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ON ALMOST CONTRA $\beta\theta$-CONTINUOUS FUNCTIONS WITH $(\beta\theta, \mathrm{s})$-CLOSED GRAPHS.

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Abstract. In this note, we study some other properties of almost contra $\beta\theta$-continuous functions with $(\beta\theta, s)$-closed graphs by utilizing $\beta\theta$-open sets and the $\beta\theta$-closure operator.

1. Introduction and preliminaries

Is common viewpoint of many topologists that generalized open sets are important ingredients in General Topology and they are now the research topics of many topologists worldwide of which lots of important and interesting results emerged. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of β-open sets or semipreopen sets introduced by Abd El Monsef et al. [1] and Andrijević [2], respectively. In this paper, we adopt the word β-open for the sake of clarity. In 2003, Noiri [20] used this notion and the β-closure [1] of a set to introduce

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the concepts of $\beta\theta$-open and $\beta\theta$-closed sets which provide a formulation of the $\beta\theta$-closure of a set in a topological space. Caldas [5, 6, 7, 8] continued the work of Noiri and defined the concepts of almost contra $\beta\theta$-continuous functions. In this direction we shall study some other properties of almost contra $\beta\theta$-continuous functions with $(\beta\theta, s)$-closed graphs by utilizing $\beta\theta$-open sets and the $\beta\theta$-closure operator.

Throughout this paper, by (X, τ) and (Y, σ) (or X and Y) we always mean topological spaces. Let A be a subset of X. We denote the interior, the closure and the complement of a set A by $Int(A)$, $Cl(A)$ and $X \backslash A$, respectively. A subset A of X is said to be regular open [26] (resp. regular closed [26]) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A subset A of a space X is called semi-open [17] (resp. β-open [1], b-open [3](=γ-open [16])) if $A \subset Cl(Int(A))$ (resp. $A \subset Cl(Int(Cl(A)))$, $A \subset Cl(Int(A)) \cup Int(Cl(A))$). The complement of a semi-open (resp. β-open) set is said to be semi-closed [21](resp. β-closed [1]). The collection of all open (resp. semiopen, β-open) subsets of X will be denoted by $O(X)$ (resp. $SO(X)$, $\beta O(X)$). We set $\beta O(X, x) = \{U : x \in U \in \beta O(X, \tau)\}$. We denote the collection of all subsets of X which is both β-open and β-closed, by $\beta R(X)$.

The b-θ-closure of A [23], denoted by b-θ-$Cl(A)$, is defined to be the set of all $x \in X$ such that b-$Cl(O) \cap A \neq \emptyset$ for every b-open set O containing x. A subset A is said to be b-θ-closed if A=b-θ-$Cl(A)$.

The $\beta\theta$-closure of A [20], denoted by $\beta Cl_\theta(A)$, is defined to be the set of all $x \in X$ such that $\beta Cl(O) \cap A \neq \emptyset$ for every $O \in \beta O(X, \tau)$ with $x \in O$. The set $\{x \in X : \beta Cl_\theta(O) \subset A \ for \ some \ O \in \beta(X, x)\}$ is called the $\beta\theta$-interior of A and is denoted by $\beta Int_\theta(A)$. A subset A is said to be $\beta\theta$-closed [20] if $A = \beta Cl_\theta(A)$. The complement of a $\beta\theta$-closed set is said to be $\beta\theta$-open. The family of all $\beta\theta$-open (resp. $\beta\theta$-closed) subsets of X is denoted by $\beta\theta O(X, \tau)$ or $\beta\theta O(X)$ (resp. $\beta\theta C(X, \tau)$). We set $\beta\theta O(X, x) = \{U : x \in U \in \beta\theta O(X, \tau)\}$ and $\beta\theta C(X, x) = \{U : x \in U \in \beta\theta C(X, \tau)\}$.

Definition 1.1. A function $f : X \to Y$ is said to be:

(1) $\beta\theta$-continuous [20] If $f^{-1}(V)$ is $\beta\theta$-closed for every closed set V of Y, equivalently if the inverse image of every open set V of Y is $\beta\theta$-open in X.

(2) almost $\beta\theta$-continuous if $f^{-1}(V)$ is $\beta\theta$-closed in X for every regular closed set V in Y.

(3) contra R-maps [13] (resp. contra-continuous [10], contra $\beta\theta$-continuous [9], contra b-θ-continuous [15]) if $f^{-1}(V)$ is regular closed (resp. closed, $\beta\theta$-closed, b-θ-closed) in X for every regular open (resp. open, open, open) set V of Y.

(4) almost contra-precontinuous [11] (resp. almost contra β-continuous [4], almost contra -continuous [4]) if $f^{-1}(V)$ is preclosed (resp. β-closed, closed) in X for every regular open set V of Y.

2. Functions with $(\beta\theta, s)$-closed graphs

Recall that, function $f : X \to Y$ is said to be almost contra $\beta\theta$-continuous [7] if $f^{-1}(V)$ is $\beta\theta$-closed in X for each regular open set V of Y.

Theorem 2.1. *For a function $f : X \to Y$, the following properties are equivalent:*

(1) f is almost contra $\beta\theta$-continuous;

(2) The inverse image of each regular closed set in Y is $\beta\theta$-open in X;

(3) For each point x in X and each regular closed set V in Y containing $f(x)$, there is a $\beta\theta$-open set U in X containing x such that $f(U) \subset V$;

(4) For each point x in X and each semiopen set V in Y containing $f(x)$, there is a $\beta\theta$-open set U in X containing x such that $f(U) \subset Cl(V)$.

Proof. It follows from ([7]: Theorem 2.1).

Recall, that a function $f : X \to Y$, is called (γ, s)-continuous [12] at a point x in X if for each semiopen set V in Y containing $f(x)$, there exists a γ-open set U in

X containing x such that $f(U) \subset Cl(V)$. (γ, s)-continuous if it has this property at each point of X.

The following implications are hold for a function $f : X \to Y$:

$$\begin{array}{ccccccc}
 & & & & J & \leftarrow & I \\
 & & & \nearrow & \uparrow & \nwarrow & \uparrow \\
 & & A & \leftarrow & B & \leftarrow & C \\
 & \nearrow & \uparrow & \nwarrow & & & \downarrow \\
H & \leftarrow & D & \rightarrow & E & & G \\
 & \nwarrow & \uparrow & \nearrow & & & \\
 & & F & & & &
\end{array}$$

Notation: A = almost contra β-continuity, B = almost contra $\beta\theta$-continuity, C = contra $\beta\theta$-continuity, D = almost contra-continuity, E = almost contra pre-continuity, F = contra R-map, G = contra β-continuity, H = almost contra semi-continuity, I = contra b-θ-continuity, $J = (\gamma, s)$-continuity.

Example 2.1. *Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Clearly $\beta\theta O(X, \tau) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Let $f : X \to X$ be defined by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then f is almost contra $\beta\theta$-continuous but f is not contra $\beta\theta$-continuous.*

Example 2.2. *Let (X, τ) be a topological space such that $X = Y = \{a, b, c\}$ $\tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f : (X, \tau) \to (Y, \sigma)$*

be the identity function. Then f is almost contra β-continuous but f is not almost contra $\beta\theta$-continuous.

Other implications not reversible are shown in [9], [12], [11], [13], [15], [21].

Definition 2.1. [14, 6, 9] A topological space (X, τ) is said to be:

1) $\beta\theta$-T_0 (resp. $\beta\theta$-T_1) if for any distinct pair of points x and y in X, there is a $\beta\theta$-open U in X containing x but not y or (resp. and) a $\beta\theta$-open set V in X containing y but not x.

2) $\beta\theta$-T_2 [14] (resp. β-T_2 [18]) if for every pair of distinct points x and y, there exist two $\beta\theta$-open (resp. β-open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 2.2. *For a topological space (X, τ), the following properties are equivalent:*

1) (X, τ) is $\beta\theta$-T_0;

2) (X, τ) is $\beta\theta$-T_1;

3) (X, τ) is $\beta\theta$-T_2;

4) (X, τ) is β-T_2;

5) For every pair of distinct points $x, y \in X$, there exist $U, V \in \beta O(X)$ such that $x \in U$, $y \in V$ and $\beta Cl(U) \cap \beta Cl(V) = \emptyset$;

6) For every pair of distinct points $x, y \in X$, there exist $U, V \in \beta R(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$;

7) For every pair of distinct points $x, y \in X$, there exist $\beta\theta$-open and $\beta\theta$-closed sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$;

8) For every pair of distinct points $x, y \in X$, there exist $U \in \beta\theta O(X, x)$ and $V \in \beta\theta O(X, y)$ such that $\beta Cl_\theta(U) \cap \beta Cl_\theta(V) = \emptyset$.

Proof. It follows from [14] and ([9]: Remark 3.2 and Theorem 3.4).

Definition 2.2. A function $f : X \to Y$ has a ($\beta\theta$, s)-closed graph if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in \beta\theta O(X, x)$ and $V \in SO(Y, y)$ such that $(U \times Cl(V)) \cap G(f) = \emptyset$.

Proposition 2.1. *The following properties are equivalent for a function $f : X \to Y$:*
1) $G(f)$ is ($\beta\theta$, s)-closed:
2) For each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in \beta\theta O(X)$ containing x and $V \in SO(Y)$ containing y such that $f(U) \cap Cl(V) = \emptyset$.
3) For each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in \beta\theta O(X)$ containing x and $V \in RC(Y)$ containing y such that $f(U) \cap V = \emptyset$.

Theorem 2.3. *If $f : X \to Y$ is $\beta\theta$-continuous and Y is Hausdorff, then $G(f)$ is ($\beta\theta$, s)-closed.*

Proof. Let $(x, y) \in X \times Y\backslash G(f)$. Since Y is Hausdorff, then there exists a set $V \in O(Y, y)$ such that $f(x) \notin Cl(V)$. Now $Y\backslash Cl(V) \in O(Y, f(x))$. Therefore, by the $\beta\theta$-continuity of f there exists $U \in \beta\theta O(X, x)$ such that $f(U) \subset Y\backslash Cl(V)$. Consequently, $f(U) \cap Cl(V) = \emptyset$ where $Cl(V)$ is a regular closed set since V is open. By Proposition 2.1, $G(f)$ is ($\beta\theta$, s)-closed.

Theorem 2.4. *Let $f : X \to Y$ has a ($\beta\theta$, s)-closed graph. If f is injective, then X is $\beta\theta$-T_1.*

Proof. Let x_1 and x_2 be two distinct pair of points in X. Then $(x_1, f(x_2)) \in (X \times Y)\backslash G(f)$. Since f has a $(\beta\theta, s)$-closed graph, there exist $U \in \beta\theta O(X, x_1)$ and a semiopen set V of Y containing $f(x_2)$ such that $f(U) \cap Cl(V) = \emptyset$. Then $U \cap f^{-1}(Cl(V)) = \emptyset$. Since $x_2 \in f^{-1}(Cl(V))$, $x_2 \notin U$. Therefore U is a $\beta\theta$-open set containing x_1 but not x_2, which proves that X is $\beta\theta$-T_1.

Theorem 2.5. *If $f : X \to Y$ is almost contra $\beta\theta$-continuous and Y is Urysohn, then $G(f)$ is ($\beta\theta$, s)-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$, then $y \neq f(x)$. Since Y is Urysohn there exist open sets V and W in Y such that $y \in V$, $f(x) \in W$ with $Cl(V) \cap Cl(W) = \emptyset$. Since f is almost contra $\beta\theta$-continuous, by Theorem 2.1(3) and since $Cl(W)$ is regular closed containing $f(x)$ there exists $U \in \beta\theta O(X, x)$ such that $f(U) \subset Cl(W)$. Therefore, we obtain $f(U) \cap Cl(V) = \emptyset$. Hence $G(f)$ is ($\beta\theta$, s)-closed in $X \times Y$.

Definition 2.3. A subset A of a space X is said to be S-closed relative to X [19] if for every cover $\{V_\alpha \mid \alpha \in \nabla\}$ of A by semi-open sets of X, there exists a finite subset ∇_0 of ∇ such that $A \subset \bigcup\{Cl(V_\alpha) \mid \alpha \in \nabla_0\}$. A space X is said to be S-closed if X is S-closed relative to X.

We recall, that a topological space is called a $\beta\theta$-space if the union of any two $\beta\theta$-closed sets is a $\beta\theta$-closed

Theorem 2.6. [22] *If a function $f : X \to Y$ has a (θ, s)-closed graph,then $f^{-1}(K)$ is closed in X for every subset K which is S-closed relative to Y.*

Remark 1. The collection of all $\beta\theta$-open sets of a space (X, τ) need not be a topology on X. However, if a space (X, τ) is $\beta\theta$-space, then $\beta\theta O(X)$ is a topology on X. Therefore, if a function $f : (X, \tau) \to Y$ has a $(\beta\theta, s)$-closed graph, then function $f : (X, \beta\theta O(X)) \to Y$ has a (θ, s)-closed graph ([22]). Thus the following corollary is a simple corollary of Theorem 2.6.

Corollary 2.1. *If a function $f : X \to Y$ has a ($\beta\theta$, s)-closed graph and if X is a $\beta\theta$-space, then $f^{-1}(K)$ is $\beta\theta$-closed in X for every subset K which is S-closed relative to Y.*

Definition 2.4. A topological space X is said to be:
(1) strongly $\beta\theta$C-compact [9] if every $\beta\theta$-closed cover of X has a finite subcover. (resp. $A \subset X$ is strongly $\beta\theta$C-compact if the subspace A is strongly $\beta\theta$C-compact).
(2) nearly-compact [24] if every regular open cover of X has a finite subcover.

Theorem 2.7. *If $f : X \to Y$ is an almost contra $\beta\theta$-continuous surjection and X is strongly $\beta\theta C$-compact, then Y is nearly compact.*

Proof. Let $\{V_\alpha : \alpha \in I\}$ be a regular open cover of Y. Since f is almost contra $\beta\theta$-continuous, we have that $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a cover of X by $\beta\theta$-closed sets. Since X is strongly $\beta\theta$C-compact, there exists a finite subset I_0 of I such that $X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective $Y = \bigcup\{V_\alpha : \alpha \in I_0\}$ and therefore Y is nearly compact.

A topological space X is said to be almost-regular [25] if for each regular closed set F of X and each point $x \in X \backslash F$, there exist disjoint open sets U and V such that $F \subset V$ and $x \in U$.

Theorem 2.8. *If a function $f : X \to Y$ is almost contra $\beta\theta$-continuous and Y is almost-regular, then f is almost $\beta\theta$-continuous.*

Proof. Let x be an arbitrary point of X and V an open set of Y containing $f(x)$. Since Y is almost-regular, by Theorem 2.2 of [25] there exists a regular open set W in Y containing $f(x)$ such that $Cl(W) \subset Int(Cl(V))$. Since f is almost contra $\beta\theta$-continuous, and $Cl(W)$ is regular closed in Y, by Theorem 2.1 there exists $U \in \beta\theta O(X, x)$ such that $f(U) \subset Cl(W)$. Then $f(U) \subset Cl(W) \subset Int(Cl(V))$. Hence, f is almost $\beta\theta$-continuous.

The $\beta\theta$-frontier of a subset A, denoted by $Fr_{\beta\theta}(A)$, is defined as $Fr_{\beta\theta}(A) = \beta Cl_\theta(A)\backslash\beta Int_\theta(A)$, equivalently $Fr_{\beta\theta}(A) = \beta Cl_\theta(A) \cap \beta Cl_\theta(X\backslash A)$.

Theorem 2.9. *The set of points $x \in X$ which $f : (X, \tau) \to (Y, \sigma)$ is not almost contra $\beta\theta$-continuous is identical with the union of the $\beta\theta$-frontiers of the inverse images of regular closed sets of Y containing $f(x)$.*

Proof. Necessity. Suppose that f is not almost contra $\beta\theta$-continuous at a point x of X. Then there exists a regular closed set $F \subset Y$ containing $f(x)$ such that $f(U)$ is not a subset of F for every $U \in \beta\theta O(X, x)$. Hence we have $U \cap (X \setminus f^{-1}(F)) \neq \emptyset$ for every $U \in \beta\theta O(X, x)$. It follows that $x \in \beta Cl_\theta(X \setminus f^{-1}(F))$. We also have $x \in f^{-1}(F) \subset \beta Cl_\theta(f^{-1}(F))$. This means that $x \in Fr_{\beta\theta}(f^{-1}(F))$.
Sufficiency. Suppose that $x \in Fr_{\beta\theta}(f^{-1}(F))$ for some $F \in RC(Y, f(x))$ Now, we assume that f is almost contra $\beta\theta$-continuous at $x \in X$. Then there exists $U \in \beta\theta O(X, x)$ such that $f(U) \subset F$. Therefore, we have $x \in U \subset f^{-1}(F)$ and hence $x \in \beta Int_\theta(f^{-1}(F)) \subset X \setminus Fr_{\beta\theta}(f^{-1}(F))$. This is a contradiction. This means that f is not almost contra $\beta\theta$-continuous.

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