

ON SOME COVERING PROPERTIES IN TERMS OF FUZZY GRILL

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ABSTRACT. In this paper our primary intension is to develop some fuzzy grill oriented compact-like notions through α -shading in a fuzzy topological space. Moreover, we characterize these ideas by means of α -finite union property, prefilterbases, prefilters and fuzzy grills via some pre-assigned fuzzy grill and the underlying fuzzy topology.

1. INTRODUCTION AND PRELIMINARIES

Compactness is known to have enormous applications in topology. After the introduction of fuzzy sets by Zadeh in [15], compactness for fuzzy sets and for fuzzy topological spaces has been considered by different mathematicians in different ways. Among these, possibly the most significant approach was initiated by Gantner et al. [8] by means of their brilliant concept of α -shading. After the introduction of grill in general topology by Choque [5] followed by that of fuzzy grill by Azad [1], these are treated as basic tools for different topological and fuzzy topological investigations. In Section 2 of this paper, we define a type of fuzzy compactness in terms of fuzzy grill and α -shading, and investigate its relation with the classical α -compactness notion

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given by Gantner et al. [8]. In Section 3, we characterize the newly defined compactness via α -finite union property, prefilters and fuzzy grills. In the last section, we consider α -almost compactness in our setting and study its variations when it is defined in terms of fuzzy grills. Moreover, we find that under slight restrictions, the aforesaid grill-oriented concepts of fuzzy α -compactness and fuzzy α -almost compactness remain invariant under fuzzy continuous functions.

A *fuzzy set* A in X , in the sense of Zadeh [15], is defined by a membership function. In fuzzy setting, basic fuzzy sets are the zero set, the whole set and the class of all fuzzy sets in X , to be denoted by 0_X , 1_X and I^X respectively (where $I = [0, 1]$). By a fuzzy topological space (X, τ) (henceforth abbreviated as an fts X), we mean a non-empty set X with the fuzzy topology τ , as given by Chang [3]. For two fuzzy sets A, B in X , we write $A \leq B$ if $A(x) \leq B(x)$ for each $x \in X$, whereas A is said to be *quasi-coincident* [14] with B , written as AqB , if $A(x) + B(x) > 1$ for some $x \in X$. The negations of these statements are denoted by $A \not\leq B$ and $A\bar{q}B$. A is called a *q-nbd of B* [14] if BqU for some fuzzy open set U in X , with $U \leq A$. For a fuzzy set A in an fts X , the *fuzzy complement*, *fuzzy interior* and *fuzzy closure* of A in X are written as $1 - A$, $intA$ and clA respectively. For a fuzzy set A in X , $suppA$ will mean the support of A given by $suppA = \{x \in X : A(x) \neq 0\}$. For a fuzzy point x_λ [i.e., a fuzzy set with the singleton support x and value λ at x], the set of all fuzzy open q -nbds of x_λ will be denoted by $Q(x_\lambda)$.

A non-void collection \mathcal{G} of fuzzy sets in an fts (X, τ) is called a fuzzy grill on X [1] if (i) $0_X \notin \mathcal{G}$, (ii) $A \in \mathcal{G}$, $B \in I^X$ and $A \leq B \Rightarrow B \in \mathcal{G}$ and (iii) $A, B \in I^X$ and $A \vee B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

In a fuzzy topological space with a fuzzy grill \mathcal{G} on X , an operator $\phi : I^X \rightarrow I^X$, denoted by $\phi_{\mathcal{G}}(A)$ or simply by $\phi(A)$ [where A is a fuzzy set in X] is defined [in [13]]

as the union of all fuzzy points x_λ of X such that if $U \in Q(x_\lambda)$, then $A * U \in \mathcal{G}$, where $A * B$ is the Lukasiewicz conjunction [12] on the power set I^X , given by $A * B = \max(0, A + B - 1_X)$, for $A, B \in I^X$, where $(A * B)(x) = A(x) + B(x) - 1$ if $A(x) + B(x) > 1$ and $(A * B)(x) = 0$ otherwise.

Throughout this paper, by a *fuzzy \mathcal{G} -space* (X, τ, \mathcal{G}) , we mean an fts (X, τ) endowed with a fuzzy grill \mathcal{G} . Some necessary definitions and results are listed below:

Definition 1.1. [13] In an fts (X, τ) , corresponding to a fuzzy grill \mathcal{G} , there exists a unique fuzzy topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \in I^X / \psi(1_X - U) = 1_X - U\}$, where for any $A \in I^X$, $\psi(A) = A \bigvee \phi(A) = \tau_{\mathcal{G}\text{-cl}}(A)$.

Theorem 1.1. [13] Let (X, τ) be an fts and \mathcal{G} be a fuzzy grill on X . Then $\mathcal{B}(\mathcal{G}, \tau) = \{V - A : V \in \tau \text{ and } A \notin \mathcal{G}\}$ is a fuzzy open base for $\tau_{\mathcal{G}}$.

Definition 1.2. Let (X, τ) be an fts and let $0 < \alpha < 1$.

- (i) A collection \mathcal{U} of fuzzy sets will be called an α -shading of X if for each $x \in X$, there exists some $U \in \mathcal{U}$ with $U(x) > \alpha$. If each member of \mathcal{U} is fuzzy open, then \mathcal{U} is called an open α -shading of X .
- (ii) A subcollection \mathcal{V} of an α -shading \mathcal{U} of X , that is also an α -shading of X is called an α -subshading of \mathcal{U} .
- (iii) (X, τ) is called α -compact if each open α -shading of X has a finite α -subshading.

2. ALPHA COMPACTNESS VIA FUZZY GRILL

As already mentioned, our concern in this section is to introduce and study a kind of fuzzy covering property for a fuzzy \mathcal{G} -space (X, τ, \mathcal{G}) , the notion being defined in terms of the idea of α -shading having suitable bearing on the grill \mathcal{G} on the underlying space.

Definition 2.1. Let (X, τ, \mathcal{G}) be a fuzzy \mathcal{G} -space and \mathcal{U} be any open α -shading of X . A finite subcollection \mathcal{U}_0 of \mathcal{U} is said to be a \mathcal{G}_α -shading of X if $1 - \bigvee \mathcal{U}_0 \notin \mathcal{G}$.

Remark 1. The above defined \mathcal{G}_α -shading of X may not be an α -subshading of a given open α -shading of X . We show it by the following example:

Example 2.1. Let X be any infinite set and suppose $\tau = \{0_X, 1_X\} \cup \{A \in I^X : 0.3 \leq A(x) < 1, x \in X\}$. Then τ is a fuzzy topology on X . Now $\mathcal{G} = \{G \in I^X : 0.7 < G(x) \leq 1, x \in X\}$ is a fuzzy grill on X . Let us take $\alpha = 0.5$. Now for each $x \in X$, let us choose $U_x \in \tau$ such that $U_x(x) = 0.6$ and $0.3 < U_x(y) < 0.5$, for all $y \in X \setminus \{x\}$. Then $\mathcal{U} = \{U_x : x \in X\}$ is an open α -shading of X . Let us take a finite subcollection $\mathcal{U}_0 = \{U_{x_1}, U_{x_2}, \dots, U_{x_n}\}$ (say) of \mathcal{U} , for any finite number of points $x_1, x_2, \dots, x_n \in X$. Then we see that although \mathcal{U}_0 is a finite subcollection of \mathcal{U} , it is not an α -subshading of \mathcal{U} . Indeed, for any $y \in X \setminus \{x_1, x_2, \dots, x_n\}$, there does not exist any $U_y \in \mathcal{U}_0$, for which $U_y(y) > 0.5$. But \mathcal{U}_0 is a \mathcal{G}_α -shading of X since $1 - \bigvee_{i=1}^n U_{x_i} \notin \mathcal{G}$ as $(1 - \bigvee_{i=1}^n U_{x_i})(x) < 0.7$, for all $x \in X$.

Definition 2.2. A fuzzy \mathcal{G} -space (X, τ, \mathcal{G}) is said to be fuzzy \mathcal{G}_α -compact if every open α -shading of X has a \mathcal{G}_α -shading.

We now give examples of fuzzy \mathcal{G}_α -compact spaces. For this we define as follows:

Definition 2.3. A fuzzy set A in X is called infinite (resp. countable, uncountable) if $\text{supp}(A)$ is infinite (resp. countable, uncountable).

Example 2.2. Let us take an uncountable set X . Consider $\tau_C = \{A \in I^X : (1_X - A) \text{ is countable}\}$ together with 0_X . Then τ_C is a fuzzy topology on X . Let \mathcal{G} be a fuzzy grill on X consisting of all uncountable fuzzy sets in X . Then (X, τ_C, \mathcal{G}) is \mathcal{G}_α -compact.

In fact, for any open α -shading $\{U_\lambda : \lambda \in \Lambda\}$ of X and for any finite subset Λ_0 of Λ , $1 - \bigvee_{\lambda \in \Lambda_0} U_\lambda = \bigwedge_{\lambda \in \Lambda_0} (1 - U_\lambda) \notin \mathcal{G}$ (since each $(1 - U_\lambda)$ is countable).

Next we give an example of a fuzzy \mathcal{G} -space (X, τ, \mathcal{G}) , for which the space $(X, \tau_{\mathcal{G}}, \mathcal{G})$ is fuzzy \mathcal{G}_α -compact.

Example 2.3. Let X be an uncountable set. Now $\tau = \{A \in I^X : (1_X - A) \text{ is of finite support in } X\}$ together with 0_X , is a fuzzy topology on X . If we consider the fuzzy grill \mathcal{G} on X consisting of all uncountable fuzzy sets in X , then we claim that $\tau_C = \tau_{\mathcal{G}}$, where τ_C is the fuzzy topology defined in Example 2.2. In fact, for any $B \in I^X$, for which $\text{supp}(1_X - B)$ is countable, $(1_X - B) = G$ (say) $\notin \mathcal{G}$. Then $B = 1_X - G$, where $1_X \in \tau$ and $G \notin \mathcal{G}$. So, $1_X - G \in \tau_{\mathcal{G}}$ (by Theorem 1.1). i.e., $\tau_C \subseteq \tau_{\mathcal{G}}$. Again, for any $\tau_{\mathcal{G}}$ -basic open set $H (\neq 0_X)$, we have $H = U - A$, where $U \in \tau$ and $A \notin \mathcal{G}$. So $(1 - U)$ is of finite support and A is countable. We want to show that $\text{supp}(1_X - H)$ is countable. In fact, for $x \in X \setminus \text{supp}(A)$, $A(x) = 0$ so that $H(x) = U(x)$ and hence $(1 - H)(x) = (1 - U)(x)$. As $\text{supp}(1 - U)$ is finite, so is $[\text{supp}(1 - H)] \cap [X \setminus \text{supp}(A)]$. Also as $\text{supp}(A)$ is countable, so is $[\text{supp}(1 - H)] \cap \text{supp}(A)$. Thus $\text{supp}(1 - H)$ is countable. Hence $H \in \tau_C$ and consequently $\tau_C = \tau_{\mathcal{G}}$. In Example 2.2, we have already seen that (X, τ_C) is \mathcal{G}_α -compact. Thus $(X, \tau_{\mathcal{G}})$ is \mathcal{G}_α -compact.

However we note at this point that α -compactness and \mathcal{G}_α -compactness are independent concepts. We show this by the following examples:

Example 2.4. Let X be an infinite set with the fuzzy topology $\tau = \{K_\beta \in I^X : K_\beta(x) = \beta, \text{ for each } x \in X; \beta \in [0, 1]\}$ and suppose $\mathcal{G} = \{G \in I^X : 0.2 \leq G(x) \leq 1; x \in X\}$. Then (X, τ, \mathcal{G}) is a fuzzy \mathcal{G} -space. Let $\alpha = 0.7$. Given any point $a \in X$ there exists $U \in \mathcal{U}$ such that $U(a) > \alpha = 0.7$ so that there exists $K_\beta \in \mathcal{U}$ such that $\beta > \alpha$, and then $\{K_\beta\}$ is a finite α -subshading of \mathcal{U} . Hence X is α -compact. Now, consider the open α -shading $\mathcal{U} = \{U_\beta : 0.6 \leq \beta \leq 0.8\}$. Then for any finite

subcollection \mathcal{U}_0 of \mathcal{U} , we clearly have $0.2 \leq 1 - \bigvee \mathcal{U}_0 \leq 0.4$ so that $1 - \bigvee \mathcal{U}_0 \in \mathcal{G}$. Thus X is not \mathcal{G}_α -compact.

Example 2.5. We consider X to be an infinite set and let us choose and fix some point p in X . Let β be a real number such that $\max(\frac{1}{2}, \alpha) \leq \beta < 1$. Let the fuzzy topology τ on X be given by $\tau = \{U \in I^X : U(p) > \beta\}$ together with 0_X , and the fuzzy grill on X be $\mathcal{G} = \{G \in I^X : 0.6 < G(x) \leq 1; x \in X\}$. Now for each $x \in X \setminus \{p\}$, we select $U_x \in \tau$ such that $U_x(x) > \alpha$ and $0 < U_x(y) < \alpha$, for each $y \in X \setminus \{p, x\}$. Then $\mathcal{U} = \{U_x : x \in X \setminus \{p\}\}$ is an open α -shading of X having no finite α -subshading. So X is not α -compact. But X is \mathcal{G}_α -compact since for any open α -shading \mathcal{V} of X and for any finite subcollection \mathcal{V}_0 of \mathcal{V} , $1 - \bigvee \mathcal{V}_0 \notin \mathcal{G}$. Indeed, for any $V \in \mathcal{V}_0$, $V(p) > \beta \Rightarrow (\bigvee \mathcal{V}_0)(p) > \max(\frac{1}{2}, \alpha) \geq \frac{1}{2} \Rightarrow 1 - (\bigvee \mathcal{V}_0)(p) < \frac{1}{2} \Rightarrow (1 - \bigvee \mathcal{V}_0) \notin \mathcal{G}$.

3. CHARACTERIZATIONS OF \mathcal{G}_α -COMPACTNESS

Under this section our aim is to achieve some characterizing conditions for \mathcal{G}_α -compactness by using different tools. To start with we first like to mention a well known property, termed α -finite union property, as follows:

Definition 3.1. [9] Let X be a non-empty set and $\alpha \in (0, 1]$. Then a family $\mathcal{U} = \{U_j : j \in \Lambda\}$ of fuzzy sets is said to have α -finite union property (α -FUP, for short), if for each finite subfamily $\mathcal{U}_0 = \{U_{j_k} : k = 1, 2, \dots, n\}$ of \mathcal{U} , $\bigvee_{k=1}^n U_{j_k}(x) \leq \alpha$ for some $x \in X$.

Definition 3.2. A fuzzy point p_λ in an fts (X, τ) is said to be an adherent point of a family \mathcal{F} of fuzzy sets, if for each q -nbd V of p_λ and for each $F \in \mathcal{F}$, VqF .

Definition 3.3. For any $\alpha \in (0, 1)$, the collection $\mathcal{G} = \{G \in I^X : G(x) \geq \alpha \text{ for at least one } x \in X\}$ is clearly a fuzzy grill on X which we call an α -grill.

Theorem 3.1. *Let (X, τ, \mathcal{G}) be a fuzzy \mathcal{G} -space where \mathcal{G} is an α -grill. Then the following are equivalent:*

- (i) *X is fuzzy \mathcal{G}_α -compact.*
- (ii) *For every family $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of fuzzy open sets with $(1 - \alpha)$ -FUP, $\bigvee_{\lambda \in \Lambda} U_\lambda(p) \leq \alpha$, for some $p \in X$.*
- (iii) *Every family $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ of fuzzy closed sets with the property that for each finite subset Λ_0 of Λ , $\bigwedge_{\lambda \in \Lambda_0} F_\lambda(x) \geq \alpha$ for some $x \in X$, has an adherent point with value at least $1 - \alpha$.*

Proof. (i) \Rightarrow (ii): Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\} \subseteq \tau$ have $(1 - \alpha)$ -FUP. Then for each finite subfamily $\mathcal{U}_0 = \{U_{\lambda_k} : k = 1, 2, \dots, n\}$ of \mathcal{U} , we have, for some $x \in X$,

$$(3.1) \quad \bigvee_{i=1}^n U_{\lambda_k}(x) \leq 1 - \alpha$$

If possible, let $\bigvee_{\lambda \in \Lambda} U_\lambda(p) > \alpha$, for each $p \in X$. Then the family $\{U_\lambda : \lambda \in \Lambda\}$ is an open α -shading of X and since X is fuzzy \mathcal{G}_α -compact, there exists a finite subset Λ_0 of Λ such that $1 - \bigvee_{\lambda \in \Lambda_0} U_\lambda \notin \mathcal{G}$. Since \mathcal{G} is an α -grill, $1 - \bigvee_{\lambda \in \Lambda_0} U_\lambda < \alpha$ so that $\bigvee_{\lambda \in \Lambda_0} U_\lambda > 1 - \alpha$, contradicting Equation(3.1).

(ii) \Rightarrow (i): If possible, let $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ be an open α -shading of X , which has no finite subcollection \mathcal{V}_0 for which $1 - \bigvee \mathcal{V}_0 \notin \mathcal{G}$. Thus for each finite subcollection $\mathcal{V}_n = \{V_k : k = 1, 2, \dots, n\}$ of \mathcal{V} , $1 - \bigvee_{i=1}^n V_k \in \mathcal{G}$. Thus $1 - \bigvee_{i=1}^n V_k(x) \geq \alpha$, for some $x \in X$ (since \mathcal{G} is an α -grill). So $\bigvee_{i=1}^n V_k(x) \leq 1 - \alpha$, for some $x \in X$. Therefore the family \mathcal{V} of fuzzy open sets has $(1 - \alpha)$ -FUP and hence by (ii), $\bigvee_{\lambda \in \Lambda} V_\lambda(x) \leq \alpha$, for some $x \in X$, contradicts the fact that $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is an α -shading of X .

(ii) \Rightarrow (iii): Let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ be a family of fuzzy closed sets on X and Λ_0

be a finite subset of Λ . Then for some $x \in X$, $\bigwedge_{\lambda \in \Lambda_0} F_\lambda(x) \geq \alpha \Rightarrow 1 - \bigwedge_{\lambda \in \Lambda_0} F_\lambda(x) \leq 1 - \alpha$. i.e., $\bigvee_{\lambda \in \Lambda_0} (1 - F_\lambda) \leq 1 - \alpha$. Thus $\{1 - F_\lambda : \lambda \in \Lambda\}$ is a family of fuzzy open sets satisfying $(1 - \alpha)$ -FUP. By (ii) we have for some $p \in X$, $\bigvee_{\lambda \in \Lambda} (1 - F_\lambda)(p) \leq \alpha \Rightarrow (1 - \bigwedge_{\lambda \in \Lambda} F_\lambda)(p) \leq \alpha \Rightarrow 1 - \alpha \leq \bigwedge_{\lambda \in \Lambda} F_\lambda(p)$. Thus $p_{1-\alpha} \leq \bigwedge_{\lambda \in \Lambda} F_\lambda \Rightarrow p_{1-\alpha} \leq F_\lambda$, for each $\lambda \in \Lambda$. Thus for each open q -nbd V of $p_{1-\alpha}$, VqF_λ , for each $F_\lambda \in \mathcal{F} \Rightarrow p_{1-\alpha}$ is an adherent point of \mathcal{F} .

(iii) \Rightarrow (i): Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open α -shading of X . If possible, suppose \mathcal{U} has no finite subcollection \mathcal{U}_F for which $1 - \mathcal{U}_F \notin \mathcal{G}$. Then for each finite subset Λ_0 of Λ , $(1 - \bigvee_{\lambda \in \Lambda_0} U_\lambda) \in \mathcal{G} \Rightarrow (1 - \bigvee_{\lambda \in \Lambda_0} U_\lambda)(x) \geq \alpha$, for some $x \in X$. $\Rightarrow \bigwedge_{\lambda \in \Lambda_0} (1 - U_\lambda)(x) \geq \alpha$, for some $x \in X$. Then $\mathcal{V} = \{1 - U_\lambda : \lambda \in \Lambda\}$ is a family of fuzzy closed sets satisfying the conditions in (iii). Now by (iii), \mathcal{V} has an adherent point $p_{1-\alpha}$. Then for all q -nbds W of $p_{1-\alpha}$ and for each $V \in \mathcal{V}$,

$$(3.2) \quad WqV$$

As \mathcal{U} is an α -shading of X , there exists $U \in \mathcal{U}$ such that $U(p) > \alpha$. Then $U(p) + 1 - \alpha > 1$, i.e., U is a q -nbd of $p_{1-\alpha}$ and also $1 - U \in \mathcal{V}$. Then by Equation (3.2), $Uq(1 - U)$ which is not possible. The contradiction establishes (i). \square

Our next intention is to characterize fuzzy \mathcal{G}_α -compactness via prefilters, prefilterbases and fuzzy grills. To accomplish this goal, we recall certain existing concepts and introduce a few others. We will also study these concepts to some extent and finally will apply them to the investigations of fuzzy \mathcal{G}_α -compactness.

Definition 3.4. Let (X, τ) be a fuzzy topological space and Ω be the family of all fuzzy closed sets in X .

- (a) [11] A non-empty family \mathcal{F} of fuzzy sets in X is called a prefilterbase on X , if
- (i) $0_X \notin \mathcal{F}$

(ii) for any $U, V \in \mathcal{F}$, there exists $W \in \mathcal{F}$ such that $W \leq U \wedge V$.

If in addition,

(iii) $A \in \mathcal{F}$ and $A \leq B \in I^X \Rightarrow B \in \mathcal{F}$ holds, then \mathcal{F} is called a prefilter on X .

(b) [2] A collection \mathcal{F} of fuzzy open sets in X , is called an open prefilter on X if the above conditions(a)(i) and (ii) hold and in addition

(iii)' ($A \in \mathcal{F}$ and $A \leq B \in \tau \Rightarrow B \in \mathcal{F}$) is satisfied.

Similarly, one defines a closed prefilter. An open or a closed prefilterbase can similarly be defined.

(c) [2] A non-empty subcollection of Ω (resp. of τ) in a fuzzy topological space (X, τ) is said to form a closed(resp. open) fuzzy grill on X if

(i) $0_X \notin \mathcal{G}$

(ii) $A \in \mathcal{G}$ and $C \in \Omega$ (resp. $C \in \tau$) with $A \leq C \Rightarrow C \in \mathcal{G}$ and

(iii) $A, B \in \Omega$ (resp. $A, B \in \tau$) and $A \vee B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Remark 2. [2] For any fuzzy grill \mathcal{G} on a fuzzy topological space (X, τ) , $\mathcal{G} \cap \tau$ and $\mathcal{G} \cap \Omega$ are open and closed fuzzy grills respectively. Clearly the same observation applies to prefilters as well.

Result 3.1. *If \mathcal{F} is a fuzzy prefilter on X , then $\mathcal{F} \cap \Omega$ (resp. $\mathcal{F} \cap \tau$) is a closed(resp. open) prefilter on X .*

Definition 3.5. [2] For any fuzzy grill or a prefilter \mathcal{G} , a collection $Sec\mathcal{G}$ of fuzzy sets is defined as $Sec\mathcal{G} = \{A \in I^X : AqG, \text{ for each } G \in \mathcal{G}\}$.

Theorem 3.2. [2] *Let (X, τ) be an fts.*

(a) *If \mathcal{G} is a fuzzy grill or a fuzzy prefilter on X , then $A \in Sec\mathcal{G} \Leftrightarrow 1 - A \notin \mathcal{G}$.*

(b) *If \mathcal{G} is a fuzzy grill(prefilter) on X , then $Sec\mathcal{G}$ is a prefilter(resp. fuzzy grill) on X .*

(c) *If \mathcal{G} is a closed(resp. open) fuzzy grill on X , then $Sec\mathcal{G} \cap \tau$ (resp. $Sec\mathcal{G} \cap \Omega$) is*

an open (resp. a closed) prefilter on X .

(d) If \mathcal{F} is an open (resp. closed) prefilter on X , then $\text{Sec}\mathcal{G} \cap \Omega$ (resp. $\text{Sec}\mathcal{G} \cap \tau$) is a closed (resp. an open) fuzzy grill on X .

Definition 3.6. In a fuzzy \mathcal{G} -spaces (X, τ, \mathcal{G}) , a fuzzy open set U is said to be an α -nbd (where $0 < \alpha < 1$) of a point $x \in X$, if $U(x) > \alpha$. The collection of all α -nbds of x in X will be denoted by $\Sigma(x)$.

Definition 3.7. For any constant α ($0 < \alpha < 1$), we define the constant fuzzy set K_α given by $K_\alpha(x) = \alpha$, for all $x \in X$.

Definition 3.8. In any fts (X, τ) ,

(a) a prefilterbase or a prefilter \mathcal{F} is said to α -adhere (resp. α_θ -adhere) at some crisp point $x \in X$ if for each $U \in \Sigma(x)$ and for each $F \in \mathcal{F}$, UqF (resp. $clUqF$).

(b) a fuzzy grill \mathcal{G} is said to α -converge (resp. α_θ -converge) to some point y of X , if corresponding to each $V \in \Sigma(y)$ there exists $G \in \mathcal{G}$ such that $G \leq V$ (resp. $G \leq clV$).

Remark 3. We note that a point $x \in X$ is an α -adherent point or an α_θ -adherent point of a prefilterbase \mathcal{F} iff x is respectively so for the prefilter generated by \mathcal{F} .

Lemma 3.1. (a) For an fts X , every prefilter α -adheres in X iff every closed prefilter α -adheres in X .

(b) In a fuzzy \mathcal{G} -space (X, τ, \mathcal{G}) , every prefilter $\subseteq \mathcal{G}$, α -adheres in X iff every closed prefilter $\subseteq \mathcal{G}$, α -adheres in X .

Proof. (a) First let each prefilter α -adhere in X . Now since every closed prefilter on X forms a base of some prefilter on X then 'if' part follows.

Conversely let every closed prefilter α -adhere in X . Let \mathcal{F} be any prefilter on X . Then by Result 3.1, $\mathcal{F} \cap \Omega$ is a closed prefilter on X . By hypothesis, $\mathcal{F} \cap \Omega$ α -adheres at some point $y \in X$. Then for each $U \in \Sigma(y)$ and for each $F \in \mathcal{F} \cap \Omega$, FqU . We claim

that for each $U \in \Sigma(y)$ and for each $F \in \mathcal{F}$, FqU . If not, then there exist $F_1 \in \mathcal{F}$ and $U \in \Sigma(y)$ such that $F_1\bar{q}U \Rightarrow F_1 \leq (1 - U) \in \mathcal{F}$, as \mathcal{F} is a prefilter. Also $(1 - U) \in \Omega \Rightarrow (1 - U) \in \mathcal{F} \cap \Omega$. Thus for $U \in \Sigma(y)$, $Uq(1 - U)$, which is a contradiction. Hence \mathcal{F} , α -adheres at $y \in X$.

(b) Similar to (a) and left. \square

Lemma 3.2. (a) In an fts (X, τ) , every fuzzy grill α -converges (resp. α_θ -converges) if every closed fuzzy grill α -converges (resp. α_θ -converges) in X .

(b) In a fuzzy \mathcal{G} -space (X, τ, \mathcal{G}) , each fuzzy grill $\mathcal{H} \subseteq \mathcal{G}$, α -converges (resp. α_θ -converges) if every closed fuzzy grill $\mathcal{H} \subseteq \mathcal{G}$, α -converges (resp. α_θ -converges) in X .

Proof. (a) Let every closed fuzzy grill α -converge (resp. α_θ -converge) in X and suppose \mathcal{G} is any fuzzy grill in X . Then by Remark 2, $\mathcal{G} \cap \Omega$ is a closed fuzzy grill in X . By hypothesis, $\mathcal{G} \cap \Omega$ α -converges (resp. α_θ -converges) to some point y in X . Then for each $U \in \Sigma(y)$, $G \leq U$ (resp. $G \leq clU$) for some $G \in \mathcal{G} \cap \Omega$. Thus for each $U \in \Sigma(y)$ $G \leq U$ (resp. $G \leq clU$) for some $G \in \mathcal{G}$. Hence \mathcal{G} , α -converges (resp. α_θ -converges) to some point y in X .

(b) The proof is quite similar to that of (a) and is omitted. \square

Lemma 3.3. (a) In an fts (X, τ) , every fuzzy grill α -converges (resp. α_θ -converges) iff every open fuzzy grill α -converges (resp. α_θ -converges) in X .

(b) In a fuzzy \mathcal{G} -space (X, τ, \mathcal{G}) , if every open fuzzy grill contained in \mathcal{G} , α -converges (resp. α_θ -converges) then every fuzzy grill contained in \mathcal{G} , α -converges (resp. α_θ -converges) in X .

Proof. (a) Let every fuzzy grill on X , α -converge (resp. α_θ -converge) in X . Let \mathcal{G} be any open fuzzy grill on X . Then by Theorem 3.2(c), $Sec\mathcal{G} \cap \Omega$ is a closed prefilter on X and hence a prefilterbase on X . Let \mathcal{F} be the prefilter generated by the prefilterbase

$\text{Sec}\mathcal{G} \cap \Omega$. Then $\text{Sec}\mathcal{F}$ is a fuzzy grill on X and by hypothesis, it α -converges (resp. α_θ -converges) to some point $x \in X$. Then for each $U \in \Sigma(x)$, there exists some $S \in \text{Sec}\mathcal{F}$ such that $S \leq U$ (resp. $S \leq clU$) $\Rightarrow U$ (resp. clU) $\in \text{Sec}\mathcal{F}$ (as $\text{Sec}\mathcal{F}$ is a fuzzy grill). Thus for each $F \in \mathcal{F}$ and each $U \in \Sigma(x)$, UqF (resp. $clUqF$). We assert that $U \in \mathcal{G}$ (resp. $clU \in \mathcal{G}$), for each $U \in \Sigma(x)$. If not, then there exists some $V \in \Sigma(x)$ such that $V \notin \mathcal{G}$ (resp. $clV \notin \mathcal{G}$) $\Rightarrow 1 - V \in \text{Sec}\mathcal{G} \cap \Omega$ (resp. $1 - clV \in \text{Sec}\mathcal{G} \cap \Omega$, (by Theorem 3.2(a)) $\Rightarrow 1 - V$ (resp. $1 - clV$) $\in \mathcal{F}$, as \mathcal{F} is the prefilter generated by $\text{Sec}\mathcal{G} \cap \Omega \Rightarrow Vq(1 - V)$ (resp. $clVq(1 - clV)$), which is a contradiction. Thus $U \in \mathcal{G}$ for each $U \in \Sigma(x)$ and consequently \mathcal{G} , α -converges (resp. α_θ -converges) to some point in X .

Conversely let us assume that each open fuzzy grill α -converges (resp. α_θ -converges) to some point in X . Let \mathcal{G} be a fuzzy grill on X . Then $\mathcal{G} \cap \tau$ is an open fuzzy grill on X (by Remark 2). By hypothesis $\mathcal{G} \cap \tau$, α -converges (resp. α_θ -converges) to some point $x \in X$. This implies that for each $V \in \Sigma(x)$, $G \leq V$ (resp. $G \leq clV$), for some $G \in \mathcal{G} \cap \tau \Rightarrow V \in \mathcal{G} \cap \tau \subseteq \mathcal{G}$ (resp. $clV \in \mathcal{G} \cap \tau \subseteq \mathcal{G}$) $\Rightarrow \mathcal{G}$, α -converges (resp. α_θ -converges) to $x \in X$.

(b) The proof can be done by proceeding in the same way as in the converse part of (a) above. □

Lemma 3.4. (a) In an fts (X, τ) , every fuzzy grill α -converges iff every closed prefilter α -adheres in X .

(b) In a fuzzy \mathcal{G} -space (X, τ, \mathcal{G}) with $\Omega \setminus \{0_X\} \subseteq \mathcal{G}$, if every closed prefilter $\subseteq \mathcal{G}$, α -adheres then each fuzzy grill $\subseteq \mathcal{G}$, α -converges in X .

Proof. (a) Let \mathcal{G} be any open fuzzy grill on X in which every closed prefilter α -adheres. Now by Theorem 3.9(c), $\text{Sec}\mathcal{G} \cap \Omega$ is a closed prefilter on X and hence

α -adheres at some point $x \in X$. Then for each $U \in \Sigma(x)$ and for each $S \in \text{Sec}\mathcal{G} \cap \Omega$,

$$(3.3) \quad UqS$$

We claim that $U \in \mathcal{G}$, for each $U \in \Sigma(x)$. If not, then $1 - U \in \text{Sec}\mathcal{G}$ (by Theorem 3.2(a)). Also $1 - U \in \Omega \Rightarrow 1 - U \in \text{Sec}\mathcal{G} \cap \Omega$. Then we get $Uq(1 - U)$ (by Equation(3.3)), which is a contradiction. Hence the open fuzzy grill \mathcal{G} α -converges to x in X and by Lemma 3.3(a), the sufficiency follows.

Conversely, let \mathcal{F} be a closed prefilter on X . Then $\text{Sec}\mathcal{F} \cap \tau$ will be an open fuzzy grill on X (by Theorem 3.2(d)). Now by hypothesis every fuzzy grill α -converges and hence by Lemma 3.3(a), $\text{Sec}\mathcal{F} \cap \tau$ α -converges to some $x \in X$. Thus for each $U \in \Sigma(x)$, there exists some $G \in \text{Sec}\mathcal{F} \cap \tau$ such that $G \leq U \Rightarrow U \in \text{Sec}\mathcal{F} \cap \tau \subseteq \text{Sec}\mathcal{F}$ for each $U \in \Sigma(x)$. Hence for each $U \in \Sigma(x)$ and for each $F \in \mathcal{F}$, FqU (by definition of $\text{Sec}\mathcal{F}$) $\Rightarrow \mathcal{F}$, α -adheres to $x \in X$.

(b) The proof goes along the same line as in the first part of the above proof and is thus omitted. \square

We are now equipped enough to deliberate on \mathcal{G}_α -compactness with regard to prefilters, prefilterbases and different type of fuzzy grills.

Theorem 3.3. *Let (X, τ, \mathcal{G}) be a fuzzy \mathcal{G} -space. Then the following implications hold:*

- (a) X is fuzzy \mathcal{G}_α -compact.
 - \Leftrightarrow (b) For every family $\{F_\lambda : \lambda \in \Lambda\}$ of fuzzy closed sets with $\bigwedge_{\lambda \in \Lambda} F_\lambda < K_{(1-\alpha)}$, there exists a finite subset Λ_0 of Λ such that $\bigwedge_{\lambda \in \Lambda_0} F_\lambda \notin \mathcal{G}$.
 - \Rightarrow (c) Every prefilter \mathcal{F} on X with $\mathcal{F} \subseteq \mathcal{G}$, α -adheres in X .
 - \Rightarrow (d) Every prefilterbase \mathcal{B} on X with $\mathcal{B} \subseteq \mathcal{G}$, α -adheres in X .
- If, in addition, \mathcal{G} is an α -grill, then $'(d) \Rightarrow (a)'$ also holds, i.e., under such a condition (a), (b), (c), (d) become equivalent.

Proof. (a) \Rightarrow (b): Let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ be a family of fuzzy closed sets such that $\bigwedge_{\lambda \in \Lambda} F_\lambda < K_{(1-\alpha)}$. Then $1 - \bigwedge_{\lambda \in \Lambda} F_\lambda > K_\alpha$, i.e., $\bigvee_{\lambda \in \Lambda} (1 - F_\lambda) > K_\alpha$. Thus $\{1 - F_\lambda : \lambda \in \Lambda\}$ is an open α -shading of X . Since X is fuzzy \mathcal{G}_α -compact, there exists a finite subset Λ_0 of Λ such that $1 - \bigvee_{\lambda \in \Lambda_0} (1 - F_\lambda) \notin \mathcal{G} \Rightarrow \bigwedge_{\lambda \in \Lambda_0} F_\lambda \notin \mathcal{G}$.

(b) \Rightarrow (a): Suppose $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open α -shading of X . Then $\mathcal{U}' = \{1 - U_\lambda : \lambda \in \Lambda\}$ is a family of fuzzy closed sets in X such that $\bigwedge_{\lambda \in \Lambda} (1 - U_\lambda) < K_{(1-\alpha)}$ (since \mathcal{U} is an α -shading of X , $\bigvee_{\lambda \in \Lambda} U_\lambda > K_\alpha$). By hypothesis, there exists some finite subset Λ_0 of Λ such that $\bigwedge_{\lambda \in \Lambda_0} (1 - U_\lambda) \notin \mathcal{G}$, i.e., $1 - \bigvee_{\lambda \in \Lambda_0} U_\lambda \notin \mathcal{G}$. Hence X is fuzzy \mathcal{G}_α -compact.

(a) \Rightarrow (c): Let \mathcal{F} be a prefilter on X such that $\mathcal{F} \subseteq \mathcal{G}$. If possible, let \mathcal{F} have no α -adherent point in X . Then for each point $x \in X$, there exist a $U_x \in \Sigma(x)$ and an $F_x \in \mathcal{F}$ such that $U_x \bar{q} F_x$. Then $\mathcal{U} = \{U_x : x \in X\}$ is an open α -shading of X . Since X is fuzzy \mathcal{G}_α -compact, there exist finitely many members $U_{x_1}, U_{x_2}, \dots, U_{x_k}$ of \mathcal{U} , for some $x_1, x_2, \dots, x_k \in X$ such that $1 - \bigvee_{i=1}^k U_{x_i} \notin \mathcal{G}$. Then $\bigwedge_{i=1}^k F_{x_i} \leq \bigwedge_{i=1}^k (1 - U_{x_i}) \notin \mathcal{G}$. Now \mathcal{F} being a prefilter, there exists $F \in \mathcal{F}$ such that $F \leq \bigwedge_{i=1}^k F_{x_i}$ and then $F \notin \mathcal{G}$ which contradicts the fact that $\mathcal{F} \subseteq \mathcal{G}$. Hence \mathcal{F} has an α -adherent point in X .

(c) \Rightarrow (d): Let \mathcal{B} be a prefilterbase on X such that $\mathcal{B} \subseteq \mathcal{G}$. Let \mathcal{P} be the prefilter on X generated by \mathcal{B} . As $\mathcal{B} \subseteq \mathcal{G}$, then $\mathcal{P} \subseteq \mathcal{G}$. By (b), the prefilter \mathcal{P} , α -adheres at some point $x \in X$. Then for any $U \in \Sigma(x)$ and $P \in \mathcal{P}$, $U q P$. Now for all $B \in \mathcal{B}$, $B \in \mathcal{P}$ so that $U q B$. Thus the prefilterbase \mathcal{B} , α -adheres at $x \in X$.

(d) \Rightarrow (a): If possible, let \mathcal{U} be an open α -shading of X such that there is no finite subcollection \mathcal{U}_0 of \mathcal{U} for which $1 - \bigvee \mathcal{U}_0 \notin \mathcal{G}$. This means that for each finite subcollection \mathcal{U}_0 of \mathcal{U} , since \mathcal{G} is an α -grill, $(1 - \bigvee \mathcal{U}_0)(x) \geq \alpha$, for some $x \in X$. Let us take $\mathcal{F} = \{1 - \bigvee \mathcal{U}_0 : \mathcal{U}_0 \text{ is a finite subcollection of } \mathcal{U}\}$. Then clearly \mathcal{F} is a prefilterbase on X . Also $\mathcal{F} \subseteq \mathcal{G}$. Now by (c), \mathcal{F} , α -adheres at some point $x \in X$.

\mathcal{U} being an α -shading of X , there exists $U \in \mathcal{U}$ such that $U(x) > \alpha$. Then U is an α -nbd of x , for which $1 - U \in \mathcal{F}$ such that $Uq(1 - U)$, which is a contradiction. \square

A necessary condition for fuzzy \mathcal{G}_α -compactness in terms of fuzzy grill, is given as follows:

Theorem 3.4. *In a fuzzy \mathcal{G} -space (X, τ, \mathcal{G}) with $\Omega \setminus \{0_X\} \subseteq \mathcal{G}$, X is fuzzy \mathcal{G}_α -compact \Rightarrow every fuzzy grill contained in \mathcal{G} , α -converges in X .*

Proof. X is fuzzy \mathcal{G}_α -compact \Rightarrow every prefilter \mathcal{F} on X with $\mathcal{F} \subseteq \mathcal{G}$, α -adheres in X (by Theorem 3.3 (a) \Rightarrow (c)) \Rightarrow every closed prefilter \mathcal{F} on X with $\mathcal{F} \subseteq \mathcal{G}$, α -adheres in X (by Lemma 3.1(b)) \Rightarrow every fuzzy grill \mathcal{F} on X with $\mathcal{F} \subseteq \mathcal{G}$, α -converges in X (by Lemma 3.4(b)). \square

Our next target in this section is to show that the fuzzy continuous image of any fuzzy \mathcal{G}_α -compact space is \mathcal{G}_α -compact. Before that we have to prove the following result:

Lemma 3.5. *Let (X, τ, \mathcal{G}) and (Y, σ, \mathcal{H}) be two fuzzy \mathcal{G} -spaces and $f : X \rightarrow Y$ be a function.*

- (i) *If f is a surjection, then $f(\mathcal{G}) = \{f(G) : G \in \mathcal{G}\}$ is a fuzzy grill on Y .*
- (ii) *If f is a bijection, then $f^{-1}(\mathcal{H}) = \{f^{-1}(H) : H \in \mathcal{H}\}$ is a fuzzy grill on X .*

Proof. (i) Clearly $0_X \notin f(\mathcal{G})$ as $0_X \notin \mathcal{G}$. Next let $A \in f(\mathcal{G})$ and $B \geq A$, for any $B \in I^Y$. Now $A \in f(\mathcal{G}) \Rightarrow A = f(G)$, for some $G \in \mathcal{G}$. Also, $f(G) \leq B \Rightarrow G \leq f^{-1}(B) \in \mathcal{G} \Rightarrow ff^{-1}(B) = B \in f(\mathcal{G})$ (since f is surjective). Lastly $A \vee B \in f(\mathcal{G}) \Rightarrow$ there exists $C \in \mathcal{G}$ such that $A \vee B = f(C) \Rightarrow C \in f^{-1}f(C) = f^{-1}(A \vee B) = f^{-1}(A) \vee f^{-1}(B) \in \mathcal{G} \Rightarrow$ either $f^{-1}(A) \in \mathcal{G}$ or $f^{-1}(B) \in \mathcal{G} \Rightarrow$ either $ff^{-1}(A) = A \in f(\mathcal{G})$ or $ff^{-1}(B) = B \in f(\mathcal{G})$. Thus $f(\mathcal{G})$ is a fuzzy grill on Y .

(ii) Obviously $0_X \notin f^{-1}(\mathcal{H})$. Now let $A \in f^{-1}(\mathcal{H})$ and $B \geq A$ for any $B \in I^X$.

Then $A = f^{-1}(H)$, for some $H \in \mathcal{H}$ and $f^{-1}(H) \leq B \Rightarrow ff^{-1}(H) \leq f(B) \Rightarrow H \leq f(B) \in \mathcal{H}$ (as f is a bijection). Thus $f(B) = H_1 \in \mathcal{H} \Rightarrow B = f^{-1}(H_1) \in f^{-1}(\mathcal{H})$. Next, for any two $U, V \in I^X$, $U \vee V \in f^{-1}(\mathcal{H}) \Rightarrow U \vee V = f^{-1}(H)$ for $H \in \mathcal{H}$. $\Rightarrow f(U \vee V) = ff^{-1}(H) = H \Rightarrow f(U) \vee f(V) = H \in \mathcal{H} \Rightarrow$ either $f(U) \in \mathcal{H}$ or $f(V) \in \mathcal{H} \Rightarrow$ either $U \in f^{-1}(\mathcal{H})$ or $V \in f^{-1}(\mathcal{H})$. \square

Theorem 3.5. *Let (X, τ, \mathcal{G}) and (Y, σ, \mathcal{H}) be two fuzzy \mathcal{G} -spaces and $f : X \rightarrow Y$ be a function.*

- (i) *If f is a fuzzy continuous surjection and X is fuzzy \mathcal{G}_α -compact then Y is fuzzy $[f(\mathcal{G})]_\alpha$ -compact.*
- (ii) *If f is a fuzzy open bijection and Y is fuzzy \mathcal{H}_α -compact then X is fuzzy $[f^{-1}(\mathcal{H})]_\alpha$ -compact.*

Proof. (i) Suppose $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ be an open α -shading of Y . Since f is a fuzzy continuous, $\{f^{-1}(V) : V \in \mathcal{V}\}$ is an open α -shading of X . In fact, if $x \in X$, then $f(x) \in Y$. So there exists $V \in \mathcal{V}$ with $V(f(x)) > \alpha$, i.e., $(f^{-1}(V))(x) > \alpha$. Since X is fuzzy \mathcal{G}_α -compact, there exists a finite subset Λ_0 of Λ such that

$$(3.4) \quad 1 - \bigvee_{\lambda \in \Lambda_0} f^{-1}(V_\lambda) \notin \mathcal{G}$$

Now by Lemma 3.5(a), $f(\mathcal{G})$ is a fuzzy grill on Y . Thus $f(1 - \bigvee_{\lambda \in \Lambda_0} f^{-1}(V_\lambda)) = f(1 - f^{-1} \bigvee_{\lambda \in \Lambda_0} (V_\lambda)) = f(f^{-1}(1 - \bigvee_{\lambda \in \Lambda_0} V_\lambda)) = 1_Y - \bigvee_{\lambda \in \Lambda_0} V_\lambda \notin f(\mathcal{G})$ [by Equation(3.4)]. Hence Y is $[f(\mathcal{G})]_\alpha$ -compact.

(ii) Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open α -shading of X . Then $\{f(U_\lambda) : \lambda \in \Lambda\}$ is an open α -shading of Y . Now Y is fuzzy \mathcal{H}_α -compact implies that there exists a finite subset Λ_0 of Λ such that $1_Y - \bigvee_{\lambda \in \Lambda_0} f(U_\lambda) \notin \mathcal{H}$. Then $f^{-1}(1_Y - \bigvee_{\lambda \in \Lambda_0} f(U_\lambda)) \notin f^{-1}(\mathcal{H})$. Now $1_X - f^{-1}(\bigvee_{\lambda \in \Lambda_0} f(U_\lambda)) = 1_X - f^{-1}(f(\bigvee_{\lambda \in \Lambda_0} U_\lambda)) = 1_X - \bigvee_{\lambda \in \Lambda_0} U_\lambda \notin f^{-1}(\mathcal{H})$ (since f is bijective). Hence X is fuzzy $[f^{-1}(\mathcal{H})]_\alpha$ -compact. \square

4. FUZZY \mathcal{G}_α -ALMOST COMPACTNESS

The notion of fuzzy almost compactness, first given by Concilio and Gerla [6], is recognized as one of the most well-known weaker form of fuzzy compactness. Our purpose in this section is to discuss about fuzzy almost compactness via fuzzy grill and through α -shading. The concept of fuzzy almost compactness through α -shading, termed α -almost compactness, was first defined by M. E. Abd. El-Monsef, and M. H. Ghanim [9] in the following way:

Definition 4.1. An fts (X, τ) is said to be α -almost compact if each open α -shading \mathcal{U} of X has a finite proximate α -subshading. i.e., there exists a finite subcollection \mathcal{U}_0 of \mathcal{U} such that $\{clU : U \in \mathcal{U}_0\}$ is again an α -shading of X .

Theorem 4.1. Let (X, τ, \mathcal{G}) be a fuzzy \mathcal{G} -space with $\tau \setminus \{0_X\} \subseteq \mathcal{G}$. If X is fuzzy \mathcal{G}_α -compact then X is α -almost compact.

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open α -shading of X . As X is fuzzy \mathcal{G}_α -compact, there exists a finite subset Λ_0 of Λ such that $(1 - \bigvee_{\lambda \in \Lambda_0} U_\lambda) \notin \mathcal{G} \Rightarrow int(1 - \bigvee_{\lambda \in \Lambda_0} U_\lambda) \notin \mathcal{G}$
 $\Rightarrow int(1 - \bigvee_{\lambda \in \Lambda_0} U_\lambda) = 0_X$ (as $\tau \setminus \{0_X\} \subseteq \mathcal{G}$) $\Rightarrow cl(\bigvee_{\lambda \in \Lambda_0} U_\lambda) = 1_X > \alpha$ for any $\alpha \in (0, 1)$
 $\Rightarrow \bigvee_{\lambda \in \Lambda_0} clU_\lambda = 1_X > \alpha \Rightarrow X$ becomes α -almost compact. \square

However, the converse of the above theorem may not be true in general, as we show below:

Example 4.1. Let $X = \{a, b\}$ and let us define fuzzy sets U_n in X as follows:

$U_n(a) = \frac{n}{n+1}$ and $U_n(b) = \frac{n+1}{n+2}$, $n \in \mathbb{N}$. Then one can easily check that the collection $\tau = \{0_X, 1_X\} \cup \{U_n : n \in \mathbb{N}\}$ is a fuzzy topology on X . Let $\mathcal{G} = \{G \in I^X / 0 < G(x) \leq 1, x \in X\}$. Then \mathcal{G} is a fuzzy grill on X with $\tau \setminus \{0_X\} \subseteq \mathcal{G}$. Let us take $\alpha = 0.5$. Then the family $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an open α -shading of X , but there is no finite

subset \mathbb{N}_0 of \mathbb{N} such that $1 - \bigvee_{n \in \mathbb{N}_0} U_n \notin \mathcal{G}$. [Indeed, for any finite subcollection \mathbb{N}_0 of \mathbb{N} , $(\bigvee_{n \in \mathbb{N}_0} U_n)(x) < 1$ for each $x \in X$]. Thus X is not fuzzy \mathcal{G}_α -compact. But for each $n \in \mathbb{N}$, we claim that $cl(U_n) = 1_X$. In fact, $U_1(a) = \frac{1}{2}$, $U_1(b) = \frac{2}{3} \Rightarrow (1 - U_1)(a) = \frac{1}{2}$, $(1 - U_1)(b) = \frac{1}{3}$. Again, $U_2(a) = \frac{2}{3}$, $U_2(b) = \frac{3}{4} \Rightarrow (1 - U_2)(a) = \frac{1}{3}$, $(1 - U_2)(b) = \frac{1}{4}$; $U_3(a) = \frac{3}{4}$, $U_3(b) = \frac{4}{5} \Rightarrow (1 - U_3)(a) = \frac{1}{4}$, $(1 - U_3)(b) = \frac{1}{5}$ and so on.

Thus we observe that for any given $m \in \mathbb{N}$, $U_m \not\leq 1 - U_m$. Thus the only fuzzy closed set containing U_m ($m \in \mathbb{N}$) is 1_X . So $cl(U_m) = 1_X$, for each $m \in \mathbb{N}$. Thus X is α -almost compact.

Definition 4.2. Let (X, τ, \mathcal{G}) be a fuzzy \mathcal{G} -space. X is said to be fuzzy \mathcal{G}_α -almost compact if for any open α -shading of X , there exists a finite subset Λ_0 of Λ such that $1 - \bigvee_{\lambda \in \Lambda_0} cl(U_\lambda) \notin \mathcal{G}$.

Here we notice from the above definition that fuzzy \mathcal{G}_α -compactness \Rightarrow fuzzy \mathcal{G}_α -almost compactness. That the converse is false, is seen from the following example:

Example 4.2. Let X be any infinite set. Let $z \in X$ be any arbitrarily chosen fixed element. We consider the fuzzy topology τ on X given by $\tau = \{U \in I^X : U(z) > 0.6\}$ together with 0_X , and consider the fuzzy grill $\mathcal{G} = \{G \in I^X / 0 < G(x) \leq 1, x \in X\}$. Let $\alpha = 0.4$, and \mathcal{U} be any open α -shading of X . Since for each $U \in \tau$, $clU = 1_X$, we see that for each finite subcollection \mathcal{U}_0 of \mathcal{U} , $\bigvee_{U \in \mathcal{U}_0} clU = 1_X \Rightarrow 1 - \bigvee_{U \in \mathcal{U}_0} clU = 0_X \notin \mathcal{G} \Rightarrow X$ is fuzzy \mathcal{G}_α -almost compact.

But for each $x \in X \setminus \{z\}$, if we choose $V_x \in \tau$ in such a way that $0.4 < V_x < 1$ and $0 < V_x(y) < 0.4$ for each $y \in X \setminus \{x, z\}$, then $\mathcal{V} = \{V_x : x \in X \setminus \{z\}\}$ is an open α -shading of X . Now for each subcollection \mathcal{V}_0 of \mathcal{V} , $(1 - \bigvee \mathcal{V}_0)(x) \neq 0$ for all $x \in X \Rightarrow 1 - \bigvee \mathcal{V}_0 \in \mathcal{G}$ and hence X is not fuzzy \mathcal{G}_α -compact.

Theorem 4.2. *For any fuzzy \mathcal{G} -space (X, τ, \mathcal{G}) , the following are equivalent:*

- (a) *X is fuzzy \mathcal{G}_α -almost compact.*
- (b) *For each family $\{F_\lambda : \lambda \in \Lambda\}$ of fuzzy closed sets such that $\bigwedge_{\lambda \in \Lambda} F_\lambda < K_{(1-\alpha)}$, there exists a finite subset Λ_0 of Λ , for which $\bigwedge_{\lambda \in \Lambda_0} \text{int} F_\lambda \notin \mathcal{G}$.*
- (c) *Every prefilterbase \mathcal{B} on X with $\mathcal{B} \subseteq \mathcal{G}$, α_θ -adheres in X .*
- (d) *Every prefilter \mathcal{F} on X with $\mathcal{F} \subseteq \mathcal{G}$, α_θ -adheres in X .*
- (e) *For any prefilterbase \mathcal{B} on X with $\mathcal{B} \subseteq \mathcal{G}$, $\bigcap_{B \in \mathcal{B}} B(\mathcal{G}) \neq \Phi$, where for each $B \in \mathcal{B}$, $B(\mathcal{G}) = \{x \in X : \text{for each } U \in \Sigma(x), BqclU\}$.*

Proof. (a) \Rightarrow (b): Let $\{F_\lambda : \lambda \in \Lambda\}$ be a family of fuzzy closed sets such that $\bigwedge_{\lambda \in \Lambda} F_\lambda < K_{(1-\alpha)}$. Thus $1 - \bigwedge_{\lambda \in \Lambda} F_\lambda > K_\alpha$, i.e., $\bigvee_{\lambda \in \Lambda} (1 - F_\lambda) > K_\alpha$. Thus $\{(1 - F_\lambda) : \lambda \in \Lambda\}$ is an open α -shading of X . By fuzzy \mathcal{G}_α -compactness of X , there exists a finite subset

Λ_0 of Λ such that $1 - \bigvee_{\lambda \in \Lambda} cl(1 - F_\lambda) \notin \mathcal{G}$ i.e., $\bigwedge_{\lambda \in \Lambda_0} \text{int} F_\lambda \notin \mathcal{G}$.

(b) \Rightarrow (a): Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open α -shading of X . Then $\{1 - U_\lambda : \lambda \in \Lambda\}$ is a family of fuzzy closed sets such that $\bigwedge_{\lambda \in \Lambda} (1 - U_\lambda) < K_{(1-\alpha)}$. By hypothesis, there exists a finite subset Λ_0 of Λ such that $\bigwedge_{\lambda \in \Lambda_0} \text{int}(1 - U_\lambda) \notin \mathcal{G} \Rightarrow 1 - \bigvee_{\lambda \in \Lambda_0} clU_\lambda \notin \mathcal{G}$.

(a) \Rightarrow (c): Let \mathcal{B} be a filterbase on X with $\mathcal{B} \subseteq \mathcal{G}$, and if possible, let \mathcal{B} do not α_θ -adhere at any point of X . Then for each point x of X , there exists a $U_x \in \Sigma(x)$ such that $clU_x \bar{q} B_x$, for some $B_x \in \mathcal{B}$. Now $\{U_x : x \in X\}$ is an open α -shading of X . So there exist finitely many points x_1, x_2, \dots, x_n in X such that $1 - \bigvee_{i=1}^n clU_{x_i} \notin \mathcal{G}$. Again,

$clU_x \bar{q} B_x \Rightarrow B_x \leq 1 - clU_x \Rightarrow \bigwedge_{i=1}^n B_{x_i} \leq \bigwedge_{i=1}^n (1 - clU_{x_i}) \notin \mathcal{G}$. But \mathcal{B} being a prefilterbase, there exists $B \in \mathcal{B}$ such that $B \leq \bigwedge_{i=1}^n B_{x_i}$ and then $B \notin \mathcal{G}$ which contradicts the fact that $\mathcal{B} \subseteq \mathcal{G}$.

(c) \Rightarrow (b): Suppose $\mathcal{C} = \{F_\lambda : \lambda \in \Lambda\}$ is a collection of fuzzy closed sets in X

with $\bigwedge_{\lambda \in \Lambda} F_\lambda < K_{(1-\alpha)}$. If possible, let \mathcal{C} be such that for every finite subset Λ_0 of Λ , $\bigwedge_{\lambda \in \Lambda_0} \text{int} F_\lambda \in \mathcal{G}$. Then clearly $\bigwedge_{\lambda \in \Lambda_0} \text{int} F_\lambda \neq 0_X$ for every finite subset Λ_0 of Λ . Let us consider $\mathcal{F} = \{ \bigwedge_{\lambda \in \Lambda_0} \text{int} F_\lambda : \Lambda_0 \text{ is a finite subset of } \Lambda \}$. It is easy to check that \mathcal{F} is a prefilterbase on X and $\mathcal{F} \subseteq \mathcal{G}$. So \mathcal{F} α_θ -adheres at some point x_0 in X . i.e., for each $U \in \Sigma(x_0)$ and for each $F \in \mathcal{F}$, $clUqF$. Since $(1 - \bigwedge_{\lambda \in \Lambda} F_\lambda) > K_\alpha$, i.e., $\bigvee_{\lambda \in \Lambda} (1 - F_\lambda) > K_\alpha$, then there exists $\beta \in \Lambda$ for which $(1 - F_\beta)(x_0) > \alpha$. Since $\text{int} F_\beta \in \mathcal{F}$ and $1 - F_\beta \in \Sigma(x_0)$, we get $\text{int} F_\beta qcl(1 - F_\beta)$ which is a contradiction.

(c) \Rightarrow (d): It is clear from the fact that each prefilter is also a prefilterbase.

(d) \Rightarrow (c): For a given prefilterbase \mathcal{F} on X with $\mathcal{F} \subseteq \mathcal{G}$, let \mathcal{F}^* be the prefilter generated by \mathcal{F} . As $\mathcal{F} \subseteq \mathcal{G}$, $\mathcal{F}^* \subseteq \mathcal{G}$. By (d), \mathcal{F}^* α_θ -adheres at some point x in X . Then for any $U \in \Sigma(x)$ and for each $F^* \in \mathcal{F}^*$, $clUqF^*$. Then for each $F \in \mathcal{F}$, $F \in \mathcal{F}^*$ so that $FqclU$. Hence \mathcal{F} α_θ -adheres at x in X .

(a) \Rightarrow (e): Let \mathcal{B} be a prefilterbase on X with $\mathcal{B} \subseteq \mathcal{G}$. If possible, let $\bigcap_{B \in \mathcal{B}} B_\mathcal{G} = \Phi$. Then for each $x \in X$, there exist $B_x \in \mathcal{B}$ and $U_x \in \Sigma(x)$ such that $B_x \bar{q}clU_x$. Then $B_x + clU_x \leq 1_X \Rightarrow B_x \leq (1 - clU_x)$. Now $\{U_x : x \in X\}$ being an open α -shading of X , by (a), there exist finitely many points x_1, x_2, \dots, x_n of X such that $1 - \bigvee_{i=1}^n clU_{x_i} \notin \mathcal{G}$.

$\Rightarrow \bigwedge_{i=1}^n (1 - clU_{x_i}) \notin \mathcal{G}$. Thus $\bigwedge_{i=1}^n B_{x_i} \leq \bigwedge_{i=1}^n (1 - clU_{x_i}) \notin \mathcal{G}$. Since \mathcal{B} is a prefilterbase, there exists $B \in \mathcal{B}$ such that $B \leq \bigwedge_{i=1}^n B_{x_i} \notin \mathcal{G} \Rightarrow B \notin \mathcal{G}$, contradicting the fact that $\mathcal{B} \subseteq \mathcal{G}$.

(e) \Rightarrow (c): Let \mathcal{F} be a prefilterbase on X with $\mathcal{F} \subseteq \mathcal{G}$. By (e), $\bigcap_{F \in \mathcal{F}} F(\mathcal{G}) \neq \Phi$. Let $y \in \bigcap_{F \in \mathcal{F}} F(\mathcal{G})$ i.e., $y \in F(\mathcal{G})$, for all $F \in \mathcal{F}$. Thus for all $F \in \mathcal{F}$ and for each $U \in \Sigma(y)$, $FqclU \Rightarrow$ the prefilterbase \mathcal{F} , α_θ -adheres at the point y in X . \square

Theorem 4.3. *In a fuzzy \mathcal{G} -space (X, τ, \mathcal{G}) with $\tau \setminus \{0_X\} \subseteq \mathcal{G}$, X is fuzzy \mathcal{G}_α -almost compact iff every fuzzy grill α_θ -converges in X .*

Proof. First let X be fuzzy \mathcal{G}_α -almost compact. Let \mathcal{H} be any closed fuzzy grill on X . Then by Theorem 3.2(c), $\text{Sec}\mathcal{H} \cap \tau$ is an open prefilter on X with $\text{Sec}\mathcal{H} \cap \tau \subseteq \mathcal{G}$ (since $\tau \setminus \{0_X\} \subseteq \mathcal{G}$). Now since each open prefilter on X forms a base for some prefilter on X , $\text{Sec}\mathcal{H} \cap \tau$, α_θ -adheres at some point $x \in X$ (by Theorem 4.2, (a) \Rightarrow (d)). Thus for each $P \in \text{Sec}\mathcal{H} \cap \tau$ and for each $U \in \Sigma(x)$, $PqclU$. Thus $clU \in \mathcal{H}$, otherwise, $clU \notin \mathcal{H} \Rightarrow 1 - clU \in \text{Sec}\mathcal{H} \cap \tau \Rightarrow clUq(1 - clU)$, which is a contradiction. Thus \mathcal{H} α_θ -converges in X and then by virtue of Lemma 3.2(a), each fuzzy grill α_θ -converges in X .

Conversely, Let \mathcal{F} be any closed prefilter on X . Then by Theorem 3.2(d), $\text{Sec}\mathcal{F} \cap \tau$ is an open fuzzy grill on X . By hypothesis each fuzzy grill α_θ -converges in X . Then by Lemma 3.3(a), each open fuzzy grill α_θ -converges in X . In particular, $\text{Sec}\mathcal{F} \cap \tau$, α_θ -converges to some point $x \in X$. Thus for each $U \in \Sigma(x)$, $G \leq clU$, for some $G \in \text{Sec}\mathcal{F} \cap \tau \Rightarrow clU \in \text{Sec}\mathcal{F} \cap \tau$, for each $U \in \Sigma(x)$. So for each $F \in \mathcal{F}$, $FqclU$, for each $U \in \Sigma(x)$ and thus \mathcal{F} , α_θ -adhere at x . Hence by Lemma 3.1(a), every prefilter α_θ -adheres in X . In particular, every prefilter \mathcal{F} on X with $\mathcal{F} \subseteq \mathcal{G}$, α_θ -adheres in X . Thus by Theorem 4.2, X is fuzzy \mathcal{G}_α -almost compact. \square

Finally we exhibit the invariance property of \mathcal{G}_α -almost compactness under fuzzy continuous and fuzzy open maps.

Theorem 4.4. *Let (X, τ, \mathcal{G}) and (Y, σ, \mathcal{H}) be two fuzzy \mathcal{G} -spaces and $f : X \rightarrow Y$ be a function.*

(a) *If f is a fuzzy continuous surjection and X is fuzzy \mathcal{G}_α -almost compact then Y is fuzzy $[f(\mathcal{G})]_\alpha$ -almost compact.*

(b) If f is a fuzzy open bijection and Y is fuzzy \mathcal{H}_α -almost compact then X is fuzzy $[f^{-1}(\mathcal{H})]_\alpha$ -almost compact.

Proof. (a) Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open α -shading of Y . By continuity of f , $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is an open α -shading of X . Since X is fuzzy \mathcal{G}_α -almost compact, there exists a finite subset Λ_0 of Λ such that $1 - \bigvee_{\lambda \in \Lambda_0} cl f^{-1}(U_\lambda) \notin \mathcal{G}$. So $f(1_X - \bigvee_{\lambda \in \Lambda_0} cl f^{-1}(U_\lambda)) = 1_Y - f(\bigvee_{\lambda \in \Lambda_0} cl f^{-1}(U_\lambda)) = 1_Y - \bigvee_{\lambda \in \Lambda_0} f(cl f^{-1}(U_\lambda)) \notin f(\mathcal{G})$
 $\Rightarrow 1_Y - \bigvee_{\lambda \in \Lambda_0} cl(ff^{-1}U_\lambda) \leq 1_Y - \bigvee_{\lambda \in \Lambda_0} f(cl f^{-1}(U_\lambda)) \notin f(\mathcal{G})$ i.e., $1_Y - \bigvee_{\lambda \in \Lambda_0} cl(U_\lambda) \notin f(\mathcal{G})$ (since f is surjective). Thus Y is $[f(\mathcal{G})]_\alpha$ -almost compact.

(b) Let $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ be an open α -shading of X . Since f is an open map, $\{f(V_\lambda) : \lambda \in \Lambda\}$ is an open α -shading of Y . Now Y is fuzzy \mathcal{H}_α -almost compact. So there exists a finite subset Λ_0 of Λ such that $1_Y - \bigvee_{\lambda \in \Lambda_0} cl f(V_\lambda) \notin \mathcal{H} \Rightarrow f^{-1}(1_Y - \bigvee_{\lambda \in \Lambda_0} cl f(V_\lambda)) \notin f^{-1}(\mathcal{H})$. Now $1_X - f^{-1}(\bigvee_{\lambda \in \Lambda_0} cl f(V_\lambda)) = 1_X - \bigvee_{\lambda \in \Lambda_0} f^{-1}(cl f(V_\lambda))$. Thus $1_X - \bigvee_{\lambda \in \Lambda_0} cl(f^{-1}f(V_\lambda)) \leq 1_X - \bigvee_{\lambda \in \Lambda_0} f^{-1}(cl f(V_\lambda)) \notin f^{-1}(\mathcal{H})$, i.e., $1_X - \bigvee_{\lambda \in \Lambda_0} cl V_\lambda \notin f^{-1}(\mathcal{H})$ (since f is bijective and open). Hence X is fuzzy $[f^{-1}(\mathcal{H})]_\alpha$ -almost compact. \square

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