

COMPLEMENT GRAPHS FOR ZERO - DIVISORS OF $C(X)$

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ABSTRACT. Let X be a completely regular Hausdorff space and let $C(X)$ be the ring of all continuous real valued functions defined on X . The complement graph for the zero-divisors in $C(X)$ is a simple graph in which two zero-divisor functions are adjacent if their product is non-zero.

In this article, the complement graph for the zero-divisor graph of $C(X)$ and its line graph are studied. It is shown that if X has more than 2 points, then these graphs are connected with radius 2, and diameter less than or equal to 3. The girth is also calculated for them to be 3, and it is shown that they are always triangulated and hypertriangulated. Bounds for the dominating number and clique number are also found for them in terms of the density number of X .

1. INTRODUCTION

Let X be a completely regular Hausdorff space, βX the Stone-Čech compactification of X , $C(X)$ the ring of all continuous real valued functions defined on X and $C^*(X)$ the subring of all bounded functions in $C(X)$.

For each $f \in C(X)$, let $Z(f) = \{x \in X : f(x) = 0\}$, $\text{coz}(f) = X \setminus Z(f)$, and $\text{Supp}(f) = \text{Cl}_X \text{coz}(f)$ and let $\text{Ann}(f) = \{g \in C(X) : fg = 0\}$. For each $A \subseteq \beta X$,

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let $O^A = \{f \in C(X) : A \subseteq \text{Int}_{\beta X} Cl_{\beta X} Z(f)\}$, and let $M^A = \{f \in C(X) : A \subseteq Cl_{\beta X} Z(f)\}$. For $p \in X$, $O^p = O_p = \{f \in C(X) : p \in \text{Int}_X Z(f)\}$ and $M^p = M_p = \{f \in C(X) : f(p) = 0\}$. A zero-set Z in X is said to be a **middle zero-set** if there exist two proper zero-sets E and F such that $Z = E \cap F$ and $E \cup F = X$. A space X is **middle P-space** if every non-empty middle zero-set in X has non-empty interior. The **density** of X , denoted by $d(X)$, is the smallest cardinal number of the form $|A|$, where A is a dense subset of X . The **weight** of X , denoted by $\varpi(X)$, is the smallest cardinality of the form $|\beta|$, where β is a base for X . The **cellularity** of the space X is $c(X) = \sup \{|F| : F \text{ is a family of pairwise disjoint non-empty open subsets of } X\}$. Let \aleph_0 denotes the infinite countable cardinal number, and let \aleph_1 denotes the first uncountable cardinal number. If \aleph_n is an infinite cardinal number, then let $\aleph_{n+1} = 2^{\aleph_n}$.

For all notations and undefined terms concerning the ring $C(X)$ in this paper the reader may consult [7].

If $|X| = 1$, then $C(X)$ is a field isomorphic to \mathbb{R} . So, in this article we will always assume that $|X| > 1$.

Let R be a commutative ring with unity 1. Let $\text{Spec}(R)$ be the spectrum of the prime ideals. For each $a \in R$ and any ideal I of R , let $V(a) = \{P \in \text{Spec}(R) : a \in P\}$, $D(a) = \text{Spec}(R) \setminus V(a)$ and $V(I) = \bigcap_{a \in I} V(a)$. A proper prime ideal in a ring R that contains no smaller prime ideal is called a **minimal prime ideal**, and the set of all minimal prime ideals in R will be denoted by $\text{Min}(R)$. For any subset S of a ring R , we define the **hull** of S to be $h(S) = \{P \in \text{Min}(R) : S \subset P\}$. An ideal I in R is called **pure** if for each $f \in I$, there exists $g \in I$ such that $f = fg$. The ring R is called **von Neuman regular ring** if for every $f \in R$, then there exists $g \in R$ such that $f = f^2g$. While if for every non-unit $f \in R$ there exist $g \in R \setminus \{1\}$ such that $f = fg$, R will be called an **almost regular ring**. The following are well-known facts:

The ring $C(X)$ is von Neuman regular if and only if X is a **P-space** (every G_δ -set is open) if and only if $Z(f)$ is open for every $f \in C(X)$.

The ring $C(X)$ is almost regular if and only if X is an **almost P-space** (every non-empty G_δ -set has non-empty interior) if and only if $\text{Int}_X Z(f) \neq \phi$, whenever $Z(f) \neq \phi$.

Let $Z(R)$ be the set of zero-divisors of R , and $Z^*(R) = Z(R) \setminus \{0\}$. The zero-divisor graph of R , $\Gamma(Z^*(R))$, usually written $\Gamma(R)$, is the simple graph in which each element of $Z^*(R)$ is a vertex, i.e. $V(\Gamma(R)) = Z^*(R)$, and two distinct vertices f and g are adjacent if and only if $fg = 0$. This graph was studied and characterized in [3] and [4]. If G is a graph, then the complement graph \overline{G} is defined on the same vertex set, but two distinct vertices f and g are adjacent if and only if f and g are not adjacent in G . The **line graph** of a graph G , denoted by $L(G)$, is a graph whose vertices are the edges of G and two vertices of $L(G)$ are adjacent wherever the corresponding edges of G are incident to a common vertex in G , see [8]. In this case, if a, b are adjacent vertices in G , then $[a, b]$ is a vertex in $L(G)$.

Let G be a graph and let u and v be two distinct vertices in G . The **distance** $d(u, v)$ between u and v is the length of a shortest path joining them in G ; if no such path exists, we set $d(u, v) = \infty$. The greatest distance between any two vertices in G is the **diameter** of G , denoted by $\text{diam}(G)$. The **associate number** $e(u)$ of a vertex u of G is defined to be $e(u) = \sup \{d(u, v) : u \neq v\}$. A vertex is **center** in G if its greatest distance from any other vertex is as small as possible, this distance is the **radius** of G , denoted by $\rho(G)$. G is **connected** if any two of its vertices are linked by a path in G , otherwise G is **disconnected**.

A **cycle** in G is a closed walk such that no vertex, except the initial and the final vertex, appears more than once. While the **girth** of G , which is denoted by $gr(G)$, is the length of the shortest cycle in G . If G has no cycles, then we set $gr(G) = \infty$. If u and v are two vertices in G , by $c(u, v)$, we mean the length of the smallest cycle

containing u and v . We write $c(u, v) = \infty$ if there is no cycle containing u and v . An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a **chord** of that cycle. The graph G is **chordal** if every cycle of length greater than three has a chord. G is called **triangulated** if each vertex in G is a vertex of a triangle. G is called **hypertriangulated** if each edge in G is an edge of a triangle.

A simple graph G in which all the vertices of G are pairwise adjacent is called **complete graph**. A complete graph on n vertices is denoted by K_n . A maximal complete subgraph of G is called a **clique**. The **clique number** of G , denoted by $\omega(G)$, is defined by $\omega(G) = \sup \{|H| : H \text{ is a complete subgraph of } G\}$. A **dominating set** in G is a set of vertices A such that every vertex outside A is adjacent to at least one vertex in A . The **dominating number** of G , denoted by $dt(G)$, is the smallest number of the form $|A|$, where A is a dominating set. For distinct vertices a and b in G , we say that a and b are **orthogonal**, written $a \perp b$ if a and b are adjacent and there is no vertex c of G which is adjacent to both a and b . G is called **complemented** if for each vertex a of G , there is a vertex b of G (called a complement of a) such that $a \perp b$. For all notations and undefined terms concerning graph theory in this paper the reader may consult [10].

This article is a continuation of the work done in [5] and [2], where the zero-divisor graph of $C(X)$ and the line graph of the zero-divisor graph of $C(X)$ were studied respectively.

In this article, we start with $\overline{\Gamma(C(X))}$, the complement graph of the zero-divisor graph of $C(X)$. We will study the cases at which $\overline{\Gamma(C(X))}$ is connected, and we will find its diameter, radius and girth. We will show that it is always triangulated and hypertriangulated. We will also show that it is chordal if and only if $|X|$ is 2 or 3. We will find bounds for the clique number, and show that its dominating number is 2.

Then we will study $L(\overline{\Gamma(C(X))})$, the line graph for the complement graph of the zero-divisor graph of $C(X)$. We will study when $L(\overline{\Gamma(C(X))})$ is connected, find its diameter, radius and girth. We will show that it is never chordal, but it is always triangulated and hypertriangulated. We will find bounds for its clique number and dominating number.

This study shows the deep relations between algebra, topology and graph theory.

2. THE COMPLEMENT GRAPH FOR ZERO-DIVISORS OF $C(X)$

Note first that $f \in Z^*(C(X))$ if and only if $\text{Int}_X Z(f) \neq \emptyset$. In this section, we study $\overline{\Gamma(C(X))}$, the complement graph of the zero-divisor graph of $C(X)$. The vertex set of $\overline{\Gamma(C(X))}$ is $Z^*(C(X))$ and two distinct vertices $f, g \in Z^*(C(X))$ are adjacent in $\overline{\Gamma(C(X))}$ if and only if $fg \neq 0$.

2.1. Connectedness. Unlike the case of the zero-divisor graph, which is always connected, the complement of the zero-divisor graph may not be. So, in this section we will study when $\overline{\Gamma(C(X))}$ is connected, we will also calculate the diameter and radius of $\overline{\Gamma(C(X))}$.

Theorem 2.1. *For any space X , $\overline{\Gamma(C(X))}$ is connected with $\text{diam}(\overline{\Gamma(C(X))}) = 2$ if and only if $|X| > 2$.*

Proof. If $|X| = 1$, then $C(X)$ is a field and $\overline{\Gamma(C(X))}$ is empty. If $X = \{a, b\}$, then f is a vertex in $\overline{\Gamma(C(X))}$ if $f(a) = 0$ and $f(b) \neq 0$ or conversely. Thus $\overline{\Gamma(C(X))}$ is the disjoint union of the two complete subgraphs $A = \{f \in C(X) : f(a) = 0 \text{ and } f(b) \neq 0\}$ and $B = \{f \in C(X) : f(a) \neq 0 \text{ and } f(b) = 0\}$. Hence $\text{diam}(\overline{\Gamma(C(X))}) = \infty$. Now assume that $|X| > 2$ and let $f, g \in Z^*(C(X))$ such that $fg = 0$. Let $a \in \text{coz}(f)$, $b \in \text{coz}(g)$, and $c \notin \{a, b\}$. By regularity of X , there exists an open set U such that $c \in U \subseteq \text{Cl}_X U \subseteq X \setminus \{a, b\}$. By complete regularity of X there exist $h_1, h_2 \in C(X)$ such that $h_1(\text{Cl}_X U) = 0$, $h_1(a) = 1$, $h_2(\text{Cl}_X U) = 0$, and $h_2(b) = 1$. Let $h = h_1^2 + h_2^2$.

Then $h \in Z^*(C(X)) \setminus \{f, g\}$ and $f - h - g$ is a path of length 2 between f and g . Thus $\overline{\Gamma(C(X))}$ is connected with diameter 2. \square

Corollary 2.1. *If $|X| > 2$, then for any distinct vertices $f, g \in Z^*(C(X))$, there exists $h \in Z^*(C(X))$ such that h is adjacent to both f and g .*

Proof. If $fg = 0$, then Theorem 2.1 shows that there exists $h \in Z^*(C(X))$ such that $f - h - g$ is a path in $\overline{\Gamma(C(X))}$. If $fg \neq 0$, then let $h = 2f$ if $g \neq 2f$ and otherwise let $h = 3f$. Then $h \in Z^*(C(X)) \setminus \{f, g\}$ and $f - h - g$ is a path in $\overline{\Gamma(C(X))}$. \square

Corollary 2.2. *Let f, g be vertices in $\overline{\Gamma(C(X))}$. Then*

- (1) $d(f, g) = 1$ if and only if $\text{coz}(f) \cap \text{coz}(g) \neq \phi$ if and only if $Z(f) \cup Z(g) \neq X$.
- (2) $d(f, g) = 2$ if and only if $\text{coz}(f) \cap \text{coz}(g) = \phi$ if and only if $Z(f) \cup Z(g) = X$.

Using Proposition 2.2 of [9], one can conclude the following:

Corollary 2.3. *Let f, g be vertices in $\overline{\Gamma(C(X))}$. Then*

- (1) $d(f, g) = 1$ if and only if $D(f) \cap D(g) \neq \phi$.
- (2) $d(f, g) = 2$ if and only if $D(f) \cap D(g) = \phi$.

Now, we find the radius of $\overline{\Gamma(C(X))}$. If $|X| = 2$, then $e(f) = \infty$, for each $f \in Z^*(C(X))$, since $\overline{\Gamma(C(X))}$ is disconnected; and so $\rho(\overline{\Gamma(C(X))}) = \infty$.

Theorem 2.2. *For any space X with $|X| > 2$, $\rho(\overline{\Gamma(C(X))}) = 2$.*

Proof. It is clear that for any vertex f in $\overline{\Gamma(C(X))}$, we have $1 \leq e(f) \leq 2$, and so $1 \leq \rho(\overline{\Gamma(C(X))}) \leq 2$. Let $f \in C(X) \setminus \{0\}$. Since the set $\{\text{coz}(g) : g \in C(X) \setminus \{0\}\}$ is a base for open sets in X , we have: $fg \neq 0$ for each $g \in C(X) \setminus \{0\}$ if and only if $\text{coz}(f) \cap \text{coz}(g) \neq \phi$ for each $g \in C(X) \setminus \{0\}$ if and only if $\text{coz}(f)$ is dense in X if and only if $\text{Supp}(f) = X$ if and only if $\text{Int}_X Z(f) = \phi$ if and only if $f \notin Z^*(C(X))$. Thus for each $f \in Z^*(C(X))$, there exists $g \in Z^*(C(X))$ such that $fg = 0$ and so $2 \geq e(f) \geq d(f, g) = 2$. Thus $\rho(\overline{\Gamma(C(X))}) = 2$. \square

Note that in $\overline{\Gamma(C(X))}$, every vertex is a center.

2.2. Cycles. In this section, we will calculate the girth of $\overline{\Gamma(C(X))}$. We will show that it is always triangulated and hypertriangulated, and we will find the length of the shortest cycle joining any two non-adjacent vertices. Finally we will characterize the case in which $\overline{\Gamma(C(X))}$ is chordal.

It was shown in [5] that $\Gamma(C(X))$ is triangulated if and only if X has no isolated points, while $\Gamma(C(X))$ is hypertriangulated if and only if X is a connected middle P-space.

Theorem 2.3. *For each space X with $|X| > 1$, the graph $\overline{\Gamma(C(X))}$ is both triangulated and hypertriangulated.*

Proof. Let $f \in Z^*(C(X))$. Then $f - 2f - 3f - f$ is a cycle of length 3 in $\overline{\Gamma(C(X))}$, hence $\overline{\Gamma(C(X))}$ is triangulated.

Now let $f - g$ be any edge in $\overline{\Gamma(C(X))}$. Let $h = 2f$ if $g \neq 2f$ and otherwise let $h = 3f$. Then $h \in Z^*(C(X)) \setminus \{f, g\}$ and $f - g - h - f$ is a triangle in $\overline{\Gamma(C(X))}$, hence the edge $f - g$ is an edge in a triangle. Therefore $\overline{\Gamma(C(X))}$ is hypertriangulated. \square

The following corollary follows immediately concerning the girth of $\overline{\Gamma(C(X))}$.

Corollary 2.4. *If $|X| > 1$, then $gr(\overline{\Gamma(C(X))}) = 3$.*

Now, we find the length of the shortest cycle joining any two vertices in $\overline{\Gamma(C(X))}$.

Theorem 2.4. *Let f and g be two distinct vertices in $\overline{\Gamma(C(X))}$. Then*

- (1) $c(f, g) = 3$ if and only if $\text{coz}(f) \cap \text{coz}(g) \neq \phi$ if and only if $Z(f) \cup Z(g) \neq X$.
- (2) $c(f, g) = 4$ if and only if $\text{coz}(f) \cap \text{coz}(g) = \phi$ if and only if $Z(f) \cup Z(g) = X$.

Proof. (1) If $c(f, g) = 3$, then obviously, f and g are adjacent in $\overline{\Gamma(C(X))}$, i.e. $\text{coz}(f) \cap \text{coz}(g) \neq \phi$ and $Z(f) \cup Z(g) \neq X$. Conversely, if $\text{coz}(f) \cap \text{coz}(g) \neq \phi$ then f and g

are adjacent in $\overline{\Gamma(C(X))}$, and by Corollary 2.1 there exists $h \in Z^*(C(X))$ such that h is adjacent to both f and g , thus $f - g - h - f$ is a cycle containing f and g , i.e. $c(f, g) = 3$.

(2) If $c(f, g) = 4$, then it follows by (1) that $fg = 0$, which says that $\text{coz}(f) \cap \text{coz}(g) = \emptyset$. Conversely, given that $\text{coz}(f) \cap \text{coz}(g) = \emptyset$, then by (1) there is no cycle of length 3 containing f and g . Again by Corollary 2.1 there exists $h \in Z^*(C(X))$ such that h is adjacent to both f and g . Let $r \in \mathbb{R} \setminus \{0, 1\}$ such that $rh \notin \{f, g, h\}$, thus $f - h - g - rh - f$ is a cycle of length 4 and it is the smallest cycle containing f and g , hence $c(f, g) = 4$. \square

We now characterize the cases at which $\overline{\Gamma(C(X))}$ is chordal.

Theorem 2.5. *The graph $\overline{\Gamma(C(X))}$ is chordal if and only if $|X| \leq 3$.*

Proof. Assume $\overline{\Gamma(C(X))}$ is chordal graph and $|X| \geq 4$. Pick four distinct points x_1, x_2, x_3 , and x_4 in X . Since X is a completely regular Hausdorff space there exist four mutually disjoint open sets U_i , where $i \in \{1, 2, 3, 4\}$ and $x_i \in U_i$. Consider the functions $h_i \in Z^*(C(X))$ such that $h_i(x_i) = 1$, and $h_i(X \setminus U_i) = 0$ for each $i \in \{1, 2, 3, 4\}$. Clearly $h_i h_j = 0$, whenever $i \neq j$. Consider the functions $f_1 = h_1 + h_4$, $f_2 = h_1 + h_2$, $f_3 = h_2 + h_3$ and $f_4 = h_3 + h_4$. Then clearly $f_i \in Z^*(C(X))$ and $f_1 - f_2 - f_3 - f_4 - f_1$ is a chordless cycle since $f_1 f_3 = 0$ and $f_4 f_2 = 0$, a contradiction. Hence whenever $|X| \geq 4$, $\overline{\Gamma(C(X))}$ is never chordal. Conversely, we have two cases:

Case I: $X = \{a, b\}$. Then f is a vertex in $\overline{\Gamma(C(X))}$ if $f(a) = 0$ and $f(b) \neq 0$ or conversely. Thus $\overline{\Gamma(C(X))}$ is the disjoint union of the two complete subgraphs $A = \{f \in C(X) : f(a) = 0 \text{ and } f(b) \neq 0\}$ and $B = \{f \in C(X) : f(a) \neq 0 \text{ and } f(b) = 0\}$. Hence if C is an induced cycle in $\overline{\Gamma(C(X))}$, then it is contained in either the complete graph induced by A or B . Thus C has a chord if it is of length greater than 3. Hence $\overline{\Gamma(C(X))}$ is chordal.

Case II: $X = \{a, b, c\}$. Let C be an induced cycle in $\overline{\Gamma(C(X))}$ of length greater than 3.

Let $f_1 - f_2 - f_3 - f_4$ be a path in C with f_1, f_2, f_3 and f_4 are distinct. If $\text{coz}(f_1) \cap \text{coz}(f_3) \neq \phi$ or $\text{coz}(f_2) \cap \text{coz}(f_4) \neq \phi$, then we have a chord joining f_1 and f_3 or a chord joining f_2 and f_4 . So assume that $\text{coz}(f_1) \cap \text{coz}(f_3) = \phi$ and $\text{coz}(f_2) \cap \text{coz}(f_4) = \phi$. But in this case we must have $|\text{coz}(f_2)| = 2 = |\text{coz}(f_3)|$. If f_1 and f_4 are adjacent, then there must be a chord joining f_2 and f_4 . So assume that f_1 and f_4 are not adjacent, and hence there is a vertex f_5 which is adjacent to f_4 . If $\text{coz}(f_5)$ has only one element, then there is a chord joining f_5 and f_3 , while if $\text{coz}(f_5)$ has two elements, then either f_5 is adjacent to f_1 and f_2 or it is adjacent to f_2 and f_3 . In each case the cycle C has a chord and $\overline{\Gamma(C(X))}$ is chordal. \square

2.3. Dominating Sets. In this section, we will calculate the dominating number for $\overline{\Gamma(C(X))}$ and characterize the dominating sets. Further, we will give an estimation for the clique number.

Theorem 2.6. *If $|X| > 1$, then $dt(\overline{\Gamma(C(X))}) = 2$.*

Proof. It is clear that $dt(\overline{\Gamma(C(X))}) \neq 1$. Let $f \in Z^*(C(X))$ and let $a \in \text{Int}_X Z(f)$. So $f \in O^a$, which is a pure ideal, see [1]. Thus there exists $g \in O^a \subseteq Z(C(X))$, such that $f = fg$, i.e. $g = 1$ on $\text{Supp}(f)$. Therefore, $1 - g \in Z^*(C(X))$ and $\{g, 1 - g\}$ dominates $Z^*(C(X))$. \square

Note that if $f \in Z^*(C(X))$ such that $f = 1$ on a non-empty set with empty interior, then $1 - f \notin Z^*(C(X))$.

Theorem 2.7. *If $|X| > 1$, then the following statements are equivalent for $f, g \in Z^*(C(X))$:*

- (1) $\{f, g\}$ dominates $\overline{\Gamma(C(X))}$.
- (2) $\text{Supp}(f) \cup \text{Supp}(g) = X$.

$$(3) \text{ } Ann(f) \cap Ann(g) = \{0\}.$$

$$(4) \text{ } V(Ann(f)) \cup V(Ann(g)) = Spec(C(X)).$$

Proof. (1) \Rightarrow (2) Assume that $y \in X \setminus (Supp(f) \cup Supp(g))$. Let $h \in C(X)$ such that $h(y) = 1$ and $h(Supp(f) \cup Supp(g)) = 0$. Then $h \in Z^*(C(X))$, $h \notin \{f, g\}$, $fh = 0$ and $gh = 0$, a contradiction.

(2) \Rightarrow (3) Let $h \in C(X) \setminus \{0\}$. Then $\phi \neq coz(h) = coz(h) \cap X = coz(h) \cap (Supp(f) \cup Supp(g))$. Thus $coz(h) \cap Supp(f) \neq \phi$ or $coz(h) \cap Supp(g) \neq \phi$. Suppose that $coz(h) \cap Supp(f) \neq \phi$, and hence $\phi \neq Cl_X(coz(h) \cap Supp(f)) = Cl_X(coz(h) \cap coz(f))$, which implies that $coz(h) \cap coz(f) \neq \phi$. So $h \notin Ann(f)$. Hence for any $h \in C(X) \setminus \{0\}$, $h \notin Ann(f)$ or $h \notin Ann(g)$. Thus $Ann(f) \cap Ann(g) = \{0\}$.

(3) \Rightarrow (4) If $P \in Spec(C(X)) \setminus \{V(Ann(f)) \cup V(Ann(g))\}$, then there exist $h_1 \in Ann(f) \setminus P$ and $h_2 \in Ann(g) \setminus P$, and so $h_1 h_2 \in Ann(f) \cap Ann(g) \setminus \{0\}$, since $h_1 h_2 \notin P$, a contradiction.

(4) \Rightarrow (1) If $h \in C(X)$ such that $fh = 0$ and $gh = 0$, then for each $P \in Spec(C(X)) = V(Ann(f)) \cup V(Ann(g))$, either $Ann(f) \subseteq P$ or $Ann(g) \subseteq P$, and therefore, $h \in P$ for each $P \in Spec(C(X))$. Hence $h = 0$, since $C(X)$ is a reduced ring, and so $\{f, g\}$ dominates $\overline{\Gamma(C(X))}$. \square

Example 2.1. Consider the function

$$f(x) = \begin{cases} 1 & x \geq 1 \\ x & 0 < x < 1 \\ 0 & x \leq 0 \end{cases}$$

Then $f \in C(\mathbb{R})$, and $\{f, 1 - f\}$ dominates $\overline{\Gamma(C(X))}$.

Theorem 2.8. The ring $C(X)$ is a von Neuman regular ring if and only if $C(X)$ is an almost regular ring and for all $f \in Z^*(C(X))$ there exists $g \in Z^*(C(X))$ such that $fg = 0$ and $\{f, g\}$ dominates $\overline{\Gamma(C(X))}$.

Proof. Assume that $C(X)$ is a von Neuman regular ring. Then clearly $C(X)$ is an almost regular ring. For each $f \in Z^*(C(X))$, $Z(f)$ is open, and so the function

$$g(x) = \begin{cases} 1 & x \in Z(f) \\ 0 & x \notin Z(f) \end{cases}$$

belongs to $Z^*(C(X))$, and $Supp(f) \cup Supp(g) = coz(f) \cup Z(f) = X$. Hence it follows by Theorem 2.7, that $\{f, g\}$ dominates $\overline{\Gamma(C(X))}$.

Conversely, assume that $C(X)$ is an almost regular ring and for all $f \in Z^*(C(X))$ there exists $g \in Z^*(C(X))$ such that $fg = 0$ and $\{f, g\}$ dominates $\overline{\Gamma(C(X))}$. Every non-unit element in $C(X)$ has a zero-set with non-empty interior, meaning that every element in $C(X)$ is either a unit or a zero-divisor. If $f \in C(X)$ is a unit, then $f = f^2 f^{-1}$. So assume that $f \in Z^*(C(X))$. By the hypothesis, there exists $g \in Z^*(C(X))$ such that $Int_X Z(f) \cap Int_X Z(g) = \phi$ and $Z(f) \cup Z(g) = X$. Assume $a \in Z(f) \cap Z(g)$. Now $h = f^2 + g^2$ is a function in $C(X)$, with $Z(h) = Z(f) \cap Z(g)$. But $C(X)$ is an almost regular ring and $a \in Z(h)$, thus $Int_X Z(h) \neq \phi$, and hence $Int_X Z(f) \cap Int_X Z(g) \neq \phi$, a contradiction. So $Z(f) \cap Z(g) = \phi$ which implies that $Z(f)$ is open, since $Z(f) = X \setminus Z(g)$. Hence $C(X)$ is a von Neuman regular ring. \square

Using Theorem 2.7 together with Proposition 2.2 and Corollary 2.5 in [5], we get:

Theorem 2.9. *The following are equivalent:*

- (1) $\Gamma(C(X))$ is complemented.
- (2) For all $f \in Z^*(C(X))$, there exists $g \in Z^*(C(X))$ such that $Z(f) \cup Z(g) = X$ and $Int_X Z(f) \cap Int_X Z(g) = \phi$.
- (3) For all $f \in Z^*(C(X))$, there exists $g \in Z^*(C(X))$ such that $h(g) = h(Ann(f))$
- (4) For all $f \in Z^*(C(X))$, there exists $g \in Z^*(C(X))$ such that $fg = 0$ and $\{f, g\}$ dominates $\overline{\Gamma(C(X))}$.
- (5) $Min(C(X))$ is compact.

Now, we estimate the clique number for $\overline{\Gamma(C(X))}$.

Lemma 2.1. *If $|X| > 1$, then $\omega(\overline{\Gamma(C(X))}) \geq \aleph_1$.*

Proof. Let $f \in Z^*(C(X))$, and consider the set $M = \{rf : r \in \mathbb{R} \setminus \{0\}\}$. Then M is a complete subgraph in $\overline{\Gamma(C(X))}$, and $\aleph_1 \leq |M| \leq \omega(\overline{\Gamma(C(X))})$. \square

Theorem 2.10. *If $2 \leq |X| \leq \aleph_0$, then $\omega(\overline{\Gamma(C(X))}) = \aleph_1$, otherwise $2^{c(X)} \leq \omega(\overline{\Gamma(C(X))}) \leq 2^{d(X)}$.*

Proof. If $2 \leq |X| \leq \aleph_0$, then by the above Lemma, $\aleph_1 \leq |M| \leq \omega(\overline{\Gamma(C(X))}) \leq |C(X)| \leq 2^{\aleph_0} = \aleph_1$.

Assume $|X| > \aleph_0$, and let $F = \{O_i : i \in \Lambda\}$ be a family of pairwise disjoint non-empty open subsets of X with $|F| = c(X)$. For all $O_i \in F$ there exists a non-zero continuous function f_i such that $\text{Supp}(f_i) \subset O_i$. Let Ω be the power set of $F \setminus \{O_1\}$. Then for every $\beta \in \Omega \setminus \{F \setminus \{O_1\}, \emptyset\}$, the function g_β defined by

$$g_\beta(x) = \begin{cases} f_1(x) & x \in O_1 \\ f_i(x) & x \in O_i \in \beta \\ 0 & \text{otherwise} \end{cases}$$

is a well-defined continuous zero-divisor function, and $g_\alpha \neq g_\beta$ for $\alpha \neq \beta$. Hence we have 2^Λ distinct continuous functions. Consider the induced subgraph H of $\{g_\beta : \beta \in \Omega \setminus \{F \setminus \{O_1\}, \emptyset\}\}$ in $\overline{\Gamma(C(X))}$. Obviously H is a complete subgraph in $\overline{\Gamma(C(X))}$, hence $2^{c(X)} \leq \omega(\overline{\Gamma(C(X))})$. Since $|C(X)| \leq 2^{d(X)}$, see [6], then we'll have the inequality $2^{c(X)} \leq \omega(\overline{\Gamma(C(X))}) \leq 2^{d(X)}$. \square

Example 2.2. *Let $X = \beta\mathbb{N}$. Then $d(X) = |\mathbb{N}| = \aleph_0$. Thus $\aleph_1 \leq \omega(\overline{\Gamma(C(X))}) \leq 2^{d(X)} = 2^{\aleph_0} = \aleph_1$, and so $\omega(\overline{\Gamma(C(\beta\mathbb{N}))}) = \aleph_1$. Similarly $\omega(\overline{\Gamma(C(\mathbb{R}))}) = \aleph_1$. If $X = \beta\mathbb{R}$, then $d(X) = |\mathbb{Q}| = \aleph_0$, hence $\omega(\overline{\Gamma(C(\beta\mathbb{R}))}) = \aleph_1$.*

3. THE LINE GRAPH OF THE COMPLEMENT GRAPH FOR ZERO-DIVISORS OF $C(X)$

Let $f, g \in Z^*(C(X))$. Then $[f, g]$ is a vertex in $L(\overline{\Gamma(C(X))})$ if $fg \neq 0$. Since $\overline{\Gamma(C(X))}$ is an undirected graph, then $[f, g] = [g, f]$. It is clear that for distinct vertices $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\overline{\Gamma(C(X))})$, $[f_1, f_2]$ is adjacent to $[g_1, g_2]$ if $f_i = g_j$ for some $i, j \in \{1, 2\}$.

3.1. Connectedness. Unlike the case of $L(\Gamma(C(X)))$, the graph $L(\overline{\Gamma(C(X))})$ needs not be connected. We will characterize in this section the case at which $L(\overline{\Gamma(C(X))})$ is connected and calculate its diameter and radius.

Lemma 3.1. *Let $[f_1, f_2]$ and $[g_1, g_2]$ be distinct vertices in $L(\overline{\Gamma(C(X))})$. Then there is a vertex $[h_1, h_2]$ that is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\overline{\Gamma(C(X))})$ if and only if $f_i g_j \neq 0$ for some $i, j \in \{1, 2\}$.*

Proof. Assume there is a vertex $[h_1, h_2]$ that is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\overline{\Gamma(C(X))})$. If $f_i = g_j$ for some $i, j \in \{1, 2\}$, then $f_i g_j = f_i^2 \neq 0$. So assume that $f_i \neq g_j$ for all $i, j \in \{1, 2\}$. If $[f_1, f_2] - [h_1, h_2] - [g_1, g_2]$ is a path in $L(\overline{\Gamma(C(X))})$, then $h_1 = f_i$ for some $i \in \{1, 2\}$ and $h_2 = g_j$ for some $j \in \{1, 2\}$. Thus $f_i g_j = h_1 h_2 \neq 0$.

Conversely, assume that $f_1 g_1 \neq 0$. If $f_1 \neq g_1$, then $[f_1, g_1]$ is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\overline{\Gamma(C(X))})$. If $f_1 = g_1$, then there exists $r \in \mathbb{R} \setminus \{0, 1\}$ such that $g_2 \neq r g_1$ and $f_2 \neq r f_1$, so $[f_1, r f_1]$ is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\overline{\Gamma(C(X))})$. \square

Theorem 3.1. *Assume that $|X| \geq 3$, $[f_1, f_2]$ and $[g_1, g_2]$ are distinct vertices in $L(\overline{\Gamma(C(X))})$. Then*

- (1) $d([f_1, f_2], [g_1, g_2]) = 1$ if and only if $f_i = g_j$ for some $i, j \in \{1, 2\}$.
- (2) $d([f_1, f_2], [g_1, g_2]) = 2$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $f_i g_j \neq 0$ for some $i, j \in \{1, 2\}$.

(3) $d([f_1, f_2], [g_1, g_2]) = 3$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $f_i g_j = 0$ for all $i, j \in \{1, 2\}$.

Proof. (1) Clear.

(2) The result follows immediately by Lemma 3.1.

(3) Using (1) and (2), we get $d([f_1, f_2], [g_1, g_2]) > 2$. It follows by Corollary 2.1 that there exists $h \in Z^*(C(X))$ such that $f_1 - h - g_1$ is a path in $\overline{\Gamma(C(X))}$. Clearly one can choose h such that $h \notin \{f_2, g_2\}$. Thus we have the path $[f_1, f_2] - [f_1, h] - [h, g_1] - [g_1, g_2]$ in $L(\overline{\Gamma(C(X))})$. Hence $d([f_1, f_2], [g_1, g_2]) = 3$. \square

Corollary 3.1. *If $|X| \geq 3$, then $L(\overline{\Gamma(C(X))})$ is connected with $\text{diam}(L(\overline{\Gamma(C(X))})) \leq 3$.*

Proof. It follows by Theorem 2.1 that if $|X| = 2$, then $\overline{\Gamma(C(X))}$ is disconnected, and so $L(\overline{\Gamma(C(X))})$ is disconnected too. Now the result follows by Theorem 3.1. \square

We now find the radius of $L(\overline{\Gamma(C(X))})$.

Theorem 3.2. *Assume $|X| > 2$ and $[f_1, f_2]$ is a vertex in $L(\overline{\Gamma(C(X))})$. Then*

$$e([f_1, f_2]) = \begin{cases} 2 & \text{Supp}(f_1) \cup \text{Supp}(f_2) = X \\ 3 & \text{otherwise} \end{cases}$$

Proof. Since $\text{diam}(L(\overline{\Gamma(C(X))})) \leq 3$, we have $1 < e([f_1, f_2]) \leq 3$. Assume that $[g_1, g_2]$ is a vertex that is not adjacent to $[f_1, f_2]$ in $L(\overline{\Gamma(C(X))})$. So $g_i \neq f_j$ for all $i, j \in \{1, 2\}$. But $\phi \neq \text{coz}(g_1) = \text{coz}(g_1) \cap X = \text{coz}(g_1) \cap (\text{Supp}(f_1) \cup \text{Supp}(f_2))$. Thus $\text{coz}(g_1) \cap \text{Supp}(f_1) \neq \phi$ or $\text{coz}(g_1) \cap \text{Supp}(f_2) \neq \phi$. Suppose that $\text{coz}(g_1) \cap \text{Supp}(f_1) \neq \phi$, and hence $\phi \neq Cl_X(\text{coz}(g_1) \cap \text{Supp}(f_1)) = Cl_X(\text{coz}(g_1) \cap \text{coz}(f_1))$, which implies that $\text{coz}(g_1) \cap \text{coz}(f_1) \neq \phi$. Hence $[f_1, f_2] - [f_1, g_1] - [g_1, g_2]$ is a path in $L(\overline{\Gamma(C(X))})$, and therefor $d([f_1, f_2], [g_1, g_2]) = 2$. Thus $e([f_1, f_2]) = 2$.

Now, assume that $y \in X \setminus (Supp(f_1) \cup Supp(f_2))$. Let V be an open set in X such that $y \in V \subseteq Cl_X V \subseteq X \setminus (Supp(f_1) \cup Supp(f_2))$. Let $g_1, g_2 \in C(X)$ such that $g_i(y) = i$ and $g(Supp(f_1) \cup Supp(f_2)) = 0$ for $i \in \{1, 2\}$. Then $f_i \neq g_j$ and $f_i g_j = 0$ for all $i, j \in \{1, 2\}$. So it follows by Theorem 3.1 that $d([f_1, f_2], [g_1, g_2]) = 3$, and hence $e([f_1, f_2]) = 3$. \square

Corollary 3.2. *If $|X| > 2$, then $\rho(L(\overline{\Gamma(C(X))})) = 2$.*

Proof. It follows by the proof of Theorem 2.6 that there exists $g \in Z^*(C(X))$ such that $\{g, 1 - g\}$ dominates $\overline{\Gamma(C(X))}$, and clearly $Supp(g) \cup Supp(1 - g) = X$. Now, if $g(1 - g) \neq 0$, then using Theorem 3.2, we get $\rho(L(\overline{\Gamma(C(X))})) = e([g, 1 - g]) = 2$. If $g(1 - g) = 0$, then $g^2 = g$ is an idempotent, and hence $Z(g)$ and $coz(g)$ are non-empty clopen sets with $coz(1 - g) \cup coz(g) = X$. Since $|X| \geq 3$, either $|coz(g)| \geq 2$ or $|coz(1 - g)| \geq 2$. Assume that $|coz(1 - g)| \geq 2$, and let x, y be two distinct points in $coz(1 - g)$. Since X is a completely regular Hausdorff space, there exist two disjoint open sets U and V , such that $x \in U \subset coz(1 - g)$ and $y \in V \subset coz(1 - g)$. Consider $h \in C(X)$ such that $h(x) = 1$ and $h(X \setminus U) = 0$. Clearly $(g + h) \in Z^*(C(X))$, with $(g + h)(1 - g) \neq 0$ and $Supp(g + h) \cup Supp(1 - g) = X$, hence again by Theorem 3.2, $\rho(L(\overline{\Gamma(C(X))})) = e([h + g, 1 - g]) = 2$. \square

3.2. Cycles. In this section, we will show that $L(\overline{\Gamma(C(X))})$ is always triangulated and hypertriangulated, we will also find the length of the shortest cycle containing two distinct vertices and show that $L(\overline{\Gamma(C(X))})$ is never chordal.

Theorem 3.3. *If $|X| > 1$, then $gr(L(\overline{\Gamma(C(X))})) = 3$.*

Proof. Let $[f, g]$ be a vertex in $L(\overline{\Gamma(C(X))})$. Then there exists $r \in \mathbb{R} \setminus \{0, 1\}$ such that $f \neq rg$. Thus we have the cycle $[f, g] - [g, rg] - [rg, f] - [f, g]$. Hence $gr(L(\overline{\Gamma(C(X))})) = 3$. \square

Theorem 3.4. *If $|X| > 1$, then $L(\overline{\Gamma(C(X))})$ is both triangulated and hypertriangulated.*

Proof. We have showed in the proof of Theorem 3.3 that $L(\overline{\Gamma(C(X))})$ is triangulated. Let $[f_1, f_2] - [f_2, g]$ be an edge in $L(\overline{\Gamma(C(X))})$. Then there exists $r \in \mathbb{R} \setminus \{0, 1\}$ such that $f_1 \neq rg$ and $f_2 \neq rg$. Hence $[f_1, f_2] - [f_2, g] - [f_2, rg] - [f_1, f_2]$ is a cycle in $L(\overline{\Gamma(C(X))})$, which implies that $L(\overline{\Gamma(C(X))})$ is hypertriangulated. \square

Theorem 3.5. *Let $[f_1, f_2]$ and $[g_1, g_2]$ be two distinct vertices in $L(\overline{\Gamma(C(X))})$. Then*

- (1) $c([f_1, f_2], [g_1, g_2]) = 3$ if and only if $f_i = g_j$ for some $i, j \in \{1, 2\}$.
- (2) $c([f_1, f_2], [g_1, g_2]) = 4$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and ((for some $i \in \{1, 2\}$, $f_i g_j \neq 0$ for all $j \in \{1, 2\}$) or $(f_1 g_i \neq 0$ and $f_2 g_j \neq 0$ where $\{i, j\} = \{1, 2\}$)).
- (3) $c([f_1, f_2], [g_1, g_2]) = 5$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and for only one $i \in \{1, 2\}$, $f_i g_j \neq 0$ for only one $j \in \{1, 2\}$.
- (4) $c([f_1, f_2], [g_1, g_2]) = 6$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $f_i g_j = 0$ for all $i, j \in \{1, 2\}$.

Proof. (1) Assume $f_1 = g_1$. Then there exists $r \in \mathbb{R} \setminus \{0\}$ such that $rg_2 \notin \{f_1, f_2, g_2\}$, and so

$$[f_1, f_2] - [f_1, g_2] - [f_1, rg_2] - [f_1, f_2]$$

is a cycle of length 3 in $L(\overline{\Gamma(C(X))})$ containing $[f_1, f_2]$ and $[g_1, g_2]$. It is clear that if a triangle contains $[f_1, f_2]$ and $[g_1, g_2]$, then $f_i = g_j$ for some $i, j \in \{1, 2\}$.

(2) Since $f_i \neq g_j$ for all $i, j \in \{1, 2\}$, then there is no triangle containing $[f_1, f_2]$ and $[g_1, g_2]$, because they are non-adjacent. Now, assume that $f_1 g_j \neq 0$ for all $j \in \{1, 2\}$. Then

$$[f_1, f_2] - [f_1, g_2] - [g_1, g_2] - [g_1, f_1] - [f_1, f_2]$$

is a cycle of length 4 in $L(\overline{\Gamma(C(X))})$ containing $[f_1, f_2]$ and $[g_1, g_2]$. Also if $f_1g_1 \neq 0$ and $f_2g_2 \neq 0$, then

$$[f_1, f_2] - [f_1, g_1] - [g_1, g_2] - [g_2, f_2] - [f_1, f_2]$$

is a cycle of length 4 in $L(\overline{\Gamma(C(X))})$ containing $[f_1, f_2]$ and $[g_1, g_2]$. Hence in both cases we have $c([f_1, f_2], [g_1, g_2]) = 4$.

Assume that $c([f_1, f_2], [g_1, g_2]) = 4$. Then it follows by (1) that $f_i \neq g_j$ for all $i, j \in \{1, 2\}$. Hence we have the cycle

$$[f_1, f_2] - [a, b] - [g_1, g_2] - [c, d] - [f_1, f_2],$$

where $a, c \in \{f_1, f_2\}$ and $b, d \in \{g_1, g_2\}$. Assume that $a = f_1$, $c = f_2$, $b = g_1$ and $d = g_2$, which implies that $f_1g_1 \neq 0$ and $f_2g_2 \neq 0$. Now if $a = c = f_1$, $b = g_1$ and $d = g_2$, then $f_1g_1 \neq 0$ and $f_1g_2 \neq 0$. Finally, if $a = c = f_1$, $b = d = g_2$, then $[a, b] = [c, d]$, which is a contradiction.

(3) Assume $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and for only one $i \in \{1, 2\}$, $f_i g_j \neq 0$ for only one $j \in \{1, 2\}$, say $f_1g_1 \neq 0$. By (1) and (2), there is no cycle of length 3 or 4 containing both $[f_1, f_2]$ and $[g_1, g_2]$. There exists $r \in \mathbb{R} \setminus \{0\}$ such that $rg_1 \notin \{f_1, g_1, g_2\}$, and so the cycle

$$[f_1, f_2] - [f_1, g_1] - [g_1, g_2] - [g_2, rg_1] - [rg_1, f_1] - [f_1, f_2]$$

is of length 5 in $L(\overline{\Gamma(C(X))})$ containing $[f_1, f_2]$ and $[g_1, g_2]$.

Hence we have $c([f_1, f_2], [g_1, g_2]) = 5$.

Now, assume that $c([f_1, f_2], [g_1, g_2]) = 5$. Then $f_i \neq g_j$ for all $i, j \in \{1, 2\}$. If $f_i g_j = 0$ for all $i, j \in \{1, 2\}$ and we have the cycle

$$[f_1, f_2] - [k_1, l_1] - [g_1, g_2] - [k_2, l_2] - [k_3, l_3] - [f_1, f_2]$$

of length 5 in $L(\overline{\Gamma(C(X))})$, then $k_1 \in \{f_1, f_2\}$ and $l_1 \in \{g_1, g_2\}$. But $k_1 l_1 \neq 0$, contradicting the assumption. Similarly if we have the cycle

$$[f_1, f_2] - [k_1, l_1] - [k_2, l_2] - [g_1, g_2] - [k_3, l_3] - [f_1, f_2],$$

we will have a contradiction. Thus we must have $f_i g_j \neq 0$ for only one $i \in \{1, 2\}$ and only one $j \in \{1, 2\}$.

(4) Assume that $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $f_i g_j = 0$ for all $i, j \in \{1, 2\}$. By the previous steps $c([f_1, f_2], [g_1, g_2]) > 5$. It follows by Corollary 2.1 that there exists $h \in Z^*(C(X))$ such that $f_1 - h - g_1$ is a path in $\overline{\Gamma(C(X))}$. Let $r \in \mathbb{R} \setminus \{0, 1\}$, then the cycle

$$[f_1, f_2] - [f_1, h] - [h, g_1] - [g_1, g_2] - [g_1, rh] - [rh, f_1] - [f_1, f_2]$$

is of length 6 in $L(\overline{\Gamma(C(X))})$ containing $[f_1, f_2]$ and $[g_1, g_2]$.

Hence we have $c([f_1, f_2], [g_1, g_2]) = 6$.

If $c([f_1, f_2], [g_1, g_2]) = 6$, then by (1), (2), and (3) $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $f_i g_j = 0$ for all $i, j \in \{1, 2\}$. \square

Theorem 3.6. *The graph $L(\overline{\Gamma(C(X))})$ is never chordal.*

Proof. Let $f \in Z^*(C(X))$. Then $[f, 2f] - [2f, 3f] - [3f, 4f] - [4f, f] - [f, 2f]$ is a cycle of length 4 in $L(\overline{\Gamma(C(X))})$, where no chord can be added. \square

3.3. Dominating Sets. In this section, we will give bounds for the dominating number and the clique number for $L(\overline{\Gamma(C(X))})$.

Lemma 3.2. *For any dominating set D in $L(\overline{\Gamma(C(X))})$, $|D| \geq \aleph_1$.*

Proof. If for each $f \in Z^*(C(X))$, there exists $g \in Z^*(C(X)) \setminus \text{Ann}(f)$ such that $[f, g] \in D$, then $|D| \geq |Z^*(C(X))| \geq \aleph_1$. So assume that there exists $f \in Z^*(C(X))$ such that $[f, g] \notin D$ for all $g \in Z^*(C(X)) \setminus \text{Ann}(f)$. Hence $K = \{[g_r, rf] : r \in$

$\mathbb{R} \setminus \{0, 1\} \subseteq D$, where $g_r \in Z^*(C(X)) \setminus \text{Ann}(rf)$, since $[f, rf] \notin D$. Therefore $|D| \geq |K| \geq \aleph_1$. \square

Theorem 3.7. *If $|X| > 1$, then $\varpi(X) \leq dt(L(\overline{\Gamma(C(X))}))$.*

Proof. Let D be a dominating set in $L(\overline{\Gamma(C(X))})$, and let $B = \{coz(f), coz(g) : [f, g] \in D\}$. Let U be any non-empty open set in X and let $a \in U$. Let V be an open set in X such that $a \in V \subseteq Cl_X V \subseteq U$. Let $f, g \in C(X)$ such that $f(a) = 1, g(a) = 2$ and $f(X \setminus V) = g(X \setminus V) = 0$. Then $f, g \in Z^*(C(X))$ and $[f, g]$ is a vertex in $L(\overline{\Gamma(C(X))})$. We now have 3 cases:

Case I: $[f, g] \in D$, and so $a \in coz(f) \subseteq U$.

Case II: $[f, h] \in D$ for some $h \in Z^*(C(X)) \setminus \text{Ann}(f)$, and so $a \in coz(f) \subseteq U$.

Case III: $[g, k] \in D$ for some $k \in Z^*(C(X)) \setminus \text{Ann}(g)$, and so $a \in coz(g) \subseteq U$.

Thus B is a base for X , and so $\varpi(X) \leq |B| \leq 2|D| = |D|$.

Since this is true for any dominating set D , we have $\varpi(X) \leq dt(L(\overline{\Gamma(C(X))}))$. \square

Corollary 3.3. *If $|X| > 1$, then $d(X) \leq dt(L(\overline{\Gamma(C(X))}))$.*

Proof. The result is immediate, since $d(X) \leq \varpi(X)$. \square

Theorem 3.8. *If $|X| > 1$, then $dt(L(\overline{\Gamma(C(X))})) \leq 2^{d(X)}$.*

Proof. It was shown in the proof of Theorem 2.6 that there exists $g \in Z^*(C(X))$ such that $\{g, 1 - g\}$ dominates $\overline{\Gamma(C(X))}$. Let $D_1 = \{[g, h] : h \in Z^*(C(X)) \setminus \text{Ann}(g)\}$, $D_2 = \{[1 - g, k] : k \in Z^*(C(X)) \setminus \text{Ann}(1 - g)\}$, and $D = D_1 \cup D_2$. It is clear that D dominates $L(\overline{\Gamma(C(X))})$, and so $dt(L(\overline{\Gamma(C(X))})) \leq |D| \leq |Z^*(C(X)) \setminus \text{Ann}(g)| + |Z^*(C(X)) \setminus \text{Ann}(1 - g)| \leq |C(X)| \leq 2^{d(X)}$, see [6]. \square

Corollary 3.4. *If $d(X) = \aleph_n$, then $\aleph_n \leq dt(L(\overline{\Gamma(C(X))})) \leq \aleph_{n+1}$.*

We now investigate cliques and clique number in $L(\overline{\Gamma(C(X))})$.

Theorem 3.9. *If $|X| > 1$, then $\omega(L(\overline{\Gamma(C(X))})) = |Z^*(C(X))|$.*

Proof. Let $\{f, g\}$ be a dominating set of $\overline{\Gamma(C(X))}$.

Let $A_f = \{[f, h] : h \in Z^*(C(X)) \text{ and } fh \neq 0\}$ and

$A_g = \{[g, h] : h \in Z^*(C(X)) \text{ and } gh \neq 0\}$. Then clearly the induced subgraphs H_1 and H_2 of A_f and A_g respectively are complete in $L(\overline{\Gamma(C(X))})$. But $|A_f| + |A_g| = |Z^*(C(X))|$. Hence $\sup\{|A_f|, |A_g|\} = |Z^*(C(X))|$. i.e. $\omega(L(\overline{\Gamma(C(X))})) = |Z^*(C(X))|$. \square

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