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RADIO MEAN NUMBER OF SOME WHEEL RELATED GRAPHS

R. PONRAJ [(1)], S. SATHISH NARAYANAN [(2)] AND R. KALA [(3)]

Abstract. A Radio Mean labeling of a connected graph G is an injective map f from the vertex set $V(G)$ to the set of natural numbers N such that for each distinct vertices u and v of G, $d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \text{diam }(G)$. The radio mean number of f, $rmn(f)$, is the maximum number assigned to any vertex of G under the labeling f. The radio mean number of G, $rmn(G)$ is the minimum value of $rmn(f)$ taken over all radio mean labelings f of G. Here we have found the radio mean number of some graphs derived from wheels.

1. Introduction

In this paper we considered finite, simple undirected and connected graphs only. $V(G)$ and $E(G)$ respectively denote the vertex set and edge set of G. Also, for a graph G, p and q denotes the number of vertices and edges respectively. The concept of radio labeling was introduced by Chatrand et al. [1] in 2001. Radio labeling is used for channel assignment problem [4]. Several authors [2, 3, 5, 6] found the radio number of numerous graphs. Motivated by the radio labeling, Ponraj et al. [8] defined the concept of radio mean labeling. A radio mean labeling is a one to one mapping

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f from $V(G)$ to N satisfying the condition

$$d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \text{diam } (G) \tag{1.1}$$

for every $u, v \in V(G)$. The span of a labeling f is the maximum integer that f maps to a vertex of graph G. The radio mean number of G, $rmn(G)$ is the lowest span taken over all radio mean labelings of the graph G. The condition (1.1) is called the radio mean condition. In [8, 9], we determined the radio mean number of some graphs like graphs with diameter three, lotus inside a circle, gear graph, Helms, Sunflower graphs, subdivision of a star, wheel, bistar, $K_2 + mK_1$. In this paper we find the radio mean number of $W_n \odot P_m$, $W_{m,n}$, $sp(W_n)$ which will be defined in the subsequent section. Let x be any real number. Then $\lceil x \rceil$ stands for smallest integer greater than or equal to x. Terms and definitions not defined here follow from Harary [7].

2. MAIN RESULTS

Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their *join* $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1 \text{ and } v \in V_2\}$.

Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

First we look into the graph $W_n \odot P_m$. A vertex of degree 3 is called rim vertex. A vertex which is adjacent to all the rim vertices is called the central vertex. The edges with one end incident with the rim and the other incident with the central vertex are called spokes. Let C_n be the cycle $u_1u_2 \ldots u_nu_1$ and $W_n = C_n + K_1$ where $V(K_1) = \{u\}$. Let $P_m^i : v_1^i v_2^i \ldots v_m^i$ $(1 \leq i \leq n+1)$ be a path. Let $V(W_n \odot P_m) = V(W_n) \cup (\bigcup_{i=1}^{n} V(P_m^i))$ and $E(W_n \odot P_m) = E(W_n) \cup (\bigcup_{i=1}^{n} E(P_m^i)) \cup \{u_i v_j^i : 1 \leq j \leq m, 1 \leq i \leq n\} \cup \{v_j^{n+1} u : 1 \leq j \leq m\}$.

Theorem 2.1. *$rmn(W_n \odot P_m) = (m+1)(n+1)$, $n > 3$ and $m \in \mathbb{Z}^+$.*

Proof. For every $n \geq 4$, the longest path of $W_n \odot P_m$ is $v_i^k u_k u u_t v_j^t$ where $t \notin \{k-1, k, k+1\}$ and hence $diam(W_n \odot P_m) = 4$. We describe a vertex labeling f of $W_n \odot P_m$ as follows. First we consider the path vertices. Assign the label 1 to v_1^1, 2 to v_1^2 and 3 to v_1^3 and in general i to v_1^i $(1 \leq i \leq n)$. Also assign the number $n+1$ to v_1^{n+1}. Note that v_1^{n+1} is a neighbour of the central vertex and it belongs to $V(P_m^{n+1})$. Next we move to the vertices v_2^i $(1 \leq i \leq n+1)$. Put the label $n+1+i$ $(1 \leq i \leq n+1)$ to v_2^i. It is clear that v_2^{n+1} is labeled by $2n+2$. Continue this process until we label all the path vertices P_m^i. Note that the vertices v_m^i received the label as follows.

v_m^1	v_m^2	$\ldots\ldots$	v_m^{n+1}
$mn+m-n$	$mn+m-n+1$	$\ldots\ldots$	$mn+m$

Now we move to the rim vertices. Assign the label $mn+m+1$ to u_1 and then put the label $mn+m+2$ to u_2 and in general $mn+m+i$ to u_i $(1 \leq i \leq n)$. Finally assign the label $(m+1)(n+1)$ to the central vertex. Next we check the radio mean condition for f.

Case 1. Check the pair (v_i^j, v_k^j).

Subcase 1. $j = 1$, $i = 1$ and $k = 2$.

$$d(v_1^1, v_2^1) + \left\lceil \frac{f(v_1^1) + f(v_2^1)}{2} \right\rceil \geq 1 + \left\lceil \frac{1+n+2}{2} \right\rceil \geq 5.$$

Subcase 2. $j \neq 1$ or $i \neq 1$ or $k \neq 2$.

$$d(v_i^j, v_k^j) + \left\lceil \frac{f(v_i^j) + f(v_k^j)}{2} \right\rceil \geq 1 + \left\lceil \frac{1+n+2}{2} \right\rceil \geq 5.$$

Case 2. Check the pair (v_i^j, v_k^r), $j \neq r$.

$$d(v_i^j, v_k^r) + \left\lceil \frac{f(v_i^j) + f(v_k^r)}{2} \right\rceil \geq 3 + \left\lceil \frac{1+2}{2} \right\rceil \geq 5.$$

Case 3. Consider the pair (u_i, v_j^k), $i \neq j$.

Subcase 1. $i = 1$.

$$d(u_1, v_j^k) + \left\lceil \frac{f(u_1) + f(v_j^k)}{2} \right\rceil \geq 1 + \left\lceil \frac{mn + m + 1 + n + 2}{2} \right\rceil \geq 7.$$

Subcase 2. $i \neq 1$.

$$d(u_i, v_j^k) + \left\lceil \frac{f(u_i) + f(v_j^k)}{2} \right\rceil \geq 1 + \left\lceil \frac{mn + m + 2 + 1}{2} \right\rceil \geq 5.$$

Case 4. Consider the pair (u_i, v_i^k).

$$d(u_i, v_i^k) + \left\lceil \frac{f(u_i) + f(v_i^k)}{2} \right\rceil \geq 1 + \left\lceil \frac{mn + m + 1 + 1}{2} \right\rceil \geq 5.$$

Case 5. Consider the pair (u_i, u_j).

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{mn + m + 1 + mn + m + 2}{2} \right\rceil \geq 7.$$

Case 6. Consider the pair (u, v_i^j).

$$d(u, v_i^j) + \left\lceil \frac{f(u + f(v_i^j)}{2} \right\rceil \geq 1 + \left\lceil \frac{(m+1)(n+1) + 1}{2} \right\rceil \geq 6.$$

Case 7. Consider the pair (u, u_i).

$$d(u, u_i) + \left\lceil \frac{f(u) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{(m+1)(n+1) + mn + m + 1}{2} \right\rceil \geq 8.$$

Therefore $rmn(W_n \odot P_m) \leq (m+1)(n+1)$. Since f is injective, $rmn(f) \geq (m+1)(n+1)$ for all radio mean labelings f and hence $rmn(W_n \odot P_m) = (m+1)(n+1)$. □

The graph $W_{m,n}$ is obtained from the wheels W_m and W_n by joining the rim vertices of the two wheels with an edge. Let $W_m = C_m + K_1$ and $W_n = C_n + K_1$ where C_m is the cycle $u_1u_2 \ldots u_mu_1$ and C_n is the cycle $v_1v_2 \ldots v_nv_1$. Let $V(W_m) = V(C_m) \cup \{u\}$, $V(W_n) = V(C_n) \cup \{v\}$ and $E(W_{m,n}) = E(W_m) \cup E(W_n) \cup \{u_1v_1\}$.

Theorem 2.2.

$$rmn\,(W_{m,n}) = \begin{cases} 10 & \text{if} \;\; m = 3,\; n = 4 \\ m+n+2 & \text{if} \;\; m = 3,\; n \neq 4 \\ m+n+3 & \text{if} \;\; m > 3,\; n > 3 \end{cases}$$

Proof. **Case 1.** $m = 3,\; n = 4$.
From Figure 1 it is easy to verify that $rmn(W_{3,4}) \leq 10$.

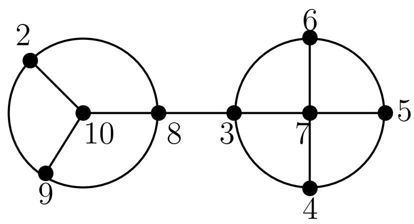

Figure 1

Claim. $rmn(W_{3,4}) > 9$.
Since the mean value of 1 and 2 is 2, they should be the labels of the vertices in the different wheels. If 1 is the label of vertices of W_3 then 2 is the label of vertices of W_4.

Subcase 1. $f(u_2) = 1$ and $f(v_2) = 2$.
Clearly v_4 should be labeled by 3 or more. Suppose $f(v_4) \geq 3$ and $f(v_4) \neq 4$. Let x be a vertex with label 4. Then $d(u_2, x) + \left\lceil \frac{f(u_2)+f(x)}{2} \right\rceil$ or $d(v_2, x) + \left\lceil \frac{f(v_2)+f(x)}{2} \right\rceil$ is less than or equal to 4. This implies 4 can not be the label of any of the vertices. Suppose $f(v_4) = 4$ and let x be a vertex with label 3 then $d(u_2, x) + \left\lceil \frac{f(u_2)+f(x)}{2} \right\rceil$ or

$d(v_2, x) + \left\lceil \frac{f(v_2)+f(x)}{2} \right\rceil$ is less than or equal to 4. Thus the radio mean condition is not satisfied.

Subcase 2. $f(u_2) = 1$ and $f(v_3) = 2$.

Let x be a vertex with label 3. Then $d(u_2, x) + \left\lceil \frac{f(u_2)+f(x)}{2} \right\rceil$ or $d(v_3, x) + \left\lceil \frac{f(v_3)+f(x)}{2} \right\rceil$ is less than or equal to 4. Hence 3 can not be a label of any of the vertices because the radio mean condition is not satisfied.

Subcase 3. $f(u_2) = 1$ and $f(v) = 2$.

Similar to subcase 2, 3 can not be a label of any of the vertices because the radio mean condition is not satisfied.

Subcase 4. $f(u_2) = 1$ and $f(v_1) = 2$.

This is not possible because the pair (u_2, v_1) does not satisfies the radio mean condition.

Subcase 5. $f(u) = 1$ and $f(u_3) = 1$.

Then the pair (u, u_3) does not satisfies the radio mean condition and hence this is impossible.

Subcase 6. $f(u_1) = 1$.

Then v_3 should be labeled by 2 or more. Suppose $f(v_3) \geq 2$ and $f(v_3) \neq 3$. Let x be a vertex with label 3. Then $d(u_1, x) + \left\lceil \frac{f(u_1)+f(x)}{2} \right\rceil$ is less than or equal to 4. This implies 3 can not be the label of any of the vertices. Suppose $f(v_3) = 3$ and let x be a vertex with label 2 then $d(u_1, x) + \left\lceil \frac{f(u_1)+f(x)}{2} \right\rceil$ is less than or equal to 4. Thus the radio mean condition is not satisfied. Hence $rmn(W_{3,4}) = 10$.

Case 2. $m = 3$, $n \neq 4$.

Subcase 1. $m = 3$, $n = 3$.

$rmn(W_{3,3}) \leq 8$ follows from Figure 2. Since f is an injective map, $rmn(f) \geq 8$ for all radio mean labelings f and hence $rmn(W_{3,3}) = 8$.

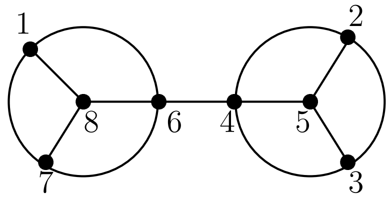

Figure 2

Subcase 2. $m = 3$, $n > 4$.

Claim. $rmn(W_{3,n}) = n + 5$, $n > 4$.

First consider the vertices of W_n. Assign the label 2 to the vertex v_2 and the label $i - 1$ to the vertex v_i $(5 \le i \le n)$. Assign the label n, $n + 1$, $n + 2$ to the vertices v_1, v_3, v respectively. Then we move to the wheel W_3. Assign the label 1, $n + 3$, $n + 4$, $n + 5$ to the vertices u_3, u_2, u_1, u respectively. We now check the radio mean condition of the above labeling f.

Case a. Check the pair (v_i, v_j).

Since $d(v_2, v_4) = d(v_2, v_5) = 2$, the pairs (v_2, v_4), (v_2, v_5) satisfy the radio mean condition. Let us check the other pairs (v_i, v_j).

Subcase 1. Check the pair (v_2, v_j).

$$d\left(v_2, v_j\right) + \left\lceil \frac{f\left(v_2\right) + f\left(v_j\right)}{2} \right\rceil \ge 1 + \left\lceil \frac{2+5}{2} \right\rceil \ge 5.$$

Subcase 2. Check the pair (v_i, v_j), $i, j \ne 2$.

$$d\left(v_i, v_j\right) + \left\lceil \frac{f\left(v_i\right) + f\left(v_j\right)}{2} \right\rceil \ge 1 + \left\lceil \frac{3+4}{2} \right\rceil \ge 5.$$

Case b. Verify the pair (v, v_i).

$$d(v, v_i) + \left\lceil \frac{f(v) + f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{n + 2 + 2}{2} \right\rceil \geq 6.$$

Case c. Consider the pairs (u_i, u_j) and (u, u_i).

Obviously the pairs satisfy the radio mean condition.

Case d. Verify the pair (u, v_i).

$$d(u, v_i) + \left\lceil \frac{f(u) + f(v_i)}{2} \right\rceil \geq 2 + \left\lceil \frac{n + 5 + 2}{2} \right\rceil \geq 7.$$

Case e. Consider the pair (u_i, v_j).

Subcase 1. Consider (u_3, v_2) and (u_3, v_4).

It is easy to check that the pairs satisfy our requirements.

Subcase 2. Verify the pair (u_3, v_j), $j \neq 2, 4$.

$$d(u_3, v_j) + \left\lceil \frac{f(u_3) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{1 + 4}{2} \right\rceil \geq 5.$$

Subcase 3. Examine the pair (u_i, v_j), $i \neq 3$ and i, j are not simultaneously 1.

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{n + 3 + 2}{2} \right\rceil \geq 6.$$

Subcase 4. Examine the pair (u_1, v_1).

$$d(u_1, v_1) + \left\lceil \frac{f(u_1) + f(v_1)}{2} \right\rceil \geq 1 + \left\lceil \frac{n + n + 4}{2} \right\rceil \geq 8.$$

Hence $rmn(W_{3,n}) \leq n + 5$. Since f is injective, $rmn(W_{3,n}) \geq n + 5$. Thus $rmn(W_{3,n}) = n + 5$.

Case 3. $m > 3$, $n > 3$.

Claim. $rmn(W_{m,n}) > m + n + 2$.

In this case $diam\,(W_{m,n}) = 5$. It follows that 1 and 2 can not be labels of the same wheel. If it is so 3 can not be a label of any of the vertices. This implies $rmn\,(W_{m,n}) > m+n+2$.

Claim. $rmn(W_{m,n}) \leq m+n+3$.

Consider the wheel W_n. Assign the label 3 to the vertex v_3 and 4 to the vertex v_1. Put the label $i+2$ to v_i $(4 \leq i \leq n)$. Finally assign the labels $n+3$, $n+4$ to the vertices v_2 and v respectively. Next we move to the vertices of the wheel W_m. Put the labels 2, 5 to the vertices u_3, u_1 respectively. Then the remaining vertices are labeled with the integers from $\{n+5, n+6, \ldots, m+n+3\}$ in any order. We check the radio mean condition

$$d\,(u,v) + \left\lceil \frac{f\,(u) + f\,(v)}{2} \right\rceil \geq 6$$

holds for every pair (u,v) with $u \neq v$.

Case a. Examine the pair (u, u_i).

$$d\,(u,u_i) + \left\lceil \frac{f(u) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{n+5+2}{2} \right\rceil \geq 7.$$

Case b. Examine the pair (u_i, u_j).

Clearly (u_3, u_1) satisfies the radio mean condition.

Subcase 1. Examine the pair (u_3, u_i), $i \neq 1$.

$$d\,(u_3,u_i) + \left\lceil \frac{f(u_3) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2+n+5}{2} \right\rceil \geq 7.$$

Subcase 2. Examine the pair (u_1, u_i), $i \neq 3$.

$$d\left(u_1, u_i\right) + \left\lceil \frac{f(u_1) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{5 + n + 5}{2} \right\rceil \geq 8.$$

Subcase 3. Check the pair (u_i, u_j), $i, j \notin \{1, 3\}$.

$$d\left(u_i, u_j\right) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{n + 5 + n + 6}{2} \right\rceil \geq 11.$$

Case c. Consider the pair (u_i, v_j).

Subcase 1. Verify the pair (u_3, v_j).

Clearly (u_3, v_3) satisfies the radio mean condition. So take $j \neq 3$.

$$d\left(u_3, v_j\right) + \left\lceil \frac{f(u_3) + f(v_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{2 + 4}{2} \right\rceil \geq 6.$$

Subcase 2. Examine the pair (u_1, v_j).

Since $d(u_1, v_3) = 3$, (u_1, v_3) satisfies the radio mean condition. Consider (u_1, v_j) with $j \neq 3$.

$$d\left(u_1, v_j\right) + \left\lceil \frac{f(u_1) + f(v_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{5 + 4}{2} \right\rceil \geq 6.$$

Subcase 3. Check the pair (u_i, v_j), $i \neq 1, 3$.

$$d\left(u_i, v_j\right) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{n + 5 + 3}{2} \right\rceil \geq 8.$$

Case d. Consider the pair (u, v_j).

Similar to sub case 3 of case c.

Case e. Consider the pair (u_i, v).

$$d\left(u_i, v\right) + \left\lceil \frac{f(u_i) + f(v)}{2} \right\rceil \geq 2 + \left\lceil \frac{2 + n + 4}{2} \right\rceil \geq 7.$$

Case f. Check the pair (u, v).

$$d\left(u,v\right)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil\geq 3+\left\lceil\frac{n+5+n+4}{2}\right\rceil\geq 12.$$

Case g. Verify the pair (v, v_i).

$$d\left(v,v_i\right)+\left\lceil\frac{f(v)+f(v_i)}{2}\right\rceil\geq 1+\left\lceil\frac{n+4+3}{2}\right\rceil\geq 7.$$

Case h. Examine the pair (v_i, v_j).

Subcase 1. Consider the pair (v_1, v_3).

It is easy to check that the pair (v_1, v_3) satisfies the radio mean condition.

Subcase 2. Verify the pair (v_3, v_j), $j \neq 1$.

$$d\left(v_3,v_j\right)+\left\lceil\frac{f(v_3)+f(v_j)}{2}\right\rceil\geq 1+\left\lceil\frac{3+6}{2}\right\rceil\geq 6.$$

Subcase 3. Examine the pair (v_1, v_j), $j \neq 3$.

$$d\left(v_1,v_j\right)+\left\lceil\frac{f(v_1)+f(v_j)}{2}\right\rceil\geq 1+\left\lceil\frac{4+6}{2}\right\rceil\geq 6.$$

Subcase 4. Check the pair (v_i, v_j), $i \neq 1, 3$ and $j \neq 1, 3$.

$$d\left(v_i,v_j\right)+\left\lceil\frac{f(v_i)+f(v_j)}{2}\right\rceil\geq 1+\left\lceil\frac{6+7}{2}\right\rceil\geq 8.$$

Therefore $rmn(W_{m,n}) \leq m + n + 3$. Hence we conclude that since $rmn(W_{m,n}) > m + n + 2$ and $rmn(W_{m,n}) \leq m + n + 3$ for this case, it follows that $rmn(W_{m,n}) = m + n + 3$. □

The next investigation is about $sp(W_n)$. The graph $sp(W_n)$ is obtained from the wheel W_n by subdividing each spokes by a vertex. Let $W_n = C_n + K_1$ where C_n :

$u_1u_2 \ldots u_nu_1$ and $V(K_1) = \{u\}$ and the spokes are subdivided by the vertices v_i $(1 \leq i \leq n)$. Note that the diameter of $sp(W_n)$ is 4.

Theorem 2.3. $rmn(sp(W_n)) = 2n + 1$.

Proof. Since $p = 2n + 1$ and the labels are unique, it follows that $rmn(sp(W_n)) \geq 2n + 1$. The vertices of $sp(W_n)$ where $3 \leq n \leq 8$ are labeled as described in Table 1 so that $rmn(sp(W_n)) \leq 2n + 1$.

n	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u
3	1	2	3						4	5	6						7
4	1	2	3	4					5	6	7	8					9
5	1	2	3	4	5				6	7	8	9	10				11
6	1	2	3	4	5	6			7	8	9	10	11	12			13
7	1	2	3	4	5	6	7		8	9	10	11	12	13	14		15
8	10	11	12	13	14	15	16	5	1	7	8	2	9	3	4	6	17

TABLE 1

Assume $n \geq 9$. Consider the rim vertices u_i $(1 \leq i \leq n)$. Put the label 1 to the vertex u_1 and 2 to the vertex u_4. Next we move to the vertex u_6. It is labeled by 3. Allocate the label 4 to the vertex u_7. Continue this process until we reach the vertex u_n. Note that u_n is labeled with $n - 3$. The unlabeled vertices u_2, u_3 and u_5 are labeled by $n - 2$, $n - 1$, n respectively. The remaining vertices are labeled with the integers from $\{n + 1, n + 2, \ldots, 2n, 2n + 1\}$ in any order. Now we check the radio mean condition for the above labeling f.

Case 1. Consider the pair (u_i, u_j).

Subcase a. Verify the pair (u_1, u_i).

It is easy to verify that the pairs (u_1, u_i) $(i = 4, 6, 7, 8)$. For $i \notin \{4, 6, 7, 8\}$, we have,

$$d(u_1, u_i) + \left\lceil \frac{f(u_1) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{1+6}{2} \right\rceil \geq 5.$$

Subcase b. Verify the pair (u_4, u_i).

The pairs (u_4, u_6), (u_4, u_7) satisfy the radio mean condition. For $i \neq 1, 6, 7$

$$d(u_4, u_i) + \left\lceil \frac{f(u_4) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2+5}{2} \right\rceil \geq 5.$$

Subcase c. Check the pair (u_i, u_j), $i, j \notin \{1, 4\}$.

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{3+4}{2} \right\rceil \geq 5.$$

Case 2. Check the pair (v_i, v_j).

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{n+1+n+2}{2} \right\rceil \geq 13.$$

Case 3. Check the pair (u_i, v_j).

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{1+n+1}{2} \right\rceil \geq 7.$$

Case 4. Examine the pair (v, v_i).

$$d(v, v_i) + \left\lceil \frac{f(v) + f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{n+1+n+2}{2} \right\rceil \geq 12.$$

Since the lower and upper bound for the radio mean number match, the radio mean number of $sp(W_n)$ is $2n + 1$. □

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(1, 2) Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412. India.

E-mail address: (1)ponrajmaths@gmail.com

E-mail address: (2)sathishrvss@gmail.com

(3) Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli - 627012, India.

E-mail address: karthipyi91@yahoo.co.in.