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BI-SHADOWING OF CONTRACTIVE SET-VALUED MAPPINGS WITH APPLICATION TO IFS'S: THE NON-CONVEX CASE

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Abstract. In this paper we prove that systems generated by uniformly locally contractive set-valued mappings are bi-shadowing with respect to comparison set-valued contractive mappings that have nonempty and compact (not necessarily convex) values. An application to iterated function systems (IFSs) in a complete metric space is given.

1. Introduction

The theory of shadowing for finite or infinite dimensional dynamical systems generated by single-valued mappings has been developed rapidly in the last two decades, see for example [6, 13, 14, 15, 16, 18, 20] and the references therein. Naturally this is due to the need of approximating systems numerically either using computers or arithmetic means. Shadowing plays an important role in determining the validity of computations in the sense that whether to every computed trajectory of the system

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there always exists a true trajectory nearby. Inverse shadowing also considered and studied by many authors, see [5].

In the meanwhile, the concept of bi-shadowing was introduced in [7], see also [6], [8] and [12]. It was proved in [6] that finite dimensional systems under certain assumptions, such as semi-hyperbolicity are bi-shadowing. Bi-shadowing was extended to infinite dimensional systems in [1, 2] and applied to delay differential equations of neutral type. The concept of bi-shadowing is a combination of the usual (direct) shadowing and inverse shadowing and consists of a comparison mappings that usually taken from a certain class of mappings. However, some shadowing results on set-valued systems was obtained in [10, 11]. It was also considered in [17, 19] where set-valued mappings with a stability condition are proved to satisfy the shadowing and inverse shadowing properties with an application is given to the T-flow of a differential inclusion. In [4], weak contractive mappings on compact metric spaces are shown to have the pseudo orbit tracing property and that the pseudo orbits approximate the Barnsley operator.

The purpose of the present paper is to discuss bi-shadowing for set-valued mappings and show that uniformly locally contractive set-valued mappings with compact values but not necessarily convex are bi-shadowing with respect to a class of comparison mappings consisting of set-valued contractive mappings. We apply this result to iterated function systems in a complete metric space.

The paper is organized as follows: In Section 2, we give some definitions and preliminaries needed throughout the paper. The result of bi-shadowing for set-valued contractive mappings and its proof will be given in Section 3. In Section 4, an application to the iterated function system (IFS) is given.

2. Definitions and Preliminaries

Let (X, d) be a complete metric space and let $\mathcal{P}(X)$ denote the collection of all nonempty subsets of X and $\mathcal{P}_c(X)$ the collection of all nonempty compact subsets of X. The Hausdorff separation $\mathcal{H}^*(A, B)$ of two subsets $A, B \in \mathcal{P}(X)$ is defined by

$$\mathcal{H}^*(A, B) = \sup_{x \in A} d(x, B)$$

where $d(x, B) = \inf_{y \in B} d(x, y)$, and the Hausdorff distance $\mathcal{H}(A, B)$ of $A, B \in \mathcal{P}(X)$ is defined by

$$\mathcal{H}(A, B) = \max\{\mathcal{H}^*(A, B), \mathcal{H}^*(B, A)\}.$$

It is well known that the Hausdorff distance $\mathcal{H}$ is a metric on $\mathcal{P}_c(X)$ and if (X, d) is complete then the metric space $(\mathcal{P}_c(X), \mathcal{H})$ is also complete, see [3].

We shall consider the set-valued dynamical system on X generated by a set-valued mapping $F : X \to \mathcal{P}_c(X)$ along with its iterates. A sequence $\{x_n\}_{n\in\mathbb{Z}} \subset X$ satisfying $x_{n+1} \in F(x_n)$, for $n \in \mathbb{Z}$ is called a bi-infinite trajectory of F, where $\mathbb{Z}$ denotes the set of integer numbers. While a sequence $\{y_n\}_{n\in\mathbb{Z}} \subset X$ satisfying

$$d(y_{n+1}, F(y_n)) \leq \gamma$$

for $n \in \mathbb{Z}$ and for $\gamma > 0$ is called a bi-infinite γ-pseudo-trajectory. Similarly, we define the infinite trajectory $\{x_n\}_{n\in\mathbb{Z}^+}$ (respectively, the infinite γ-pseudo-trajectory $\{y_n\}_{n\in\mathbb{Z}^+}$) of F by replacing $\mathbb{Z}$ by $\mathbb{Z}^+$ where $\mathbb{Z}^+ = \{0, 1, 2, \cdots\}$. By a trajectory (respectively, a γ-pseudo-trajectory) we mean either bi-infinite trajectory or infinite trajectory (respectively, bi-infinite γ-pseudo-trajectory or infinite γ-pseudo-trajectory) and we denote it simply by $\{x_n\}$ (respectively, $\{y_n\}$). Note that a true trajectory is also a pseudo-trajectory with $\gamma = 0$.

Definition 2.1. A set-valued mapping $F : X \to \mathcal{P}_c(X)$ is called (δ, λ)-uniformly locally contractive if there exist constants $\delta > 0$ and $0 < \lambda < 1$ such that

$$\mathcal{H}((F(x)), F(y)) \leq \lambda\, d(x, y)$$

whenever $d(x, y) < \delta$, for all $x, y \in X$.

We slightly modify the definition of bi-shadowing that was introduced in [6, 7] for single-valued mappings and consider the following definition of bi-shadowing for set-valued mappings in a metric space X.

Definition 2.2. A set-valued mapping $F : X \to \mathcal{P}_c(X)$ is said to be bi-shadowing on a subset K of X with positive parameters α and β if for any given γ-pseudo-trajectory $\mathbf{y} = \{y_n\}$ of F in the set K with $0 \leq \gamma \leq \beta$ and any comparison (δ, λ_1)-uniformly locally contractive set-valued mapping $\Phi : X \to \mathcal{P}_c(X)$ satisfying

$$\gamma + \sup_{x \in X} \mathcal{H}(F(x), \Phi(x)) \leq \beta, \tag{2.1}$$

there exists a trajectory $\mathbf{x} = \{x_n\}$ of Φ in K such that

$$d(x_n, y_n) \leq \alpha \left(\gamma + \sup_{x \in X} \mathcal{H}(F(x), \Phi(x)) \right), \qquad n \in \mathbb{Z}, \tag{2.2}$$

for all n for which x_n and y_n are defined. We also say that a system is bi-shadowing on $\mathbb{Z}$ (respectively, bi-shadowing on $\mathbb{Z}^+$), if it is bi-shadowing in the sense of bi-infinite trajectories and bi-infinite γ-pseudo-trajectories (respectively, in the sense of infinite trajectories and infinite γ-pseudo-trajectories).

It should be mentioned that contractivity of comparison mappings in the preceding definition of bi-shadowing is a restrictive requirement. Nevertheless, we shall obtain a bi-shadowing result for contractive set-valued mappings without the assumption of having convex values.

3. Bi-Shadowing of Contractive Mappings

In this section, we prove that locally contractive set-valued mappings are bi-shadowing with respect to continuous set-valued comparison mappings with compact values, but not necessarily having convex values. Recall that a set-valued mapping F defined in a metric space X has a fixed point $x \in X$ if $x \in F(x)$. We need the following fixed point theorem of [9].

Theorem 3.1. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and let*

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}$$

be the open ball with radius r and center x_0. Let $F : B_r(x_0) \to \mathcal{P}(X)$ be a set-valued mapping with closed and bounded values such that the following two conditions hold:

- **1)** *There exists $\lambda \in [0, 1)$ such that $\mathcal{H}(F(x), F(y)) \leq \lambda\, d(x, y)$, for all $x, y \in B_r(x_0)$*
- **2)** *$d(x_0, F(x_0)) \leq (1 - \lambda)r$,*

then the mapping F has a fixed point in $B_r(x_0)$.

Some parts of the proof of the following result are inspired by that of Theorem 70 in [19].

Theorem 3.2. *Let (X, d) be a complete metric space and $F : X \to \mathcal{P}_c(X)$ be a set-valued mapping which is (δ, λ)-uniformly locally contractive on a subset $K \subseteq X$. Assume that there exists a positive constant $M > 0$ such that* $\operatorname{diam} F(x) \leq M$ *for all $x \in K$. Then F is bi-shadowing on K with parameters α and β given by*

$$\alpha = \frac{2}{1 - \lambda} \quad \textit{and} \quad \beta = (1 - \lambda)\delta. \tag{3.3}$$

Proof. *We give the proof of bi-shadowing on $\mathbb{Z}$. For, let $\mathbf{y}^0 = \{y_n^0\}_{n \in \mathbb{Z}}$ be a given γ-pseudo-trajectory of the set-valued mapping F with*

$$\gamma < \frac{\delta}{2}(1 - \lambda). \tag{3.4}$$

Also, let $\Phi : X \to \mathcal{P}_c(X)$ be a comparison (δ, λ_1)-uniformly locally contractive set-valued mapping with compact values satisfying

$$\sup_{x \in X} \mathcal{H}(F(x), \Phi(x)) < \gamma. \tag{3.5}$$

It follows from the inequalities (3.4) and (3.5) that

$$\gamma + \sup_{x \in X} \mathcal{H}(F(x), \Phi(x)) \leq \beta.$$

Let

$$Z_\infty(\mathbf{y}^0) = \left\{ \mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \in X^{\mathbb{Z}} : \sup_{n \in \mathbb{Z}} d(x_n, y_n) < \infty \right\}.$$

The space $Z_\infty(\mathbf{y}^0)$ is a complete metric space with the metric d' given by

$$d'(\mathbf{x}, \mathbf{y}) = \sup_{n \in \mathbb{Z}} d(x_n, y_n),$$

for $\mathbf{x}, \mathbf{y} \in Z_\infty(\mathbf{y}^0)$. Let $A, B \subseteq Z_\infty(\mathbf{y}^0)$ and let

$$d'(A, B) := \sup_{\mathbf{x} \in A} \inf_{\mathbf{y} \in B} d'(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad d'_{\mathcal{H}}(A, B) := \max\{d'(A, B), d'(B, A)\}$$

be the Hausdorff separation and the Hausdorff distance respectively defined on non-empty subsets of $Z_\infty(\mathbf{y}^0)$. Consider the ball

$$B_{\alpha\gamma}(\mathbf{y}^0) = \{\mathbf{x} \in Z_\infty(\mathbf{y}^0) : d'(\mathbf{x}, \mathbf{y}^0) < \alpha\gamma\},$$

and let $\Psi : B_{\alpha\gamma}(\mathbf{y}^0) \to Z_\infty(\mathbf{y}^0)$ be a set-valued mapping given by

$$\Psi(\mathbf{z}) = \{\mathbf{w} : w_{n+1} \in \Phi(z_n), n \in \mathbb{Z}\},$$

for $\mathbf{z} = \{z_n\}_{n \in \mathbb{Z}} \in B_r(\mathbf{y}^0)$ and $\mathbf{w} = \{w_n\}_{n \in \mathbb{Z}} \in (\mathbb{R}^m)^{\mathbb{Z}}$.

Lemma 3.1. *Assume that there exists a positive constant $M > 0$ such that* $\operatorname{diam} F(x) \leq M$ *for all $x \in B_{\alpha\gamma}(\mathbf{y}^0)$. If $\Phi : X \to \mathcal{P}_c(X)$ is a set-valued mapping satisfying* $\sup_{x \in B_{\alpha\gamma}(\mathbf{y}^0)} \mathcal{H}(F(x), \Phi(x)) < \gamma$ *then*

$$\operatorname{diam} \Phi(x) \leq \delta(1 - \lambda) + 2M \quad \text{for all } x \in B_{\alpha\gamma}(\mathbf{y}^0).$$

Proof Let $x \in B_{\alpha\gamma}(\mathbf{y}^0)$. For $z \in F(x)$ and for any $y_1, y_2 \in \Phi(x)$ we have

$$\begin{aligned} d(y_1, y_2) &\leq d(y_1, z) + d(y_2, z) \\ &\leq d(y_1, F(x)) + d(y_2, F(x)) + 2M \\ &\leq 2 \sup_{x \in B_{\alpha\gamma}(\mathbf{y}^0)} \mathcal{H}(F(x), \Phi(x)) + 2M \\ &\leq 2\gamma + 2M < \delta(1-\lambda) + 2M. \end{aligned}$$

This ends the proof of the lemma. □

Lemma 3.2. *The mapping Ψ is well-defined and $\Psi(\mathbf{z})$ is bounded and closed subset of $Z_\infty(\mathbf{y}^0)$, for all $\mathbf{z} \in B_{\alpha\gamma}(\mathbf{y}^0)$.*

Proof. Let $\mathbf{z} \in B_{\alpha\gamma}(\mathbf{y}^0)$ and $\mathbf{w} \in \Psi(\mathbf{z})$. Boundedness of the set $\Psi(\mathbf{z})$ follows from Lemma 3.1 and the following estimates:

$$\begin{aligned} d(y^0_{n+1}, w_{n+1}) &\leq d(y^0_{n+1}, \Phi(z_n)) + \operatorname{diam} \Phi(z_n) \\ &\leq d(y^0_{n+1}, F(y^0_n)) + \mathcal{H}^*(F(y^0_n), F(z_n)) + \mathcal{H}^*(F(z_n), \Phi(z_n)) + \\ &\quad \delta(1-\lambda) + 2M \\ &\leq 2\gamma + \lambda\, d'(\mathbf{y}^0, \mathbf{z}) + \delta(1-\lambda) + 2M \\ &\leq 2\delta(1-\lambda) + \frac{2\gamma\lambda}{1-\lambda} + 2M. \end{aligned}$$

It also follows from these estimates that Ψ is well-defined. Let $\{\mathbf{w}^j\}$ be a sequence in $\Psi(\mathbf{z})$ such that $\{\mathbf{w}^j\} \to \mathbf{w}^*$ as $j \to \infty$. Fix $n \in \mathbb{Z}$. Since $\{w^j_{n+1}\} \to w^*_{n+1}$, $j \to \infty$ and $w^j_{n+1} \in \Phi(z_n)$, for $j = 1, 2, \cdots$ then $w^*_{n+1} \in \Phi(z_n)$, as the set $\Phi(z_n)$ is closed by assumption. Hence $\mathbf{w}^* \in \Psi(\mathbf{z})$ and consequently $\Psi(\mathbf{z})$ is closed for all $\mathbf{z} \in B_{\alpha\gamma}(\mathbf{y}^0)$. The lemma is proved. □

Now we show that Ψ *is a contractive mapping on* $B_{\alpha\gamma}(\mathbf{y}^0)$. *If* $\mathbf{z}' = \{z'_n\}_{n\in\mathbb{Z}}$, $\mathbf{z}'' = \{z''_n\}_{n\in\mathbb{Z}} \in B_{\alpha\gamma}(\mathbf{y}^0)$ *and if* $\mathbf{w}' = \{w'_n\}_{n\in\mathbb{Z}} \in \Psi(\mathbf{z}')$ *then*

$$\begin{aligned}
d'_{\mathcal{H}}(\Psi(\mathbf{z}'), \Psi(\mathbf{z}'')) &= \max\left\{\sup_{\mathbf{w}'\in\Psi(\mathbf{z}')} d'(\mathbf{w}', \Psi(\mathbf{z}'')), \sup_{\mathbf{w}''\in\Psi(\mathbf{z}'')} d'(\Psi(\mathbf{z}''), \mathbf{w}')\right\} \\
&\le \max\left\{\sup_{\mathbf{w}'\in\Psi(\mathbf{z}')} \sup_{n\in\mathbb{Z}} d'(w'_{n+1}, \Phi(z''_n)),\right. \\
&\qquad\qquad \left.\sup_{\mathbf{w}''\in\Psi(\mathbf{z}'')} \sup_{n\in\mathbb{Z}} d'(\Phi(z''_n), w'_{n+1})\right\} \\
&\le \max\left\{\sup_{n\in\mathbb{Z}} \mathcal{H}(\Phi(z'_n), \Phi(z''_n)), \sup_{n\in\mathbb{Z}} \mathcal{H}(\Phi(z''_n), \Phi(z'_n))\right\} \\
&\le \max\left\{\sup_{n\in\mathbb{Z}} \lambda_1 d'(z'_n, z''_n), \sup_{n\in\mathbb{Z}} \lambda_1 d'(z''_n, z'_n)\right\} = \lambda_1 d'(\mathbf{z}', \mathbf{z}'').
\end{aligned}$$

The second condition of Theorem 3.1 follows from the estimates:

$$\begin{aligned}
d'(\mathbf{y}^0, \Psi(\mathbf{y}^0)) &\le \sup_{n\in\mathbb{Z}} d(y^0_{n+1}, \Phi(y^0_n)) \\
&\le \sup_{n\in\mathbb{Z}} d(y^0_{n+1}, F(y^0_n)) + \sup_{n\in\mathbb{Z}} H^*(F(y^0_n), \Phi(y^0_n)) \\
&\le 2\gamma = (1-\lambda)\, r,
\end{aligned}$$

where $r = \alpha\gamma$. *Finally, by Theorem 3.1 the mapping* Ψ *has a fixed point* $\mathbf{x}^0 = \{x_n\}_{n\in\mathbb{Z}} \in B_{\alpha\gamma}(\mathbf{y}^0)$ *and the fixed point* $\mathbf{x}^0$ *is a trajectory of* Φ. *Moreover,*

$$\begin{aligned}
d'(\mathbf{x}^0, \mathbf{y}^0) &= \sup_{n\in\mathbb{Z}} d(x_n, y_n) \le \frac{2\gamma}{1-\lambda} \\
&\le \frac{2}{1-\lambda}\left(\gamma + \sup_{x\in X} \mathcal{H}(F(x), \Phi(x))\right) \\
&= \alpha\left(\gamma + \sup_{x\in X} \mathcal{H}(F(x), \Phi(x))\right).
\end{aligned}$$

This ends the proof of Theorem 3.2. ∎

Bi-shadowing of F *on* $\mathbb{Z}^+$ *is also possible, so we have the following result.*

Theorem 3.3. *Let (X, d) be a complete metric space and $F : X \to \mathcal{P}_c(X)$ be a set-valued mapping which is bi-shadowing on $\mathbb{Z}$ with parameters α and β and assume that*

$$\bigcup_{x \in X} F(x) = X. \tag{3.6}$$

Then F is bi-shadowing on $\mathbb{Z}^+$, with the same parameters α and β.

Proof.Given any bi-infinite γ-pseudo-trajectory $\mathbf{y} = \{y_n\}_{n\in\mathbb{Z}}$ of F in a set K with $\gamma \le \beta$ and a mapping $\Phi : X \to \mathcal{P}_c(X)$ such that

$$\gamma + \sup_{x \in X} \mathcal{H}(F(x), \Phi(x)) \le \beta. \tag{3.7}$$

By Theorem 3.2, there exists a bi-infinite trajectory $\mathbf{x} = \{x_n\}_{n\in\mathbb{Z}}$ of Φ such that $d(x_n, y_n) \le \alpha(\gamma + \sup_{x\in X} \mathcal{H}(F(x), \Phi(x)))$, for $n \in \mathbb{Z}$. *To prove bi-shadowing for infinite trajectories, let $\mathbf{y}' = \{y'_n\}_{n\in\mathbb{Z}^+}$ be a given infinite γ-pseudo-trajectory of F in K with $\gamma \le \beta$ and let $\Phi : X \to \mathcal{P}_c(X)$ be a given mapping such that the relation (3.7) holds.*

We now find an infinite trajectory $\mathbf{x}' = \{x'_n\}_{n\in\mathbb{Z}^+}$ of Φ such that

$$d(x'_n, y'_n) \le \alpha(\gamma + \sup_{x \in X} \mathcal{H}(F(x), \Phi(x))),\ n \in \mathbb{Z}^+. \tag{3.8}$$

We define a sequence $\mathbf{z}' = \{z'_n\}_{n\in\mathbb{Z}}$ by $z'_n = y'_n$ for $n \ge 0$ and using the relation (3.6) to choose $z_n \in X$ such that $z_{n+1} = F(z_n)$ for $n < 0$. Note that since $\{y'_n\}_{n\in\mathbb{Z}^+}$ is an infinite γ-pseudo-trajectory of F and since $d(z_{n+1}, F(z_n)) = 0$ for $n \le 0$ then the sequence $\mathbf{z}' = \{z'_n\}_{n\in\mathbb{Z}}$ is, in fact a bi-infinite γ-pseudo-trajectory of F and consequently by Theorem 3.2 there exists a bi-infinite trajectory $\mathbf{x}' = \{x'_n\}_{n\in\mathbb{Z}}$ of Φ such that $d({x'}_n, z'_n) \le \alpha(\gamma + \sup_{x\in X} \mathcal{H}(F(x), \Phi(x))),\ n \in \mathbb{Z}$. We also obtain that

$$d({x'}_n, z'_n) \le \alpha(\gamma + \sup_{x \in X} \mathcal{H}(F(x), \Phi(x))),\ n \in \mathbb{Z}^+.$$

So the sequence $\{x'_n\}_{n\in\mathbb{Z}^+}$ *is the infinite trajectory of* Φ *that satisfy the relation (3.8). This ends the proof of the theorem.* ∎

The condition (3.6) in the preceding theorem was used in the same context in [10], *see also* [11].

4. An Application To IFSs

Let (X,d) *be a complete metric space and consider the Hausdorff metric* $\mathcal{H}$ *on* $\mathcal{P}_c(X)$. *Let us recall the following lemma, and for the proof, see* [3].

Lemma 4.1. *Let* (X,d) *be a complete metric space. Then*

- **a)** *The metric space* $(\mathcal{P}_c(X),\mathcal{H})$ *is complete.*
- **b)** *If* $\{A_n\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $\mathcal{P}_c(X)$, *then there exists* $A\in\mathcal{P}_c(X)$ *such that* $\lim_{n\to\infty} A_n = A$. *The set* A *consists of all* $x\in X$ *for which there is a Cauchy sequence* $\{x_n\}_{n=1}^{\infty}$ *where* $x_n\in A_n$, *for all* n *and* $\lim_{n\to\infty} x_n = x$.

A mapping $f : X \to X$ *on a metric space* (X,d) *is called contractive mapping, if there exists a constant* $0\le\lambda<1$ *such that*

$$d(f(x),f(y)) \le \lambda\, d(x,y),$$

for all $x,y\in X$. *The number* λ *is called contractivity factor. The Banach Contraction Principle states that if* $f : X\to X$ *is a contractive mapping on a complete metric space* (X,d) *with contractivity factor* λ, *then* f *possesses exactly one fixed point* $x^*\in X$ *such that* $f(x^*) = x^*$. *Moreover, for any* $x\in X$ *the sequence* $\{f^n(x)\}_{n=1}^{\infty}$ *converges to* x^*, *that is* $\lim_{n\to\infty} f^n(x) = x^*$ *for each* $x\in X$. *Here we use the n-th iteration of* $f : X\to X$ *as*

$$f^0 = id \text{ and } f^{n+1} = f^n\circ f.$$

An iterated function system (IFS) consists of a complete metric space (X, d) together with a finite set of contractive mappings $f_i : X \to X$ with the corresponding contractivity factors λ_i, for $i = 1, \cdots, k$. The IFS is denoted by $\{X; f_i, i = 1, \cdots, k\}$ and its contractivity factor is $\lambda = \max\{\lambda_i : i = 1, \cdots, k\}$.

Theorem 4.1. [3] *Let $\{X; f_i, i = 1, \cdots, k\}$ be an IFS with contractivity factor*

$$\lambda = \max\{\lambda_i : i = 1, \cdots, k\}. \tag{4.9}$$

where λ_i is the contractivity factor of f_i for all $i = 1, \cdots, k$. Then the mapping $F : \mathcal{P}_c(X) \to \mathcal{P}_c(X)$ defined by

$$F(B) = \bigcup_{i=1}^{k} f_i(B) \tag{4.10}$$

for $B \in \mathcal{P}_c(X)$ is a contractive mapping on the complete metric space $(\mathcal{P}_c(X), \mathcal{H})$ with contractivity factor λ given in (4.9).

Consequently, it follows from this theorem that

$$\mathcal{H}\,(F(U), F(V)) \leq \lambda\,\mathcal{H}(U, V)$$

for every $U, V \in \mathcal{P}_c(X)$, where $\lambda = \max\{\lambda_i : i = 1, \cdots, k\}$. Note that the function F generates a dynamical system on $\mathcal{P}_c(X)$ through its iterates. The Banach Contraction Principle implies the existence of a unique fixed point $A^* \in \mathcal{P}_c(X)$ such that $A^* = F(A^*) = \bigcup_{i=1}^{k} f_i(A^*)$. This unique fixed point A^* is given by $A^* = \lim_{n\to\infty} F^n(B)$ for any $B \in \mathcal{P}_c(X)$ and is called the attractor of the IFS.

As an example, consider $X = [0, 1] \subseteq \mathbb{R}$ with the Euclidean metric and define $F : \mathcal{P}_c(X) \to \mathcal{P}_c(X)$ by

$$F(B) = f_1(B) \cup f_2(B),$$

where $f_1(x) = x/3$ *and* $f_2(x) = x/3 + 2/3$*, for* $0 \leq x \leq 1$*. Then* f_1 *and* f_2 *are contractive mappings with equal contractivity constants* $\lambda_1 = \lambda_2 = 1/3$*. The attractor* A *of this IFS is in fact the Cantor ternary subset of the interval* $[0, 1]$*.*

As a consequence of Theorem 3.2, we have the following result:

Theorem 4.2. *Let* $\{X; f_i, i = 1, \cdots, k\}$ *be an IFS with contractivity factor* $\lambda = \max\{\lambda_i : i = 1, \cdots, k\}$ *and let* $F : \mathcal{P}_c(X) \to \mathcal{P}_c(X)$ *be the mapping defined by* $F(B) = \bigcup_{i=1}^k f_i(B)$*. Assume that there exists a constant* $M > 0$ *such that* $\operatorname{diam} F(B) \leq M$*, for all* $B \in \mathcal{P}_c(X)$*. Then* F *is bi-shadowing on* $\mathbb{Z}$ *with parameters* α *and* β *given by*

$$\alpha = \frac{2}{1-\lambda} \quad \text{and} \quad \beta = (1-\lambda)\delta \tag{4.11}$$

for any fixed $\delta > 0$*.*

Proof. The proof of this theorem follows from Lemma 4.1, Theorem 4.1 and applying Theorem 3.2. ∎

Note that $F(B) = \bigcup_{i=1}^k f_i(B)$ *is not a convex set, since the union of convex sets is in general not necessarily convex. Hence, Theorem 4.2 seems convenient to apply to mappings without convex values. Indeed, we now give a result on the approximation of the unique attractor of the IFS as in* [4]*, but here we use trajectories of the comparison mapping* Φ *rather than trajectories of* F*, which is generally less restrictive than* F *itself.*

Theorem 4.3. *Let* $\{X; f_i, i = 1, \cdots, k\}$ *be an IFS with contractivity factor given in (4.9). Let also the contractive mapping* $F : \mathcal{P}_c(X) \to \mathcal{P}_c(X)$ *be defined by*

$$F(B) = \bigcup_{i=1}^{k} f_i(B)$$

and let A^* *be its unique attractor. Assume that condition (3.6) holds and that there exists a constant* $M > 0$ *such that* $\operatorname{diam} F(B) \leq M$*, for all* $B \in \mathcal{P}_c(X)$*. Then for* *every $\varepsilon > 0$ and every infinite γ-pseudo trajectory $\{B_n\}_{n=0}^{\infty}$ of F with $0 \le \gamma \le \beta$ and for every (δ, λ_1)-locally contractive mapping $\Phi : \mathcal{P}_c \to \mathcal{P}_c$ satisfying the property*

$$\gamma + \sup_{A \in \mathcal{P}_c(X)} \mathcal{H}(F(A), \Phi(A)) \le \beta,$$

there exists an infinite trajectory $\{A_n\}_{n=0}^{\infty}$ of Φ and a positive integer n_0 such that

$$\mathcal{H}(A^*, A_{n_0}) < \varepsilon.$$

Proof. Given $\varepsilon > 0$. Let $\{B_n\}_{n=0}^{\infty}$ be a γ-pseudo trajectory of F and $\Phi : \mathcal{P}_c(X) \to \mathcal{P}_c(X)$ satisfying the conditions mentioned in the theorem above. Then by Theorem 3.3 there exists a trajectory $\{A_n\}_{n=0}^{\infty}$ of Φ such that

$$\begin{aligned} \mathcal{H}(A_n, B_n) &\le \alpha(\gamma + \sup_{A \in \mathcal{P}_c(X)} \mathcal{H}(F(A), \Phi(A))) \\ &\le \alpha(\gamma + \beta - \gamma) = 2\delta. \end{aligned} \tag{4.12}$$

Moreover, since $\{B_n\}_{n=0}^{\infty}$ is a γ-pseudo trajectory of F then there exists $B \in \mathcal{P}_c(X)$ such that

$$\mathcal{H}(F^n(B), B_n) < \gamma < \beta = (1 - \lambda)\delta, \tag{4.13}$$

for every $n = 0, 1, 2, \cdots$. Since A^* is a fixed point of F then for every $B \in \mathcal{P}_c(X)$ we have

$$\lim_{n \to \infty} F^n(B) = A^*.$$

That is there exists a positive integer n_0 such that $\mathcal{H}(F^n(B), A^*) < \varepsilon/2$ for all $n \ge n_0$. In particular,

$$\mathcal{H}(F^{n_0}(B), A^*) < \varepsilon/2. \tag{4.14}$$

We now choose δ satisfying the relation

$$\delta < \frac{\varepsilon}{2(3 - \lambda)}. \tag{4.15}$$

Finally, we combine (4.12), (4.13) and (4.14) and we use (4.15) to obtain

$$\begin{aligned}\mathcal{H}(A^*, A_{n_0}) &\leq \mathcal{H}(A^*, F^{n_0}(B)) + \mathcal{H}(F^{n_0}(B), B_{n_0}) + \mathcal{H}(B_{n_0}, A_{n_0}) \\ &< \varepsilon/2 + \gamma + 2\delta < \varepsilon/2 + (1-\lambda)\delta + 2\delta \\ &= \varepsilon/2 + (3-\lambda)\delta < \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

This ends the proof of the theorem. ∎

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