > 0$ in which case $\ell_\infty(p) = \ell_\infty$, ([24, Theorem 9]). So we can state the next lemma, where for any given integer k, we denote by $\mathbb{N}_k$ the set of all integers $n \geq k$.

Lemma 4.3. *[16, Theorem 5.1.13] Let* $p,\, q \in U^+ \bigcap \ell_\infty$.

i) $A \in (c_0(p), c_0(q))$ *if and only if for all* $N \in \mathbb{N}_1$ *there is* $M \in \mathbb{N}_2$ *such that*

$$\sup_n \left(N^{1/q_n} \sum_{k=1}^{\infty} |\mathbf{a}_{nk}| \, M^{-1/p_k} \right) < \infty \text{ and } \lim_{n\to\infty} |\mathbf{a}_{nk}|^{p_n} = 0 \text{ for all } k.$$

ii) $A \in (c_0(p), \ell_\infty(q))$ *if and only if there is* $M \in \mathbb{N}_2$ *such that*

$$\sup_n \left(\sum_{k=1}^{\infty} |\mathbf{a}_{nk}| \, M^{-1/p_k} \right)^{q_n} < \infty.$$

Example 4.1. *In this way we have* $A \in (\mathbf{\Gamma}, \mathbf{\Lambda})$ *if and only if there is* $M \geq 2$ *integer such that* $\sup_n \left(\sum_{k=1}^{\infty} |\mathbf{a}_{nk}| \, M^{-k} \right)^{1/n} < \infty$, *since* $\mathbf{\Gamma} = c_0(p)$ *and* $\mathbf{\Lambda} = \ell_\infty(p)$ *with* $p_n = 1/n$.

5. The spaces of strongly bounded and summable sequences by the Cesàro method

5.1. **The sets w_∞ and w_0.** Recall that when $\lambda_n = n$ for all n, the triangle $C(\lambda)$ is the well known Cesàro operator C_1. In the following we will use the spaces of *strongly bounded and summable sequences* by *the Cesàro method of order 1* defined by

$$w_\infty = \{ y \in \omega \ : C_1 |y| \in \ell_\infty \} \text{ and } w_0 = \{ y \in \omega : C_1 |y| \in c_0 \},$$

where $|y| = (|y_n|)_n$. These spaces were studied by Maddox [17] and Malkowsky, see for instance [20]. It is well known that the sets w_∞ and w_0 are BK spaces normed by $\|y\|_{w_\infty} = \sup_n (n^{-1} \sum_{k=1}^{n} |y_k|)$. In [21] it was shown that the class (w_∞, w_∞) is a *Banach algebra* normed by $\|A\|^*_{(w_\infty, w_\infty)} = \sup_{y \neq 0} \left(\|Ay\|_{w_\infty} / \|y\|_{w_\infty} \right)$.

5.2. **Matrix transformations in the sets w_0 and w_∞.** Here we recall some results that are direct consequence of [1, Theorem 2.4]. For this we let $\chi_n = \sum_{\nu=1}^{\infty} 2^\nu \max_{2^\nu \leq k \leq 2^{\nu+1}-1} |\mathbf{a}_{nk}|$. Then we can state the following.

Lemma 5.1. *[1] (i) We have* $(w_0, \ell_\infty) = (w_\infty, \ell_\infty)$ *and* $A \in (w_\infty, \ell_\infty)$ *if and only if*

$$\sup_n \chi_n < \infty, \tag{5.1}$$

(ii) $A \in (w_\infty, c_0)$ *if and only if* $\lim_{n\to\infty} \chi_n = 0$.

(iii) $A \in (w_0, c_0)$ *if and only if (5.1) holds and* $\lim_{n\to\infty} \mathbf{a}_{nk} = 0$ *for all* k.

5.3. **The multiplier** $M(E, F)$ **where** E **and** F **are any of the sets** w_0**,** w_∞**,** c_0**,** c**,** ℓ_∞**, or** ℓ_1**.** In the following we will use the next results.

Lemma 5.2. *[11, Lemma 4.2] We have*

i) $M(w_0, F) = M(w_\infty, \ell_\infty) = s_{(1/n)_n}$ *for* $F = c_0$*,* c*, or* ℓ_∞.

ii) $M(w_\infty, c_0) = s^0_{(1/n)_n}$.

iii) $M(\ell_1, w_\infty) = s_{(n)_n}$ *and* $M(\ell_1, w_0) = s^0_{(n)_n}$.

iv) $M(E, w_0) = w_0$ *for* $E = c$*, or* ℓ_∞.

6. On the (SSE) $E_a + F_x = F_b$

In this section we apply the previous results to the solvability of the (SSE) $E_a+F_x = F_b$ with $\mathbf{1} \in F$.

6.1. **Regular sequence spaces equations.** For $b \in U^+$ and for any subset F of ω, we denote by $cl^F(b)$ the equivalent class for the equivalence relation R_F defined by

$$xR_Fy \text{ if } D_xF = D_yF \text{ for } x,\ y \in U^+.$$

It can easily be seen that $cl^F(b)$ is the set of all $x \in U^+$ such that $x/b \in M(F, F)$ and $b/x \in M(F, F)$, (cf. [14]). We then have $cl^F(b) = cl^{M(F,F)}(b)$. For instance $cl^c(b)$ is the set of all $x \in U^+$ such that $D_xc = D_bc$, that is, $s_x^{(c)} = s_b^{(c)}$. This is the set of all sequences $x \in U^+$ such that $x_n \sim Cb_n$ $(n \to \infty)$ for some $C > 0$. In [14] we denote by $cl^\infty(b)$ the class $cl^{\ell_\infty}(b)$. Recall that $cl^\infty(b)$ is the set of all $x \in U^+$, such that $K_1 \le x_n/b_n \le K_2$ for all n and for some $K_1,\ K_2 > 0$. In [8, Proposition 3.1] the class $cl^{\mathbf{\Lambda}}(b)$ is the set of all $x \in U^+$, such that $k_1^n \le x_n/b_n \le k_2^n$ for all n and for some $k_1,\ k_2 > 0$. Note that the relations $R_{\mathbf{\Lambda}}$ and $R_{\mathbf{\Gamma}}$ are equivalent, since we have $M(\mathbf{\Lambda}, \mathbf{\Lambda}) = M(\mathbf{\Gamma}, \mathbf{\Gamma}) = \mathbf{\Lambda}$.

For any given linear spaces of sequences X and Y, we have $X+Y = \{u + v : u,\ v \in \omega\}$. It can easily be seen that for any given linear subspaces X, Y and Z of ω, the inclusion $X + Y \subset Z$ holds if and only if $X \subset Z$ and $Y \subset Z$. In this way, for $a,\ b \in U^+$, we define the set

$$S(E, F) = \left\{x \in U^+ : E_a + F_x = F_b\right\},$$

where $E,\ F$ are linear subspaces of ω. For instance, $S(w_\infty, \ell_\infty)$ is the set of all sequences $x \in U^+$ that satisfy the statement: $\sup_n (|y_n|/b_n) < \infty$ if and only if there are two sequences u and v for which $y = u + v$ and

$$\sup_n \left(\frac{1}{n}\sum_{k=1}^{n}\frac{|u_k|}{a_k}\right) < \infty \text{ and } \sup_n \left(\frac{|v_n|}{x_n}\right) < \infty \text{ for all } y.$$

Definition 6.1. We say that $S(E, F)$, (or the equation $E_a + F_x = F_b$), is *regular* if

$$\mathcal{S}(E, F) = \begin{cases} cl^{M(F,F)}(b) & \text{if } a/b \in M(E, F), \\ \varnothing & \text{if } a/b \notin M(E, F). \end{cases}$$

Note that $E_a + F_x = F_b$ is not regular in general. Indeed for $E = F = \ell_\infty$ we have $M(\ell_\infty, \ell_\infty) = \ell_\infty$ and if $a/b \in \ell_\infty \diagdown c_0$ and $s_a = s_b$ we have $\mathcal{S}(\ell_\infty, \ell_\infty) = s_b \cap U^+ \neq cl^{M(F,F)}(b)$, (cf. [15, Theorem 11, pp. 916-917]). In particular the solutions of the (SSE) $\ell_\infty + s_x = \ell_\infty$ are determined by $x \in \ell_\infty \cap U^+$, that is, $0 < x_n \le M$ for all n and for some $M > 0$.

6.2. **Solvability of (SSE) of the form $E_a + F_x = F_b$.** For instance the solvability of the equation $s_a + s_x^{(c)} = s_b^{(c)}$ for $a,\ b \in U^+$ consists in determining the set of all $x \in U^+$ that satisfy the next statement: $y_n/b_n \to l\ (n \to \infty)$ if and only if there are two sequences u, v such that $y = u + v$ and

$$\frac{|u_n|}{a_n} \leq K \text{ and } \frac{v_n}{x_n} \to l' \ (n \to \infty) \text{ for all } y.$$

In the following we will use the condition

$$\chi \subset \chi(D_\alpha) \text{ for all } \alpha \in c(1), \tag{6.1}$$

where $\chi \subset \omega$ is any linear space, and $c(1)$ is the set of all sequences that tend to 1. It can easily seen that this condition is true for any of the spaces $F = c$, s_1, or $\mathbf{\Lambda}$. To state the next results we also need the next conditions:

$$\mathbf{1} \in F, \tag{6.2}$$

$$F \subset M(F, F). \tag{6.3}$$

We then recall the next result which is a direct consequence of [8, Theorem 5.1, pp. 106-107].

Lemma 6.1. *Let a, $b \in U^+$ and let E, F be two linear subspaces of ω. We assume F satisfies the conditions in (6.1), (6.2), (6.3), and that*

$$M(E, F) \subset M(E, c_0). \tag{6.4}$$

Then $S(E, F)$ is regular.

In all what follows we are interested in the study of the (SSE)

$$E + F_x = F_b.$$

In this way replacing a by $\mathbf{1}$ in the previous lemma and noticing that the conditions in (6.2) and (6.3) imply $M(F, F) = F$ we obtain the following lemma.

Lemma 6.2. *Let $b \in U^{+}$ and let E, F be two linear subspaces of ω. We assume F satisfies the conditions in (6.1), (6.2), (6.3), and (6.4). Then $S(E,F)$ is regular and we have*

$$S(E,F) = \begin{cases} cl^{F}(b) & \text{if } 1/b \in M(E,F), \\ \varnothing & \text{if } 1/b \notin M(E,F). \end{cases}$$

As a direct consequence of Lemma 6.2 we obtain the next results.

Lemma 6.3. *Let $b \in U^{+}$ and let $p \geq 1$. Then each of the next (SSE) is regular, where*

i) $\Gamma + \mathbf{\Lambda}_x = \mathbf{\Lambda}_b$.

ii) $E + c_x = c_b$, *for* $E = \mathbf{\Gamma}$, $\mathbf{\Lambda}$, c_0, ℓ_∞, w_0 *and* ℓ_p.

iii) $E + s_x = s_b$, *for* $E = \mathbf{\Gamma}$, $\mathbf{\Lambda}$, c_0, w_0, w_∞ *and* ℓ_p.

Proof. Statement i) and statements ii) and iii) with $E = \mathbf{\Gamma}$, $\mathbf{\Lambda}$, were shown in [8, Proposition 5.1]. Statements ii) with $E = c_0$, or ℓ_∞ and iii) with $E = c_0$, were shown in [14, Theorem 4.4, p. 7]. Statements ii) with $E = w_0$ and iii) with $E = w_0$, or w_∞ were shown in [11, Theorem 6.5]. Statements ii) and iii) with $E = \ell_p$ $(p \geq 1)$ were shown in [11, Remark 6.4]. □

More precisely we obtain the following lemma which is a direct consequence of Lemma 6.3.

Lemma 6.4. *Let $b \in U^{+}$. We have*

i) a)

$$S(\ell_\infty, c) = \begin{cases} cl^{c}(b) & \text{if } 1/b \in c_0, \\ \varnothing & \text{otherwise.} \end{cases}$$

b) Let F be any of the sets c, s_1, or $\mathbf{\Lambda}$. Then we have

$$S(\mathbf{\Gamma},F) = \begin{cases} cl^{F}(b) & \text{if } 1/b \in \mathbf{\Lambda}, \\ \varnothing & \text{otherwise.} \end{cases}$$

ii) For $F = c$, or s_1 we have

a) Let $p \geq 1$. We have $S(\ell_p, F) = S(c_0, F)$ and

$$S(c_0, F) = \begin{cases} cl^F(b) & \text{if } 1/b \in s_1, \\ \varnothing & \text{otherwise.} \end{cases}$$

b)

$$S(w_0, F) = \begin{cases} cl^F(b) & \text{if } 1/b \in s_{(1/n)_n}, \\ \varnothing & \text{otherwise.} \end{cases}$$

c)

$$S(\mathbf{\Lambda}, F) = \begin{cases} cl^F(b) & \text{if } 1/b \in \mathbf{\Gamma}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Remark 2. The results for $S(\ell_p, c)$ and $S(\ell_p, \ell_\infty)$ come from Lemma 3.6 where $M(\ell_p, c) = M(\ell_p, \ell_\infty) = \ell_\infty$.

Remark 3. Notice that the set $S(c, c)$ is not regular since by [8, Theorem 5.2, p. 12] we have $S(c, c) = cl^c(b)$ for $1/b \in c_0$; $S(c, c) = c_b$ for $1/b \in c \setminus c_0$, and $S(c, c) = \varnothing$ for $1/b \notin c$.

Example 6.1. *Consider the set of all $x \in U^+$ that satisfy the statement: for every sequence y we have $y_n \to l_1$ $(n \to \infty)$ if and only if there are u and $v \in \omega$ for which $y = u + v$ and*

$$|u_n|^{\frac{1}{n}} \to 0 \text{ and } x_n v_n \to l_2 \ (n \to \infty) \text{ for some } l_1 \text{ and } l_2.$$

Since this set corresponds to the equation $\mathbf{\Gamma} + s_{1/x}^{(c)} = c$, by Lemma 6.3 it is equal to the set of all sequences that tend to a positive limit.

6.3. **Application to the solvability of the (SSE)** $E_T + F_x = F_b$ **with** $\mathbf{1} \in F$**.** Let $b \in U^+$, and E, F be two subsets of ω. We deal with the (SSE) with operator

$$E_T + F_x = F_b, \tag{6.5}$$

where T is a triangle and $x \in U^+$ is the unknown. The equation in (6.5) means for every $y \in \omega$, we have $y/b \in F$ if and only if there are u, $v \in \omega$ such that $y = u + v$ such that

$$Tu \in E \text{ and } v/x \in F.$$

We assume $e = \mathbf{1} \in F$. By $S(E_T, F)$ we denote the set of all $x \in U^+$ that satisfy the (SSE) in (6.5). We obtain the next result which is a direct consequence of Lemma 6.2, where we replace E by E_T.

Proposition 6.1. *Let $b \in U^+$ and let E, F be linear vector spaces of sequences. We assume F satisfies the conditions in (6.1), (6.2), (6.3), and that*

$$M(E_T, F) \subset M(E_T, c_0). \tag{6.6}$$

Then the set $S(E_T, F)$ is regular, that is,

$$S(E_T, F) = \begin{cases} cl^F(b) & \text{if } 1/b \in M(E_T, F), \\ \varnothing & \text{if } 1/b \notin M(E_T, F). \end{cases}$$

We may adapt the previous result using the notations of matrix transformations instead of the multiplier of sequence spaces. So we obtain the following.

Corollary 6.1. *Let $b \in U^+$ and let E, F be linear vector spaces of sequences. We assume F satisfies the conditions in (6.1), (6.2), (6.3), and that*

$$D_\alpha T^{-1} \in (E, F) \text{ implies } D_\alpha T^{-1} \in (E, c_0) \text{ for all } \alpha \in \omega. \tag{6.7}$$

Then we have

$$S(E_T, F) = \begin{cases} cl^F(b) & \text{if } D_{1/b}T^{-1} \in (E, F), \\ \varnothing & \text{if } D_{1/b}T^{-1} \notin (E, F). \end{cases}$$

Proof. This result is a direct consequence of Proposition 6.1 and of the fact that the condition $1/b \in M(E_T, F)$ is equivalent to $D_{1/b} \in (E_T, F)$ and to $D_{1/b}T^{-1} \in (E, F)$. □

7. The main results. Application to the solvability of (SSE) of the form $E_\Delta + F_x = F_b$ and $E_\Sigma + F_x = F_b$

In this section we apply Proposition 6.1 and Lemma 6.4 to solve (SSE) of the form $E_T + F_x = F_b$ in each of the cases $T = \Delta$ and $T = \Sigma$. We obtain a class of (SSE) that are regular, that is, for which $S(E, F)$ is regular.

7.1. Solvability of (SSE) of the form $E_\Delta + F_x = F_b$.

7.1.1. *On the (SSE)* $(c_0)_\Delta + F_x = F_b$. Here we solve each of the (SSE) defined by $(c_0)_\Delta + c_x = c_b$, and by $(c_0)_\Delta + s_x = s_b$. The solvability of the first (SSE) means that for every $y \in \omega$ we have $y_n/b_n \to l_1$ $(n \to \infty)$ if and only if there are u, $v \in \omega$ such that $y = u + v$ and

$$u_n - u_{n-1} \to 0 \text{ and } \frac{v_n}{x_n} \to l_2 \ (n \to \infty) \text{ for some scalars } l_1 \text{ and } l_2.$$

Proposition 7.1. *Let $b \in U^+$ and let $F = c$, or ℓ_∞. We have*

$$S((c_0)_\Delta, F) = \begin{cases} cl^F(b) & \text{if } 1/b \in s_{(1/n)_n}, \\ \varnothing & \text{if } 1/b \notin s_{(1/n)_n}. \end{cases}$$

Proof. The condition $\alpha \in M((c_0)_\Delta, s_1)$ means $D_\alpha \Sigma \in (c_0, s_1) = S_1$ and is equivalent to $n\alpha_n = O(1)$ $(n \to \infty)$. So $M((c_0)_\Delta, s_1) = s_{(1/n)_n}$. On the same way by the characterization of (c_0, c_0) we obtain $M((c_0)_\Delta, c_0) = s_{(1/n)_n}$. We then have

$$s_{(1/n)_n} = M((c_0)_\Delta, c_0) \subset M((c_0)_\Delta, c) \subset M((c_0)_\Delta, s_1) = s_{(1/n)_n},$$

and $M\left((c_0)_\Delta, F\right) = s_{(1/n)_n}$ for $F = s_1$, c, or c_0. We conclude by Proposition 6.1. $\square$

Example 7.1. *Let $\alpha \geq 0$. The (SSE) defined by $(c_0)_\Delta + c_x = c_{(n^\alpha)_n}$ has solutions if and only if $\alpha \geq 1$. These solutions are determined by $\lim_{n\to\infty} x_n/n^\alpha > 0$ $(n \to \infty)$. If $0 \leq \alpha < 1$ the (SSE) has no solution, Notice that the (SSE) $(c_0)_\Delta + c_x = c$ has no solution.*

Example 7.2. *Let $u > 0$. The set of all positive sequences x that satisfy the (SSE) $(c_0)_\Delta + s_x = s_u$ is empty if $u \leq 1$, and if $u > 1$ it is equal to the set of all sequences that satisfy $K_1 u^n \leq x_n \leq K_2 u^n$ for all n and for some K_1, $K_2 > 0$.*

7.1.2. *The (SSE) with operator $bv_p + F_x = F_b$.* In this part we solve each of the (SSE) defined by $bv_p + c_x = c_b$, and by $bv_p + s_x = s_b$, where $bv_p = (\ell_p)_\Delta$, $(p > 1)$. Recall that $bv_p = \{y \in \omega : \sum_{k=1}^\infty |y_k - y_{k-1}|^p < \infty\}$ is the set of p-bounded variation sequences. The solvability of the second (SSE) consists in determining the set of all positive sequences x, such that the next statement holds. For every $y \in \omega$ we have $\sup_n (|y_n| / b_n) < \infty$ if and only if there are u, $v \in \omega$ with $y = u + v$ such that

$$\sum_{k=1}^\infty |u_n - u_{n-1}|^p < \infty \text{ and } \sup_n \left(\frac{|v_n|}{x_n}\right) < \infty.$$

We obtain the next proposition.

Proposition 7.2. *Let $b \in U^+$, and let $p > 1$, and $q = p/(p-1)$. For $F = c$, or ℓ_∞ we have*

$$S(bv_p, F) = \begin{cases} cl^F(b) & \text{if } \left(\dfrac{n^{1/q}}{b_n}\right)_n \in s_1, \\ \varnothing & \text{if } \left(\dfrac{n^{1/q}}{b_n}\right)_n \notin s_1. \end{cases}$$

Proof. We have $\alpha \in M(bv_p, \ell_\infty)$ if and only if $D_\alpha \Sigma \in (\ell_p, \ell_\infty)$, and from the characterization of (ℓ_p, ℓ_∞) given in Lemma 3.6 we have

$$n|\alpha_n|^q = O(1) \quad (n \to \infty). \tag{7.1}$$

So we have $M(bv_p, \ell_\infty) = s_{\left(n^{-1/q}\right)_n}$. Now we have $\alpha \in M(bv_p, c_0)$ if and only if (7.1) holds and

$$\alpha_n \to 0 \;\; (n \to \infty). \tag{7.2}$$

But trivially the condition in (7.1) implies the condition in (7.2). So we have

$$s_{\left(n^{-1/q}\right)_n} = M(bv_p, c_0) \subset M(bv_p, c) \subset M(bv_p, \ell_\infty) = s_{\left(n^{-1/q}\right)_n},$$

and $M(bv_p, F) = s_{\left(n^{-1/q}\right)_n}$ for $F = c_0$, c, or ℓ_∞. We may apply Proposition 6.1 where the condition $1/b \in M(bv_p, \ell_\infty) = s_{\left(n^{-1/q}\right)_n}$ means $\left(n^{1/q}/b_n\right)_n \in s_1$. This concludes the proof. □

Example 7.3. *The (SSE) defined by* $bv_2 + c_x = c$ *has no solution since* $q = 2$ *and* $\left(\sqrt{n}/b_n\right)_n \notin s_1$.

Example 7.4. *Let* $p > 1$ *and* $r > 0$. *The set* $S = S(bv_p, c)$ *of all the solutions of the (SSE)* $bv_p + c_x = c_{(n^r)_n}$ *is empty if* $r < (p-1)/p$ *and if* $r \geq (p-1)/p$, *it is determined by* $\lim_{n\to\infty}(x_n/n^r) > 0$. *For any given* $r \neq 1$, *we have* $S \neq \varnothing$ *if and only if* $p \leq 1/(1-r)$.

7.1.3. *Solvability of the (SSE) defined by* $(w_0)_\Delta + F_x = F_b$. Here a positive sequence x is a solution of the (SSE) $(w_0)_\Delta + c_x = c_b$ if the next statement holds. For every $y \in \omega$ we have $y_n/b_n \to l_1$ $(n \to \infty)$ if and only if there are u, $v \in \omega$ with $y = u + v$ such that

$$\frac{1}{n}\sum_{k=1}^{n}|u_k - u_{k-1}| \to 0 \text{ and } \frac{v_n}{x_n} \to l_2 \;\; (n \to \infty) \text{ for some scalars } l_1 \text{ and } l_2.$$

We obtain a similar statement for the (SSE) $(w_0)_\Delta + s_x = s_b$. We have the next proposition.

Proposition 7.3. *Let $b \in U^{+}$. Then for $F = c$, or ℓ_∞ we have*

$$S\left((w_0)_\Delta, F\right) = S\left((c_0)_\Delta, F\right) = S\left(w_0, F\right) = \begin{cases} cl^F(b) & \text{if } 1/b \in s_{(1/n)_n}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. We have $\alpha \in M\left((w_0)_\Delta, \ell_\infty\right)$ if and only if

$$D_\alpha \Sigma \in \left(w_0, \ell_\infty\right). \tag{7.3}$$

Now we define the integer ν_n by

$$2^{\nu_n} \leq n \leq 2^{\nu_n + 1} - 1. \tag{7.4}$$

Then from the characterization of (w_0, ℓ_∞) in Lemma 5.1 the condition in (7.3) means there is $K > 0$ such that

$$\sigma_n = \sum_{\nu=0}^{\infty} 2^{\nu} \max_{2^\nu \leq k \leq 2^{\nu+1}-1} \left|(D_\alpha \Sigma)_{nk}\right| = |\alpha_n| \sum_{\nu=0}^{\nu_n} 2^{\nu} = |\alpha_n| \left(2^{\nu_n+1} - 1\right) \leq K \text{ for all } n. \tag{7.5}$$

Then from (7.4) we have $D_\alpha \Sigma \in (w_0, \ell_\infty)$ if and only if

$$n\left|\alpha_n\right| \leq \left(2^{\nu_n+1} - 1\right)\left|\alpha_n\right| \leq K \text{ for all } n, \text{ and for some } K > 0.$$

Then we have $M\left((w_0)_\Delta, \ell_\infty\right) \subset s_{(1/n)_n}$. Now we show $s_{(1/n)_n} \subset M\left((w_0)_\Delta, \ell_\infty\right)$. Let $\alpha \in s_{(1/n)_n}$. Then we have $n\left|\alpha_n\right| \leq K$ for all n, and by (7.5) and (7.4) we have

$$\sigma_n = \left(2^{\nu_n+1} - 1\right)\left|\alpha_n\right| \leq (2n-1)\left|\alpha_n\right| \leq 2K \text{ for all } n.$$

This shows $s_{(1/n)_n} \subset M\left((w_0)_\Delta, \ell_\infty\right)$ and $M\left((w_0)_\Delta, \ell_\infty\right) = s_{(1/n)_n}$. By similar arguments we obtain $M\left((w_0)_\Delta, c_0\right) = s_{(1/n)_n}$. Then we have

$$s_{(1/n)_n} = M\left((w_0)_\Delta, c_0\right) \subset M\left((w_0)_\Delta, c\right) \subset M\left((w_0)_\Delta, \ell_\infty\right) = s_{(1/n)_n}.$$

Finally, we have $M\left((w_0)_\Delta, F\right) = s_{(1/n)_n}$ for $F = c_0$, c, or ℓ_∞. We conclude by Proposition 6.1 and Lemma 6.4. This completes the proof. □

Example 7.5. *The (SSE)* $(w_0)_\Delta + c_x = c$ *has no solution.*

Example 7.6. *The solutions of the (SSE)* $(w_0)_\Delta + s_x = s_{(n)_n}$ *are determined by* $K_1 n \leq x_n \leq K_2 n$ *for all* n *and for some* $K_1,\ K_2 > 0$.

7.1.4. *Solvability of the (SSE) with operator* $\Lambda_\Delta + F_x = F_b$.

Proposition 7.4. *Let* $b \in U^+$. *For* $F = c$, *or* s_1 *we have*

$$S(\Lambda_\Delta, F) = S(\Lambda, F) = \begin{cases} cl^F(b) & \text{if } 1/b \in \mathbf{\Gamma}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. By Lemma 4.2 we have $\Delta \in (\Lambda, \Lambda)$ bijective since $e \in \widehat{C}_\Lambda$. Indeed, we have $n \leq K^n$ for all n and for some $K > 1$. So we have $\Lambda_\Delta = \Lambda$, and by Lemma 4.1 we have $M(\Lambda_\Delta, F) = M(\Lambda, F) = \mathbf{\Gamma}$ for $F = c_0$, c, or s_1 and we may apply Lemma 6.4. This concludes the proof. □

7.2. **Solvability of (SSE) of the form** $E_\Sigma + F_x = F_b$**.** In this subsection we solve the (SSE) defined by $E_\Sigma + F_x = F_b$, where $E = c$, c_0, w_0, $\mathbf{\Lambda}$, Γ, or ℓ_p, $(p > 1)$, and $F = c$, or ℓ_∞, and the (SSE) $\Gamma_\Sigma + \mathbf{\Lambda}_x = \mathbf{\Lambda}_b$ and $(\ell_\infty)_\Sigma + c_x = c_b$.

7.2.1. *The (SSE) using the sets* cs, bs, cs_0, *or* $(\ell_p)_\Sigma$. In this subsection we deal with the (SSE) defined by $\chi + F_x = F_b$ where $\chi = cs$, bs, or cs_0, and by $(\ell_p)_\Sigma + F_x = F_b$, and $F = c$, or ℓ_∞. For instance, x is a solution of the (SSE) $cs + c_x = c_b$ if the next statement holds. For every $y \in \omega$ we have $y_n/b_n \to l_1$ $(n \to \infty)$ if and only if there are u, $v \in \omega$ with $y = u + v$ and the series $\sum_{k=1}^{\infty} u_k$ is convergent and $v_n/x_n \to l_2$ $(n \to \infty)$ for some scalars l_1 and l_2.

Proposition 7.5. *Let* $b \in U^+$. *Then*

i) we have

$$S(bs, c) = S(\ell_\infty, c) = \begin{cases} cl^c(b) & \text{if } 1/b \in c_0, \\ \varnothing & \text{otherwise.} \end{cases}$$

ii) For $F = c$, or ℓ_∞ we have

$$S(cs, F) = S(cs_0, F) = S\left((\ell_p)_\Sigma, F\right) = S(c_0, F),$$

with ($p \geq 1$)*, and*

$$S(c_0, F) = \begin{cases} cl^F(b) & \text{if } 1/b \in s_1, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. i) We have $\alpha \in M(bs, c)$ if and only if $D_\alpha \in (\ell_\infty(\Sigma), c)$ and $D_\alpha \Delta \in (\ell_\infty, c)$. The matrix $D_\alpha \Delta$ is the triangle defined by $(D_\alpha \Delta)_{nn} = -(D_\alpha \Delta)_{n,n-1} = \alpha_n$ for all n, with the convention $(D_\alpha \Delta)_{1,0} = 0$, the other entries being equal to zero. Trivially we have $\lim_{n \to \infty} (D_\alpha \Delta)_{nk} = 0$ for all k and by Lemma 3.5 we have $\lim_{n \to \infty} \sum_{k=1}^\infty |(D_\alpha \Delta)_{nk}| = 0$ which implies $M(bs, c) = c_0$. Since $bs = \ell_\infty(\Sigma) \subset \ell_\infty$, we conclude

$$c_0 = M(\ell_\infty, c_0) \subset M(bs, c_0) \subset M(bs, c) = c_0,$$

and Proposition 6.1 and Lemma 6.4 can be applied.

ii) Case of $S(cs_0, F)$. Since $c_0 \subset c \subset \ell_\infty$ and $cs_0 = (c_0)_\Sigma \subset c_0$ we obtain

$$s_1 = M(c_0, c_0) \subset M(cs_0, c_0) \subset M(cs_0, c) \subset M(cs_0, \ell_\infty). \tag{7.6}$$

Now $\alpha \in M(cs_0, \ell_\infty)$ if and only if $D_\alpha \in (cs_0, \ell_\infty)$ and $D_\alpha \Delta \in (c_0, \ell_\infty)$. Since $(c_0, \ell_\infty) = S_1$, we have $|\alpha_n| + |\alpha_{n-1}| \leq K$ for all n and for some $K > 0$, and $\alpha \in s_1$. So $M(cs_0, \ell_\infty) = s_1$. Using (7.6) we conclude $M(cs_0, F) = s_1$, for $F = c_0$, c, or ℓ_∞, and Proposition 6.1 and Lemma 6.4 can be applied. This completes the proof of i). Case of $S(cs, F)$. By similar arguments that above and noticing that $cs = c_\Sigma$, we obtain

$$s_1 = M(cs, c_0) \subset M(cs, c) \subset M(cs, \ell_\infty) = s_1. \tag{7.7}$$

Case of $S\left((\ell_p)_\Sigma, F\right)$. Let $p > 1$. We have $\alpha \in M\left((\ell_p)_\Sigma, \ell_\infty\right)$ implies $D_\alpha \Delta \in (\ell_p, \ell_\infty)$ and by Lemma 3.6 we have $|\alpha_n|^q = O(1)$ $(n \to \infty)$ and $\alpha \in s_1$. This means

$M\left((\ell_p)_\Sigma, \ell_\infty\right) \subset s_1$. We have $(\ell_p)_\Sigma \subset \ell_p$ since $\Delta \in (\ell_p, \ell_p)$ and

$$s_1 = M(\ell_p, c_0) \subset M\left((\ell_p)_\Sigma, c_0\right) \subset M\left((\ell_p)_\Sigma, c\right) \subset M\left((\ell_p)_\Sigma, \ell_\infty\right) \subset s_1.$$

So Proposition 6.1 and Lemma 6.4 can be applied. In the case $p = 1$, reasoning as above and using the characterizations of (ℓ_1, ℓ_∞) and (ℓ_1, c_0) given in Lemma 3.6 we obtain $M((\ell_1)_\Sigma, F) = s_1$ where $F = c_0$, c, or ℓ_∞. This concludes the proof of ii). $\square$

7.2.2. *Solvability of the (SSE)* $(w_0)_\Sigma + F_x = F_b$. Here we solve the (SSE) with operator defined by $(w_0)_\Sigma + s_x = s_b$ and $(w_0)_\Sigma + c_x = c_b$. Note that x is a solution of the second (SSE) if for every $y \in \omega$ we have $y_n/b_n \to l_1$ $(n \to \infty)$ if and only if there are u, $v \in \omega$ such that $y = u + v$ and

$$\frac{1}{n}\sum_{k=1}^{\infty}\left|\sum_{i=1}^{k} u_i\right| \to 0 \text{ and } \frac{v_n}{x_n} \to l_2 \ \ (n \to \infty) \text{ for some scalars } l_1 \text{ and } l_2.$$

First we state a lemma.

Lemma 7.1. *We have* $M((w_0)_\Sigma, \ell_\infty) = M((w_0)_\Sigma, c_0) = M((w_\infty)_\Sigma, \ell_\infty) = s_{(1/n)_n}$.

Proof. We have $M((w_0)_\Sigma, c_0) = M((w_0)_\Sigma, \ell_\infty)$. Indeed, we have $\alpha \in M((w_0)_\Sigma, c_0)$ if and only if $D_\alpha \Delta \in (w_0, c_0)$, but by Lemma 5.1 we have $(w_0, c_0) = (w_0, \ell_\infty)$, so we have $\alpha \in M((w_0)_\Sigma, c_0)$ if and only if $\alpha \in M((w_0)_\Sigma, \ell_\infty)$ and $M((w_0)_\Sigma, c_0) = M((w_0)_\Sigma, \ell_\infty)$. Now we show $M((w_\infty)_\Sigma, \ell_\infty) = s_{(1/n)_n}$. For this let $\alpha \in M((w_\infty)_\Sigma, \ell_\infty)$. Then we have $D_\alpha \Delta \in (w_\infty, \ell_\infty)$. If we define the integer ν_n by $2^{\nu_n} \leq n \leq 2^{\nu_n + 1} - 1$, we obtain

$$\sigma_n = \sum_{\nu=0}^{\infty} 2^\nu \max_{2^\nu \leq k \leq 2^{\nu+1}-1} |(D_\alpha \Delta)_{nk}| \geq |\alpha_n| \, 2^{\nu_n} \geq \frac{n+1}{2} |\alpha_n|.$$

But by Lemma 5.1 we have $D_\alpha \Delta \in (w_\infty, \ell_\infty)$ implies $\sigma \in \ell_\infty$ and $\alpha \in s_{(1/n)_n}$. So we have shown $M((w_\infty)_\Sigma, \ell_\infty) \subset s_{(1/n)_n}$. Conversely, show $s_{(1/n)_n} \subset M((w_\infty)_\Sigma, \ell_\infty)$. We have $w_\infty \subset s_{(n)_n}$ and since $\Delta \in \left(s_{(n)_n}, s_{(n)_n}\right)$ we obtain $\left(s_{(n)_n}\right)_\Sigma \subset s_{(n)_n}$ and

$$M((w_\infty)_\Sigma, \ell_\infty) \supset M\left(\left(s_{(n)_n}\right)_\Sigma, \ell_\infty\right) \supset M\left(s_{(n)_n}, \ell_\infty\right) = s_{(1/n)_n}.$$

We conclude $M\left((w_\infty)_\Sigma, \ell_\infty\right) = s_{(1/n)_n}$. We obtain $M\left((w_0)_\Sigma, c_0\right) = s_{(1/n)_n}$ using a similar arguments. This concludes the proof. □

Proposition 7.6. *Let $b \in U^+$ and let $F = c$, or ℓ_∞. Then we have*

$$S\left((w_0)_\Sigma, F\right) = S\left(w_0, F\right) = \begin{cases} cl^F(b) & \text{if } 1/b \in s_{(1/n)_n}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. We have $(w_0)_\Sigma \subset w_0$ implies $M\left(w_0, c_0\right) \subset M\left((w_0)_\Sigma, c_0\right)$ and by Lemma 5.2 and Lemma 7.1 we obtain

$$s_{(1/n)_n} = M\left(w_0, c_0\right) \subset M\left((w_0)_\Sigma, c_0\right) \subset M\left((w_0)_\Sigma, c\right) \subset M\left((w_0)_\Sigma, \ell_\infty\right) = s_{(1/n)_n}.$$

We then have $M\left((w_0)_\Sigma, F\right) = s_{(1/n)_n}$ for $F = c_0$, c, or ℓ_∞, and Proposition 6.1 and Lemma 6.4 can be applied. □

Remark 4. From Proposition 7.1, Proposition 7.3, and Proposition 7.6 we have $S\left(\chi, F\right) = S\left(w_0, F\right)$ for $\chi = (w_0)_\Sigma$, $(w_0)_\Delta$, or $(c_0)_\Delta$.

7.2.3. *Solvability of the (SSE) $\Gamma_\Sigma + F_x = F_b$.* We deal with the (SSE) $\Gamma_\Sigma + F_x = F_b$ where $F = c$, ℓ_∞, or Λ, and the (SSE) $\mathbf{\Lambda}_\Sigma + F_x = F_b$ for $F = c$, or ℓ_∞. A positive sequence x is a solution of the (SSE) $\Gamma_\Sigma + c_x = c_b$ if the next statement holds: $\lim_{n\to\infty} y_n/b_n = l$ if and only if there are two sequences u, v with $y = u + v$ such that $\lim_{n\to\infty} \left|\sum_{k=1}^n u_k\right|^{1/n} = 0$ and $\lim_{n\to\infty} v_n/x_n = l'$ for some scalars l and l' and for all $y \in \omega$. We obtain the next result.

Theorem 7.1. *Let $b \in U^+$. Then*

i) for $F = c$, ℓ_∞, or $\mathbf{\Lambda}$ we have

$$S\left(\mathbf{\Gamma}_\Sigma, F\right) = S\left(\mathbf{\Gamma}, F\right) = \begin{cases} cl^F(b) & \text{if } 1/b \in \mathbf{\Lambda}, \\ \varnothing & \text{otherwise.} \end{cases}$$

ii) for $F = c$, or ℓ_∞ we have

$$S(\mathbf{\Lambda}_\Sigma, F) = S(\mathbf{\Lambda}, F) = \begin{cases} cl^F(b) & \text{if } 1/b \in \mathbf{\Gamma}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. i) First let $\alpha \in M(\mathbf{\Gamma}_\Sigma, \mathbf{\Lambda})$. Then we have $D_\alpha \Delta \in (\Gamma, \mathbf{\Lambda})$, and by Lemma 4.3 we obtain

$$\left[|\alpha_n| \left(M^{-n} + M^{-n+1}\right)\right]^{\frac{1}{n}} \leq K \text{ for all } n \text{ and for some } K > 0 \text{ and } M \geq 2.$$

This implies

$$|\alpha_n|^{\frac{1}{n}} \leq \frac{KM}{(1+M)^{\frac{1}{n}}} \leq K' \text{ for all } n \text{ and for some } K' > 0.$$

We conclude $M(\mathbf{\Gamma}_\Sigma, \mathbf{\Lambda}) \subset \mathbf{\Lambda}$. Then it can easily be seen that $\mathbf{\Gamma}_\Sigma \subset \mathbf{\Gamma}$ since $\Delta \in (\mathbf{\Gamma}, \mathbf{\Gamma})$. Then by Lemma 4.1 we have

$$\mathbf{\Lambda} = M(\mathbf{\Gamma}, c_0) \subset M(\mathbf{\Gamma}_\Sigma, c_0) \subset M(\mathbf{\Gamma}_\Sigma, c) \subset M(\mathbf{\Gamma}_\Sigma, \ell_\infty) \subset M(\mathbf{\Gamma}_\Sigma, \mathbf{\Lambda}) \subset \mathbf{\Lambda}.$$

So we obtain $M(\mathbf{\Gamma}_\Sigma, F) = \mathbf{\Lambda}$ for $F = c_0$, c, ℓ_∞, or Λ and Proposition 6.1 and Lemma 6.4 can be applied. This concludes the proof of i).

ii) As we have seen in Proposition 7.4 the operator $\Delta \in (\mathbf{\Lambda}, \mathbf{\Lambda})$ is bijective and it is the same for $\Sigma \in (\mathbf{\Lambda}, \mathbf{\Lambda})$. Then we have $\mathbf{\Lambda}_\Sigma = \mathbf{\Lambda}$ and by Lemma 6.4 we conclude ii) holds. □

Example 7.7. *The solutions of the (SSE) $\mathbf{\Gamma}_\Sigma + \mathbf{\Lambda}_x = \mathbf{\Lambda}_u$ with $u > 0$ are determined by $k_1^n \leq x_n \leq k_2^n$ for all n and for some k_1, $k_2 > 0$.*

Example 7.8. *Each of the (SSE) $\mathbf{\Lambda}_\Sigma + F_x = F_u$ where $F = c$, or s_1 has no solution for any given $u > 0$.*

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