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SUBSETS IN TERMS OF $\Psi_{\mathcal{H}}$

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ABSTRACT. In this paper, we study the properties of $\Psi_{\mathcal{H}}A-$sets and $\Psi_{\mathcal{H}}C-$sets introduced by Kim and Min. Also, we characterize these sets in terms of strongly $\mu-$codense hereditary classes.

1. Introduction

A family μ of subsets of a nonempty set X is called a *generalized topology* (GT) [1] if $\emptyset \in \mu$ and the arbitrary union of members of μ is again in μ. The pair (X, μ) is called a generalized topological space (GTS) or simply a space. The elements of μ are called $\mu-open$ sets and the complements of $\mu-$open sets are called $\mu-closed$ sets. The largest $\mu-$open set contained in a subset A of X is denoted by $i_\mu(A)$ [1] and is called the $\mu-interior$ of A. The smallest $\mu-$closed set containing A is called the $\mu-closure$ of A and is denoted by $c_\mu(A)$ [1]. A GT μ is said to be a *quasi-topology* [4] on X if $M,\ N \in \mu$ implies $M \cap N \in \mu$. A subset A of a space is said to be $\mu-preopen$ [2](resp. $\mu-rare$ [3], $\mu-\alpha-open$ [2], $\mu-semiopen$ [2], $\mu-\beta-open$ [2]) if $A \subset i_\mu c_\mu(A)$(resp. $i_\mu c_\mu(A) = \emptyset,\ A \subset i_\mu c_\mu i_\mu(A),\ A \subset c_\mu i_\mu(A),\ A \subset c_\mu i_\mu c_\mu(A)$). The

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family of all $\mu-$preopen (resp. $\mu-\alpha-$open, $\mu-$semiopen) sets in (X, μ) is denoted by $\pi(\mu)$ (resp. $\alpha(\mu),\ \sigma(\mu)$).

A *hereditary class* $\mathcal{H}$ of X is a nonempty collection of subsets of X such that $A \subset B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$ [3]. A hereditary class $\mathcal{H}$ of X is an *ideal* [6] if $A \cup B \in \mathcal{H}$ whenever $A \in \mathcal{H}$ and $B \in \mathcal{H}$. With respect to the generalized topology μ of all $\mu-$open sets and a hereditary class $\mathcal{H}$, for each subset A of X, a subset $A^\star(\mathcal{H})$ or simply $A^\star$ of X is defined by $A^\star = \{x \in X \mid M \cap A \notin \mathcal{H}$ for every $M \in \mu$ containing $x\}$ [3]. $\mathcal{H}$ is said to be $\mu-$ *codense* if $\mu \cap \mathcal{H} = \{\emptyset\}$ [3] and is said to be *strongly* $\mu-$ *codense* [3] if $M \in \mu,\ N \in \mu$ and $M \cap N \in \mathcal{H}$, then $M \cap N = \emptyset$. Every strongly $\mu-$codense hereditary class is $\mu-$codense but the converse is not true [3]. If $\mathcal{H}_r$ is the collection of all $\mu-$rare sets, then $\mathcal{H}_r$ is a hereditary class and for this hereditary class, $A^\star \subset c_\mu i_\mu c_\mu(A)$ for every subset A of X [3, Proposition 2.11]. If $c^\star_\mu(A) = A \cup A^\star$ for every subset A of X, with respect to μ and a hereditary class $\mathcal{H}$ of subsets of X, then $\mu^\star = \{A \subset X \mid c^\star_\mu(X - A) = X - A\}$ is a generalized topology [3]. $\beta = \{U - H \mid U \in \mu$ and $H \in \mathcal{H}\}$ is a basis for $\mu^\star$. $i^\star_\mu(A)$ is the interior of A in $(X, \mu^\star)$. A subset of a GTS (X, μ) with a hereditary class $\mathcal{H}$ is said to be $\mathcal{H}-$ *open* [9] if $A \subset i_\mu(A^\star)$. The family of all $\mathcal{H}-$open sets in (X, μ) is denoted by $\mathcal{H}O(\mu)$ and the family of all $\mu^\star-$preopen (resp. $\mu^\star-$semiopen) sets in $(X, \mu^\star)$ is denoted by $\pi(\mu^\star)$ (resp. $\sigma(\mu^\star)$). The following lemmas will be useful in the sequel and we use some of the results without mentioning it, when the context is clear.

Lemma 1.1. [3] *Let X be a nonempty set and $\mathcal{H}$ be a hereditary class on X. If A and B are any two subsets of X, then the following hold.*

(a) *If $A \in \mathcal{H}$, then $A^\star = X - \mathcal{M}_\mu$ where $\mathcal{M}_\mu = \bigcup\{M \mid M \in \mu\}$.*

(b) *If $A \subset A^\star$, then $c_\mu(A) = A^\star = c^\star(A) = c^\star(A^\star)$.*

(c) *$A^\star$ is $\mu-$closed for every subset A of X.*

Lemma 1.2. [7, Theorem 2.4] *If (X, μ) is a quasi-topological space and $\mathcal{H}$ is a hereditary class of subsets of X, then the following statements are equivalent.*
(a) *$\mathcal{H}$ is $\mu-$codense.*
(b) *$\mathcal{H}$ is strongly $\mu-$codense.*

Lemma 1.3. [7, Theorem 2.5] *If X is a nonempty set, $\mathcal{H}$ is a hereditary class of subsets of X, then the following statements are equivalent.*
(a) *$\mathcal{H}$ is strongly $\mu-$codense.*
(b) *$M \subset M^\star$ for every $M \in \mu$.*
(c) *$c_\mu(M) = M^\star$ for every $M \in \mu$.*

Lemma 1.4. [7, Theorem 2.6] *If (X, μ) is a quasi-topological space and $\mathcal{H}$ is a hereditary class of subsets of X, then $M \cap A^\star \subset (M \cap A)^\star$ for every $M \in \mu$ and $A \subset X$.*

Lemma 1.5. [9, Theorem 2.7] *Let (X, μ) be a quasi-topological space with a hereditary class $\mathcal{H}$ on X. Then the following are equivalent.*
(a) $\pi(\gamma) \cap \mathcal{H} = \{\emptyset\}$.
(b) *$A \subset A^\star$ for every subset $A \in \pi(\gamma)$.*
(c) *$i_\pi(A) = \emptyset$ for every $A \in \mathcal{H}$.*

2. $\Psi_{\mathcal{H}}A-$SET

If $\mathcal{H}$ is a hereditary class on a space (X, μ), an operator $\Psi_{\mathcal{H}} : \wp(X) \to \wp(X)$ [8] is defined as follows: for every $A \in \wp(X), \Psi_{\mathcal{H}}(A) = \{x \in X \mid$ there exists $M \in \mu$ containing x such that $M - A \in \mathcal{H}\}$. $\Psi_{\mathcal{H}}$ is nothing but the monotonic operator $\gamma_\mu^\star : \wp(X) \to \wp(X)$ defined by $\gamma_\mu^\star(A) = X - (X - A)^\star$ for every subset A of X in [5]. A subset A of X is said to be $\Psi_{\mathcal{H}}\mathcal{A} - set$ if $A \subset i_\mu c_\mu \Psi_{\mathcal{H}}(A)$. Since $\Psi_{\mathcal{H}}(A)$ is $\mu-$open for every subset A of X and $M \subset \Psi_{\mathcal{H}}(M)$ for every $M \in \mu$ [5, Theorem 3.3], clearly every $\mu - \alpha-$open set is a $\Psi_{\mathcal{H}}\mathcal{A}-$set.

Lemma 2.1. [8, Theorem 2.4] *Let* (X,μ) *be a quasi-topological space and* $\mathcal{H}$ *be an ideal on X. If* A, $B\subset X$, *then* $\Psi_{\mathcal{H}}(A\cap B)=\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)$.

Proof. Clearly, $\Psi_{\mathcal{H}}(A\cap B)\subseteq\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)$. Let $x\in\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)$. Since $x\notin(X-A)^{\star}$, there exists $U_x\in\mu$ such that $U_x\cap(X-A)\in\mathcal{H}$ which implies that $U_x-A\in\mathcal{H}$. Since $x\notin(X-B)^{\star}$, there exists $V_x\in\mu$ such that $V_x\cap(X-B)\in\mathcal{H}$ which in turn implies that $V_x-B\in\mathcal{H}$. Since $(V_x\cap U_x)-A\subset U_x-A$, $(V_x\cap U_x)-A\in\mathcal{H}$, by heredity. Similarly, $(U_x\cap V_x)-B\in\mathcal{H}$. Therefore, $((U_x\cap V_x)-A)\cup((U_x\cap V_x)-B)\in\mathcal{H}$ which implies that $(U_x\cap V_x)\cap((X-A)\cup(X-B))\in\mathcal{H}$ and so $(U_x\cap V_x)\cap(X-(A\cap B))\in\mathcal{H}$. Since $x\in U_x\cap V_x$, $x\notin(X-(A\cap B))^{\star}$. Hence $x\in\Psi_{\mathcal{H}}(A\cap B)$. Hence $\Psi_{\mathcal{H}}(A\cap B)=\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)$. □

The following Theorem 2.1 shows that the collection of all $\Psi_{\mathcal{H}}\mathcal{A}-$sets, denoted by $\mu_{\mathcal{A}}$, is a generalized topology, if $\mathcal{H}$ is a hereditary class. Example 2.1 below shows that the conditions *quasi-topology* on μ and *ideal* on $\mathcal{H}$ cannot be dropped in Theorem 2.1. Theorem 2.2(a) below gives a characterization of $\Psi_{\mathcal{H}}\mathcal{A}-$sets. Example 2.2 shows that a $\Psi_{\mathcal{H}}\mathcal{A}-$set need not be a $\mu-$semiopen set.

Theorem 2.1. *Let* (X,μ) *be a space with a hereditary class* $\mathcal{H}$. *Then* $\mu_{\mathcal{A}}$ *is a generalized topology on* X. *Further, if* μ *is a quasi-topology and* $\mathcal{H}$ *is an ideal, then* $\mu_{\mathcal{A}}$ *is also a quasi-topology on* X.

Proof. Clearly, $\emptyset\in\mu_{\mathcal{A}}$. Let $\{A_\alpha\mid\alpha\in\triangle\}$ be a family of $\Psi_{\mathcal{H}}\mathcal{A}-$sets in (X,μ). Then for each $\alpha\in\triangle$, $A_\alpha\subset i_\mu c_\mu\Psi_{\mathcal{H}}(A_\alpha)\subset i_\mu c_\mu\Psi_{\mathcal{H}}(\cup A_\alpha)$ and so $\cup A_\alpha\subset i_\mu c_\mu\Psi_{\mathcal{H}}(\cup A_\alpha)$. Hence $\cup A_\alpha\in\mu_{\mathcal{A}}$. Let A and B be $\Psi_{\mathcal{H}}\mathcal{A}-$ sets in X. Then $A\cap B\subset i_\mu c_\mu\Psi_{\mathcal{H}}(A)\cap i_\mu c_\mu\Psi_{\mathcal{H}}(B)\subset i_\mu(c_\mu\Psi_{\mathcal{H}}(A)\cap i_\mu c_\mu\Psi_{\mathcal{H}}(B))\subset i_\mu c_\mu(\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B))=i_\mu c_\mu\Psi_{\mathcal{H}}(A\cap B)$, by Lemma 2.1. Therefore, $A\cap B\in\mu_{\mathcal{A}}$. □

Example 2.1. (a) *Let* $X=\{a,b,c,d\}$, $\mu=\{\emptyset,\{a,c\},\{b,c\},\{a,b,c\}\}$ *and* $\mathcal{H}=\{\emptyset,\{c\},\{d\},\{c,d\}\}$. *Then* $i_\mu c_\mu\Psi_{\mathcal{H}}(\{a,c\})=i_\mu c_\mu(\{a,c\})=i_\mu(X)=\{a,b,c\}\supset\{a,c\}$

and so $\{a,c\} \in \mu_{\mathcal{A}}$. *Also,* $i_\mu c_\mu \Psi_{\mathcal{H}}(\{b,c\}) = i_\mu c_\mu(\{b,c\}) = i_\mu(X) = \{a,b,c\} \supset \{b,c\}$ *implies that* $\{b,c\} \in \mu_{\mathcal{A}}$. *But* $i_\mu c_\mu \Psi_{\mathcal{H}}(\{c\}) = i_\mu c_\mu(\{\emptyset\}) = i_\mu(\{d\}) = \{\emptyset\} \not\supseteq \{c\}$. *Hence* $\{c\} \notin \mu_{\mathcal{A}}$.

(b) *Let* $X = \{a,b,c,d\}$, $\mu = \{\emptyset, \{c\}, \{a,b,c\}, \{c,d\}, X\}$ *and* $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$ *Then* $i_\mu c_\mu \Psi_{\mathcal{H}}(\{a,b,c\}) = i_\mu c_\mu\{c,d\} = i_\mu(X) = X \supset \{a,b,c\}$ *which implies that* $\{a,b,c\} \in \mu_{\mathcal{A}}$. *Also,* $i_\mu c_\mu \Psi_{\mathcal{H}}(\{a,b,d\}) = i_\mu c_\mu(\{a,b,c\}) = i_\mu(X) = X \supset \{a,b,d\}$ *which implies that* $\{a,b,d\} \in \mu_{\mathcal{A}}$. *But* $i_\mu c_\mu \Psi_{\mathcal{H}}(\{a,b\}) = i_\mu c_\mu(\{\emptyset\}) = i_\mu(\{\emptyset\}) = \{\emptyset\} \not\supseteq \{a,b\}$. *Hence* $\{a,b\} \notin \mu_{\mathcal{A}}$.

Theorem 2.2. *Let* (X,μ) *be a space with a hereditary class* $\mathcal{H}$. *Then the following hold.*

(a) $A \in \mu_{\mathcal{A}}$ *if and only if* $c_\mu i_\mu(X-A)^\star \subset X-A$.

(b) $\alpha(\mu) \subset \mu_{\mathcal{A}}$.

(c) $\mu^\star \subset \mu_{\mathcal{A}}$.

Proof. (a) $A \in \mu_{\mathcal{A}}$ if and only if $A \subset i_\mu c_\mu \Psi_{\mathcal{H}}(A)$ if and only if $A \subset i_\mu c_\mu(X-(X-A)^\star)$ if and only if $A \subset X - c_\mu i_\mu(X-A)^\star$ if and only if $c_\mu i_\mu(X-A)^\star \subset X-A$.

(b) $A \in \alpha(\mu)$ implies that $A \subset i_\mu c_\mu i_\mu(A)$ which implies that $c_\mu i_\mu c_\mu(X-A) \subset X-A$. Now $c_\mu i_\mu(X-A)^\star \subset c_\mu i_\mu c_\mu(X-A) \subset X-A$ and so $A \in \mu_{\mathcal{A}}$, by (a).

(c) Suppose $A \in \mu^\star$. Then by Theorem 3.18 of [5], $A \subset \Psi_{\mathcal{H}}(A)$. Now $A \subset \Psi_{\mathcal{H}}(A) = X-(X-A)^\star = X-c_\mu(X-A)^\star \subset X-c_\mu i_\mu(X-A)^\star = i_\mu c_\mu(X-(X-A)^\star) = i_\mu c_\mu \Psi_{\mathcal{H}}(A)$ which implies that $A \subset i_\mu c_\mu \Psi_{\mathcal{H}}(A)$ and so $A \in \mu_{\mathcal{A}}$. Thus, $\mu^\star \subset \mu_{\mathcal{A}}$. □

Example 2.2. *Consider the GTS* (X,μ) *with a hereditary class* $\mathcal{H}$ *where* $X = \{a,b,c\}$, $\mu = \{\emptyset, \{a,b\}, \{a,c\}, X\}$ *and* $\mathcal{H} = \{\emptyset, \{a\}\}$. *If* $A = \{b,c\}$, *then* $i_\mu c_\mu \Psi_{\mathcal{H}}(A) = i_\mu c_\mu(X) = X$ *and so* A *is a* $\Psi_{\mathcal{H}}\mathcal{A}-$*set. But* $c_\mu i_\mu(A) = c_\mu(\emptyset) = \emptyset$ *implies that* A *is not* $\mu-$*semiopen.*

The following Theorem 2.3 gives a characterization of strongly $\mu-$codense hereditary class in terms of $\Psi_{\mathcal{H}}\mathcal{A}-$sets. Example 2.3 below shows that the strongly $\mu-$codenseness on the hereditary class cannot be dropped in Theorem 2.4.

Lemma 2.2. [8, Theorem 2.14] *Let (X, μ) be a space with a strongly $\mu-$codense ideal $\mathcal{H}$. Then $\Psi_{\mathcal{H}}(A) \subset A^\star$ for every subset A of X. Moreover, if $A \in \mathcal{H}$, then $\Psi_{\mathcal{H}}(A) = \emptyset$.*

Proof. Suppose that $x \in \Psi_{\mathcal{H}}(A)$ and $x \notin A^\star$. Since $x \in \Psi_{\mathcal{H}}(A)$, there exists a $\mu-$open set U containing x such that $U - A \in \mathcal{H}$. Since $x \notin A^\star$, there is a $\mu-$open set V containing x such that $V \cap A \in \mathcal{H}$. Therefore, $(U \cap V) \cap A \in \mathcal{H}$ and $(U \cap V) - A \in \mathcal{H}$. By hypothesis, $\mathcal{H}$ is strongly $\mu-$codense and so $U \cap V = (U \cap V - A) \cup (U \cap V \cap A) \in \mathcal{H}$ implies that $U \cap V = \emptyset$, a contradiction to the fact that $x \in U \cap V$. Hence $x \in A^\star$ so that $\Psi_{\mathcal{H}}(A) \subset A^\star$. Since $A \in \mathcal{H}$, by Lemma 1.1(a), $\Psi_{\mathcal{H}}(A) \subset X - \mathcal{M}_\mu$ and so $\Psi_{\mathcal{H}}(A) = \emptyset$. $\square$

Theorem 2.3. *Let (X, μ) be a space with an ideal $\mathcal{H}$. Then the following are equivalent.*

(a) *$\mathcal{H}$ is strongly $\mu-$codense.*

(b) *$\mu_{\mathcal{A}} \subset \mathcal{H}O(\mu)$.*

(c) *$\mu^\star \subset \mathcal{H}O(\mu)$.*

Proof. (a)⇒(b). Suppose that $A \in \mu_{\mathcal{A}}$. Then by Lemma 2.2 and Lemma 1.1(c), $A \subset i_\mu c_\mu \Psi_{\mathcal{H}}(A) \subset i_\mu c_\mu(A^\star) = i_\mu(A^\star)$ and so A is $\mathcal{H}-$open.

(b)⇒(c). Follows from Theorem 2.2(c).

(c)⇒(a). Suppose A is $\mu-$open. Then $A \in \mu^\star$ and so $A \subset i_\mu(A^\star)$, by hypothesis. Hence $A \subset A^\star$ and so $\mathcal{H}$ is strongly $\mu-$codense. $\square$

Lemma 2.3. *Let (X, μ) be a space with a hereditary class $\mathcal{H}$. Then the following hold.*

(a) *If* $\mathcal{H}$ *is strongly* $\mu-$*codence and* $A \subset X$ *is* $\mu-$*closed, then* $i_\mu(A) = \Psi_{\mathcal{H}}(A) = i^\star_\mu(A)$.
(b) *For any subset* A *of* X, $i^\star_\mu(A) = A \cap \Psi_{\mathcal{H}}(A)$.

Proof. (a) Suppose that A is $\mu-$closed. Then by Lemma 1.3, $c^\star(X-A) = (X-A)^\star = c_\mu(X-A)$ which implies that $X - i^\star_\mu(A) = (X-A)^\star = X - i_\mu(A)$ which in turn implies that $i^\star_\mu(A) = \Psi_{\mathcal{H}}(A) = i_\mu(A)$.
(b) Let $x \in A \cap \Psi_{\mathcal{H}}(A)$. Then $x \in A$ and $x \in \Psi_{\mathcal{H}}(A)$. Since $x \in \Psi_{\mathcal{H}}(A)$, there exists $M_x \in \mu$ containing x such that $M_x - A \in \mathcal{H}$. Therefore, $x \in M_x - (M_x - A) \subset A$. Since β is a basis for $\mu^\star$ and $M_x - (M_x - A) \in \beta$, $x \in i^\star_\mu(A)$ where $i^\star_\mu$ is the interior operator in $(X, \mu^\star)$. Conversely, assume that $x \in i^\star_\mu(A)$. Then there exists a $\mu^\star-$open set M_x containing x and $H \in \mathcal{H}$ such that $x \in M_x - H \subset A$. Now $M_x - H \subset A$ implies that $M_x - A \subset H$ which in turn implies that $M_x - A \in H$ and so $x \in \mathcal{H}(A)$. Therefore, $x \in A \cap \Psi_{\mathcal{H}}(A)$. Hence $A \cap \Psi_{\mathcal{H}}(A) = i^\star_\mu(A)$. □

Theorem 2.4. *Let* (X, μ) *be a quasi-topological space with a* $\mu-$*codense ideal* $\mathcal{H}$. *Then* $\mu_{\mathcal{A}} = \alpha(\mu^\star)$.

Proof. If $A \in \mu_{\mathcal{A}}$, then $A \subset i_\mu c_\mu \Psi_{\mathcal{H}}(A)$. By Lemma 2.2, $A \subset i_\mu c_\mu(\Psi_{\mathcal{H}}(A) \cap A^\star)$ which implies that $A \subset i_\mu c_\mu(\Psi_{\mathcal{H}}(A) \cap A)^\star \subset i_\mu c^\star_\mu(i^\star_\mu(A)) \subset i^\star_\mu c^\star_\mu i^\star_\mu(A)$, by Lemma 2.3(b). Thus, $A \in \alpha(\mu^\star)$ and so $\mu_{\mathcal{A}} \subset \alpha(\mu^\star)$. Conversely, let $A \in \alpha(\mu^\star)$. Then $A \subset i^\star_\mu c^\star_\mu i^\star_\mu(A) = i^\star_\mu c^\star_\mu(A \cap \Psi_{\mathcal{H}}(A)) \subset i^\star_\mu c^\star_\mu \Psi_{\mathcal{H}}(A) = i^\star_\mu c_\mu \Psi_{\mathcal{H}}(A) = i_\mu c_\mu \Psi_{\mathcal{H}}(A)$. Therefore, $A \in \mu_{\mathcal{A}}$. Hence $\mu_{\mathcal{A}} = \alpha(\mu^\star)$. □

Example 2.3. *Consider the space* (X, μ) *where* $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{d\}, \{a, b, c\}, \{c, d\}, X\}$ *and* $\mathcal{H} = \{\emptyset, \{c\}, \{d\}\}$. *Clearly,* $\mathcal{H}$ *is not strongly* $\mu-$ *codense. Here* $\mu_{\mathcal{A}} = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$ *which is not a quasi-topology. If* $A = \{a, d\}$, *then* $A \in \mu_{\mathcal{A}}$. *Since* $i^\star_\mu c^\star_\mu i^\star_\mu(A) = i^\star_\mu c^\star_\mu(\{d\}) = i^\star_\mu(\{d\}) = \{d\} \not\supseteq \{a, d\}$, A *is not* $\alpha-$*open in* $(X, \mu^\star)$.

The following Theorem 2.5 gives characterizations of $\mu-$codense ideals in a quasi-topological space.

Theorem 2.5. *Let (X, μ) be a quasi-topological space and $\mathcal{H}$ be an ideal. Then the following are equivalent.*
(a) *$\mathcal{H}$ is $\mu-$codense.*
(b) $\mu_{\mathcal{A}} \cap \mathcal{H} = \{\emptyset\}$.
(c) *$A \subset A^\star$ for $A \in \mu_{\mathcal{A}}$.*

Proof. (a)⇒(b). Suppose $A \in \mu_{\mathcal{A}} \cap \mathcal{H}$. Then $A \in \mu_A$ and $A \in \mathcal{H}$. By Lemma 2.2, $A \in \mathcal{H}$ implies that $\Psi_{\mathcal{H}}(A) = \emptyset$. Since $A \in \mu_{\mathcal{A}}, A \subset i_\mu c_\mu \Psi_{\mathcal{H}}(A) = i_\mu c_\mu(\emptyset) = i_\mu(X - \mathcal{M}_\mu) = \emptyset$ and so $A = \emptyset$.
(b)⇒(c). Let $A \in \mu_{\mathcal{A}}$. Suppose $x \notin A^\star$. Then there exists $M \in \mu$ containing x such that $M \cap A \in \mathcal{H}$. Since $M \in \mu,\ M \in \mu_{\mathcal{A}}$ and so $M \cap A \in \mu_{\mathcal{A}}$, by Theorem 2.1. Hence $M \cap A = \emptyset$, which implies that $x \notin A$. Therefore, $A \subset A^\star$ for $A \in \mu_{\mathcal{A}}$.
(c)⇒(a). Let $A \in \mu \cap \mathcal{H}$. Then $A \in \mu$ implies that $A \subset A^\star$, by (c). Also, by Lemma 1.1(a), $A \in \mathcal{H}$ implies that $A^\star = X - \mathcal{M}_\mu$. Therefore, $A \subset X - \mathcal{M}_\mu$ so that $A \cap \mathcal{M}_\mu = \emptyset$ which implies $A = \emptyset$. Hence $\mathcal{H}$ is $\mu-$codense. □

Theorem 2.6. *Let (X, μ) be a space with a strongly $\mu-$codense hereditary class $\mathcal{H}$ and $A \subset X$. Then $\Psi_{\mathcal{H}}(A) \neq \emptyset$ if and only if $i^\star_\mu(A) \neq \emptyset$.*

Proof. Suppose $\Psi_{\mathcal{H}}(A) \neq \emptyset$. Then there exists $\emptyset \neq M \in \mu$ such that $M - A \in \mathcal{H}$. If $M - A = P$ for some $P \in \mathcal{H}$, then $M - P \subset A$. Since $M \in \mu$ and $P \in \mathcal{H}$, $M - P \in \beta$. Therefore, $M - P \in \mu^\star$ and so A has nonempty $\mu^\star-$interior. Conversely, suppose that A has nonempty $\mu^\star-$interior. If $x \in A$, then there exists $M \in \mu$ containing x and $P \in \mathcal{H}$ such that $M - P \subseteq A$. Since $M - A \subset P,\ M - A \in \mathcal{H}$ and so $\Psi_{\mathcal{H}}(A) \neq \emptyset$. □

3. $\Psi_{\mathcal{H}}C-$SET

A subset A of a space (X,μ) is said to be $\Psi_{\mathcal{H}}C-set$ if $A \subset c_\mu\Psi_{\mathcal{H}}(A)$. We denote the family of all $\Psi_{\mathcal{H}}C-$sets in (X,μ) by $\Psi_{\mathcal{H}}C(X)$. Clearly, every $\Psi_{\mathcal{H}}\mathcal{A}-set$ is a $\Psi_{\mathcal{H}}C-$ set. Every $\mu-$semiopen set is a $\Psi_{\mathcal{H}}C-$set and so every $\mu-$open set is a $\Psi_{\mathcal{H}}C-$set. But the converse need not be true as shown by the following Example 3.1.

Example 3.1. *Consider the space (X,μ) with the hereditary class $\mathcal{H}$ where $X = \{a,b,c,d\}$, $\mu = \{\phi,\{a\},\{a,b\},\{b,c\},\{a,b,c\}\}$ and $\mathcal{H} = \{\phi,\{a\},\{b\}\}$. Since $\Psi_{\mathcal{H}}(\{b\}) = X$, $\{b\}$ is a $\Psi_{\mathcal{H}}C-$set. But it is neither $\mu-$open nor $\mu-$semiopen.*

The following Theorem 3.1 shows that $\Psi_{\mathcal{H}}C-$sets are $\mu-\beta-$open sets, if $\mathcal{H}$ is a strongly $\mu-$codense ideal. Example 3.2 below shows that the converse of Theorem 3.1 need not be true and Example 3.3 below shows that the condition strongly $\mu-$codense on $\mathcal{H}$ cannot be dropped.

Theorem 3.1. *Let (X,μ) be a space with a strongly $\mu-$codense ideal $\mathcal{H}$. Then every $\Psi_{\mathcal{H}}C-$set is a $\mu-\beta-$open set.*

Proof. If A is a $\Psi_{\mathcal{H}}C-$set, then by Lemma 2.2, $A \subset c_\mu\Psi_{\mathcal{H}}(A) \subset c_\mu i_\mu(A^\star) \subset c_\mu i_\mu c_\mu(A)$. Therefore, A is a $\mu-\beta-$open set. □

Example 3.2. *Let $X = \{a,b,c,d\}$ $\mu = \{\phi,\{a,b\},\{b,c\},\{a,b,c\}\}$ and $\mathcal{H} = \{\phi,\{a\},\{c\},\{a,c\}\}$. Then $\mathcal{H}$ is a strongly $\mu-$codense ideal. If $A = \{a,c\}$, then $c_\mu i_\mu c_\mu(A) = c_\mu i_\mu(X) = c_\mu(\{a,b,c\}) \supset A$. Thus, $A \subset c_\mu i_\mu c_\mu(A)$ and so A is $\mu-\beta-$open. Again, $c_\mu\Psi_{\mathcal{H}}(A) = c_\mu(X-X) = c_\mu(\emptyset) = \{d\} \not\supseteq A$. Hence A is not a $\Psi_{\mathcal{H}}C-$set.*

Example 3.3. *Let $X = \{a,b,c,d\}$ $\mu = \{\phi,\{a,b\},\{b,c\},\{a,b,c\}\}$ and $\mathcal{H} = \{\phi,\{b\},\{c\}\}$. Here $\mathcal{H}$ is not a strongly $\mu-$codense hereditary class. If $A = \{a\}$, then $\Psi_{\mathcal{H}}(A) = \{a,b\}$ which implies that $c_\mu\Psi_{\mathcal{H}}(A) = X \supset A$. Therefore, A is a $\Psi_{\mathcal{H}}C-$set. But $c_\mu i_\mu c_\mu(A) = c_\mu i_\mu(\{a,d\}) = c_\mu(\phi) = \{d\} \not\supset A$ and so A is not a $\mu-\beta-$open set.*

In [4], it is established that the intersection of a $\mu-\alpha-$open set with a $\mu-$semiopen set is a $\mu-$semiopen set. The following Theorem 3.2 is analogous to this result.

Theorem 3.2. *Let (X, μ) be a quasi-topological space with an ideal $\mathcal{H}$. Then the intersection of a $\mu-\alpha-$open set with a $\Psi_{\mathcal{H}}C-$set is a $\Psi_{\mathcal{H}}C-$set.*

Proof. Let A be a $\mu-\alpha-$open set and B be a $\Psi_{\mathcal{H}}C-$set. Then $A \subset i_\mu c_\mu i_\mu(A)$ and $B \subset c_\mu \Psi_{\mathcal{H}}(B)$. Now $A \cap B \subset i_\mu c_\mu i_\mu(A) \cap c_\mu \Psi_{\mathcal{H}}(B) \subset c_\mu(i_\mu c_\mu i_\mu(A) \cap \Psi_{\mathcal{H}}(B)) \subset c_\mu(c_\mu i_\mu(A) \cap \Psi_{\mathcal{H}}(B)) \subset c_\mu c_\mu(i_\mu(A) \cap \Psi_{\mathcal{H}}(B)) = c_\mu(i_\mu(A) \cap \Psi_{\mathcal{H}}(B)) \subset c_\mu(\Psi_{\mathcal{H}}(i_\mu(A)) \cap \Psi_{\mathcal{H}}(B)) \subset c_\mu(\Psi_{\mathcal{H}}(A) \cap \Psi_{\mathcal{H}}(B)) = c_\mu \Psi_{\mathcal{H}}(A \cap B)$, by Lemma 2.1. Therefore, $A \cap B$ is a $\Psi_{\mathcal{H}}C-$set. □

The following Theorem 3.3 gives a characterization of $\Psi_{\mathcal{H}}C-$ sets and Theorem 3.4 below characterizes strongly $\mu-$codense ideal in terms of $\Psi_{\mathcal{H}}C-$sets.

Theorem 3.3. *Let (X, μ) be a space with a hereditary class $\mathcal{H}$. Then the following are equivalent.*

(a) *A is a $\Psi_{\mathcal{H}}C-$ set.*

(b) *$A \subset X - i_\mu(X - A)^\star$.*

(c) *$A \subset c_\mu i_\mu \Psi_{\mathcal{H}}(A)$.*

Proof. (a)⇔(b). A is a $\Psi_{\mathcal{H}}C-$set if and only if $A \subset c_\mu \Psi_{\mathcal{H}}(A)$ if and only if $A \subset c_\mu(X - (X - A)^\star)$ if and only if $A \subset X - i_\mu(X - A)^\star$.

(a)⇔(c). Follows from the fact that $\Psi_{\mathcal{H}}(A)$ is $\mu-$open. □

Theorem 3.4. *Let (X, μ) be a space with an ideal $\mathcal{H}$. Then the following are equivalent.*

(a) *$\mathcal{H}$ is strongly $\mu-$codense.*

(b) *For every $\Psi_{\mathcal{H}}C-$set A, $A \subset A^\star$.*

Proof. (a) $\Rightarrow$ (b). Suppose that A is a $\Psi_{\mathcal{H}}C-$set. Then $A \subset c_\mu \Psi_{\mathcal{H}}(A) \subset c_\mu(A^\star) = A^\star$, by Lemma 2.2 and Lemma 1.1(c) so $A \subset A^\star$.
(b) $\Rightarrow$ (a). Let A be a $\mu-$open subset of X. Then A is a $\Psi_{\mathcal{H}}C-$set and so $A \subset A^\star$, by (b). Hence $\mathcal{H}$ is strongly $\mu-$codense, by Lemma 1.3. □

The following Theorem 3.5 gives a characterization of $\mu-$codense hereditary class in terms of $\mathcal{H}-$open sets of a quasi-topological space and Theorem 3.6 shows that in a quasi-topological space (X, μ) with a $\mu-$codense ideal $\mathcal{H}$, the family of all $\Psi_{\mathcal{H}}C-$sets is nothing but the family of all $\mu^\star-$semiopen sets in X. Corollary 3.1 below follows from the fact that $\sigma(\mu) \subset \sigma(\mu^\star)$.

Theorem 3.5. *Let (X, μ) be a quasi-topological space. Then the following are equivalent.*
(a) $\mathcal{H}$ *is (strongly) $\mu-$codense.*
(b) $\mathcal{H}O(\mu) = \pi(\mu^\star)$.

Proof. (a)$\Rightarrow$(b). Suppose $\mathcal{H}$ is (strongly) $\mu-$codense. Now, $A \in \mathcal{H}O(\mu)$ implies that $A \subset i_\mu(A^\star) \subset i_\mu c^\star_\mu(A) \subset i^\star_\mu c^\star_\mu(A)$ and so $A \in \pi(\mu^\star)$. If $A \in \pi(\mu^\star)$, then $A \subset i^\star_\mu c^\star_\mu(A) \subset i^\star_\mu c_\mu(A) = i_\mu c_\mu(A)$ which implies that $A \in \pi(\mu)$. Therefore, $A \subset A^\star$, by Lemma 1.5. Hence $A \subset i_\mu c_\mu(A)$ implies that $A \subset i_\mu(A^\star)$ and so $A \in \mathcal{H}O(\mu)$.
(b)$\Rightarrow$(a). Suppose $A \in \mu$. Then $A \in \pi(\mu^\star)$ and so $A \in \mathcal{H}O(\mu)$ which implies that $A \subset i_\mu(A^\star) \subset A^\star$. Hence $\mathcal{H}$ is (strongly) $\mu-$codense. □

Theorem 3.6. *Let (X, μ) be a quasi-topological space with a $\mu-$codense ideal $\mathcal{H}$. Then $\sigma(\mu^\star) = \Psi_{\mathcal{H}}C(X)$.*

Proof. Let $A \in \sigma(\mu^\star)$. Then $A \subset c^\star_\mu i^\star_\mu(A) \subset c_\mu i^\star_\mu(A) = c_\mu(A \cap \Psi_{\mathcal{H}}(A))$, by Theorem 2.3(b) and so $A \subset c_\mu \Psi_{\mathcal{H}}(A)$. Hence $\sigma(\mu^\star) \subset \Psi_{\mathcal{H}}C(X)$. Conversely, let $A \in \Psi_{\mathcal{H}}C(X)$. If $x \notin c^\star_\mu i^\star_\mu(A)$, then $U \cap i^\star_\mu(A) = \emptyset$ for some $\mu^\star-$open set U containing x which implies that $(A \cap \Psi_{\mathcal{H}}(A)) \cap U = \emptyset$. Since $U \in \mu^\star$, there exists $G \in \mu$ and $H \in \mathcal{H}$ such that

$x \in G-H \subset U$. Now $(A \cap \Psi_{\mathcal{H}}(A)) \cap U = \emptyset$ implies that $A \cap \Psi_{\mathcal{H}}(A) \cap (G-H) = \emptyset$ which implies that $A \cap \Psi_{\mathcal{H}}(A) \cap G \subset H$ which in turn implies that $(A \cap \Psi_{\mathcal{H}}(A) \cap G)^\star \subset H^\star = X - \mathcal{M}_\mu$ and so $A^\star \cap \Psi_{\mathcal{H}}(A) \cap G \subset X - \mathcal{M}_\mu$, by Lemma 1.4. Thus, $A^\star \cap \Psi_{\mathcal{H}}(A) \cap G = \emptyset$ and so $\Psi_{\mathcal{H}}(A) \cap G = \emptyset$, by Lemma 2.2. Therefore, $x \notin c_\mu \Psi_{\mathcal{H}}(A)$ so that $x \notin A$. Thus, $A \in \sigma(\mu^\star)$ which implies that $\Psi_{\mathcal{H}}C(X) \subset \sigma(\mu^\star)$. □

Corollary 3.1. *Let (X, μ) be a quasi-topological space with a $\mu-$codense ideal $\mathcal{H}$. Then $\sigma(\mu) \subset \Psi_{\mathcal{H}}C(X)$.*

Corollary 3.2. *Let (X, μ) be a quasi-topological space with a $\mu-$codense ideal $\mathcal{H}$. Then $\mu_{\mathcal{A}} = \mathcal{H}O(\mu) \cap \Psi_{\mathcal{H}}C(X)$.*

Proof. We know that $\alpha(\mu) = \sigma(\mu) \cap \pi(\mu)$[8]. Since $\mathcal{H}$ is strongly $\mu-$codense and hence $\mu-$codense, by Theorem 3.5, $\pi(\mu^\star) = \mathcal{H}O(\mu)$ and by Theorem 3.6, $\sigma(\mu^\star) = \Psi_{\mathcal{H}}C(X)$. Therefore, the proof follows from Theorem 2.4. □

Theorem 3.7. *Let (X, μ) be a quasi-topological space with an ideal $\mathcal{H}$ and $A, B \subset X$. If $A \in \mu_A$, then $A \cap B \in \Psi_{\mathcal{H}}C(X)$ for every $B \in \Psi_{\mathcal{H}}C(X)$.*

Proof. Let $A \in \mu_A$ and $B \in \Psi_{\mathcal{H}}C(X)$. Then $A \subset i_\mu c_\mu \Psi_{\mathcal{H}}(A)$ and $B \subset c_\mu \Psi_{\mathcal{H}}(B)$. Suppose $x \in A \cap B$ and U be a $\mu-$open set containing x. Since $x \in A$ and $A \subset i_\mu c_\mu \Psi_{\mathcal{H}}(A)$, $i_\mu c_\mu \Psi_{\mathcal{H}}(A)$ is a $\mu-$open set containing x. Since μ is a quasi-topology, $U \cap i_\mu c_\mu \Psi_{\mathcal{H}}(A)$ is also a $\mu-$open set containing x. Since $x \in c_\mu \Psi_{\mathcal{H}}(B)$, $U \cap i_\mu c_\mu \Psi_{\mathcal{H}}(A) \cap \Psi_{\mathcal{H}}(B) \neq \emptyset$. Let $V = (U \cap i_\mu c_\mu \Psi_{\mathcal{H}}(A)) \cap \Psi_{\mathcal{H}}(B)$. Then V is an $\mu-$open set containing x such that $V \subseteq i_\mu c_\mu \Psi_{\mathcal{H}}(A) \subseteq c_\mu \Psi_{\mathcal{H}}(A)$. Therefore, $V \cap \Psi_{\mathcal{H}}(A) \neq \emptyset$ which implies that $U \cap i_\mu c_\mu \Psi_{\mathcal{H}}(A) \cap \Psi_{\mathcal{H}}(B) \cap \Psi_{\mathcal{H}}(A) \neq \emptyset$ which in turn implies that $U \cap \Psi_{\mathcal{H}}(A \cap B) \neq \emptyset$, by Lemma 2.1. Hence $x \in c_\mu \Psi_{\mathcal{H}}(A \cap B)$ and so $A \cap B \subseteq c_\mu \Psi_{\mathcal{H}}(A \cap B)$. Thus, $A \cap B \in \Psi_{\mathcal{H}}C(X)$. □

Theorem 3.8. *Let (X, μ) be a strong generalized space with a strong $\mu-$codense hereditary class $\mathcal{H}$ and $A, B \subset X$. If $A \cap B \in \Psi_{\mathcal{H}}C(X)$ for all $B \in \Psi_{\mathcal{H}}C(X)$,then $A \in \mu_A$.*

Proof. Since $\emptyset \in \mathcal{H}$, by Lemma 1.1(a), $c_\mu\Psi_{\mathcal{H}}(X) = c_\mu(M_\mu) = X$ which implies that $X \in \Psi_{\mathcal{H}}C(X)$ and so $A \in \Psi_{\mathcal{H}}C(X)$, by hypothesis. Suppose that $x \in A$ and $x \notin i_\mu c_\mu \Psi_{\mathcal{H}}(A)$. Then $x \in X - i_\mu c_\mu \Psi_{\mathcal{H}}(A) = c_\mu(X - c_\mu\Psi_{\mathcal{H}}(A))$. Let $B = X - c_\mu\Psi_{\mathcal{H}}(A)$, then $x \in c_\mu(B)$ so that $V_x \cap B \neq \emptyset$ for every $\mu-$open set V_x containing x. Since B is $\mu-$open, $B \subset \Psi_{\mathcal{H}}(B)$ implies that $V_x \cap B \subset V_x \cap \Psi_{\mathcal{H}}(B)$ and so $V_x \cap \Psi_{\mathcal{H}}(B) \neq \emptyset$ which implies that $x \in c_\mu\Psi_{\mathcal{H}}(B) \subset c_\mu\Psi_{\mathcal{H}}(\{x\} \cup B)$ implies that $\{x\} \subset c_\mu\Psi_{\mathcal{H}}(\{x\} \cup B)$. Also, $B \subset c_\mu\Psi_{\mathcal{H}}(B)$ implies that $B \subset c_\mu\Psi_{\mathcal{H}}(\{x\} \cup B)$. Hence $\{x\} \cup B \subset c_\mu\Psi_{\mathcal{H}}(\{x\} \cup B)$. Therefore, $\{x\} \cup B \in \psi_{\mathcal{H}}C(X)$. Therefore, by hypothesis, $A \cap (\{x\} \cup B) \in \Psi_{\mathcal{H}}C(X)$. If possible, suppose there exists $y \in X$ such that $x \neq y$ and $y \in A \cap (\{x\} \cup B)$. Then $y \in A$ and $y \in B$. Now $y \in A$ implies that $y \in c_\mu\Psi_{\mathcal{H}}(A)$, a contradiction to $y \in B$. Therefore, $A \cap (\{x\} \cup B) = \{x\}$ so that $\{x\} \in \Psi_{\mathcal{H}}C(X)$. Hence $\{x\} \subset c_\mu\Psi_{\mathcal{H}}\{x\}$. If $\Psi_{\mathcal{H}}\{x\} = \emptyset$, then $\{x\} \subset c_\mu\Psi_{\mathcal{H}}\{x\} \subset c_\mu\emptyset = \emptyset$, since μ is strong. Hence $\Psi_{\mathcal{H}}\{x\} \neq \emptyset$. Therefore, $\{x\}$ contains a nonempty $\mu^\star-$interior. Hence $\{x\} = i_\mu^\star\{x\} \subset i_\mu^\star c_\mu\Psi_{\mathcal{H}}\{x\} = i_\mu c_\mu\Psi_{\mathcal{H}}\{x\} = i_\mu c_\mu\Psi_{\mathcal{H}}(A \cap (\{x\} \cup B)) \subseteq i_\mu c_\mu\Psi_{\mathcal{H}}(A)$. Therefore, $x \in i_\mu c_\mu\Psi_{\mathcal{H}}(A)$, a contradiction to our assumption. Hence $A \subset i_\mu c_\mu\Psi_{\mathcal{H}}(A)$ and so $A \in \mu_A$. □

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