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CYCLIC CONTRACTIONS IN θ-COMPLETE PARTIAL CONE METRIC SPACES AND FIXED POINT THEOREMS

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Abstract. In this paper, we introduce the generalized cyclic contractions on θ-complete partial cone metric spaces and prove a fixed point result in such spaces without assuming the normality of cone. Our result generalizes some known results from metric and cone metric spaces in θ-complete cone metric spaces. For illustration examples are provided.

1. Introduction

There are a number of generalizations of metric space and the well-known Banach contraction principle. The spaces with vector-valued metric and the spaces in which the metric takes a value in ordered sets studied by several authors (see [1, 2, 3, 4] and the references therein). Huang and Zhang introduced the notion of cone metric spaces, which was similar as K-metric and K-normed spaces (see [4] and references therein). In addition to the known concepts, Huang and Zhang defined the Cauchy sequences and convergence of a sequence in cone metric spaces in terms of the interior points of a closed subset of Banach space, called cone. The results of Huang and Zhang were generalized by Rezapour and Hamlbarani [5] by omitting the normality of cone.

Matthews [6, 7] introduced the partial metric spaces which have the property of nonzero-self distances of points. He also proved that the Banach contraction principle

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can be extended into the partial metric spaces. Romaguera [8] generalized the notion of complete partial metric spaces and introduced the notion of 0-complete partial metric spaces and obtained a characterization of completeness of partial metric spaces.

Cirić [9] introduced the generalized contractions in metric spaces and proved that such contractions has a unique fixed point in complete metric spaces. Kirk et al. [10] introduced the notion of cyclic contractions on metric spaces and generalized the Banach contraction principle for such contractions. Note that the cyclic and generalized contractions are not necessarily continuous. Recently, Nashine et al. [12] considered various cyclic contractive conditions on partial metric spaces.

Very recently, Malhotra et al. [13] (see also [14]) generalized the concepts of cone metric and 0-complete partial metric spaces by introducing the notion of θ-complete cone metric spaces and prove some fixed point results in such spaces without assuming the normality of cone. In this paper, we introduce the generalized cyclic contractions in the setting of θ-complete cone metric spaces and prove a fixed point result in such spaces. Our result generalizes the results of Huang and Zhang [1], Matthews [6, 7], Kirk et al. [10] and some particular cases of the results of Abbas et al [11] and Nashine et al. [12] in the sense of generalized contractive condition, as well as we prove the fixed point results in a more general setting of θ-complete partial cone metric spaces. An example is presented which shows that the generalizations of this paper are proper.

2. Preliminaries

First we recall some definitions and results which will be useful in the sequel.

Definition 2.1. [1] Let E be a real Banach space and P be a subset of E. The set P is called a cone if:

(i) P is closed, nonempty and $P \neq \{\theta\}$, here θ is the zero vector of E;

(ii) $\alpha, \beta \in \mathbb{R},\ \alpha, \beta \geq 0,\ x, y \in P \Rightarrow \alpha x + \beta y \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$, we define a partial ordering " $\preceq$ " with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. While $x \ll y$ if and only if $y - x \in P^0$, where P^0 denotes the interior of P.

Let P be a cone in a real Banach space E. Then P is called normal, if there exists a constant $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \ \Rightarrow \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

The following remark will be useful in the sequel.

Remark 1. [15] Let P be a cone in a real Banach space E, and $a, b, c \in P$.

(a) If $a \preceq b$ and $b \ll c$, then $a \ll c$.

(b) If $a \ll b$ and $b \ll c$, then $a \ll c$.

(c) If $\theta \preceq u \ll c$ for every $c \in P^0$, then $u = \theta$.

(d) If $c \in P^0$ and $a_n \to \theta$, then there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ we have $a_n \ll c$.

(e) If $\theta \preceq a_n \preceq b_n$ for each n and $a_n \to a,\ b_n \to b$, then $a \preceq b$.

(f) If $a \preceq \lambda a$ where $0 \leq \lambda < 1$, then $a = \theta$.

Definition 2.2. [1] Let X be a nonempty set and P be a cone in a real Banach space E. Suppose the mapping $d : X \times X \to E$ satisfies,

(CM1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(CM3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space. If P is normal, then (X, d) is said to be a normal cone metric space.

Definition 2.3. [1] Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X, $x \in X$.

(a) If for every $c \in E$ with $\theta \ll c$ there is a positive integer n_0 such that, $d(x_n, x) \ll c$ for all $n > n_0$, then the sequence $\{x_n\}$ converges properly to x. We denote this by $x_n \to x$ (properly) as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$ (properly).

(b) If for every $c \in E$ with $\theta \ll c$ there is a positive integer n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then the sequence $\{x_n\}$ is called a proper Cauchy sequence in X.

(X, d) is called a complete cone metric space, if every proper Cauchy sequence in X is properly convergent in X.

In further discussion, we always suppose that E is a real Banach space, P is a solid cone in E, i.e., $P^0 \neq \emptyset$ and " $\preceq$ " is the partial ordering with respect to P.

Definition 2.4. [13] Let X be a nonempty set and P be a cone in real Banach space E. Suppose, a mapping $p : X \times X \to E$ satisfies, for any $x, y, z \in X$,

(PCM1) $\theta \preceq p(x, x) \preceq p(x, y)$,

(PCM2) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,

(PCM3) $p(x, y) = p(y, x)$,

(PCM4) $p(x, y) \preceq p(x, z) + p(z, y) - p(z, z)$.

Then p is called a partial cone metric and the pair (X, p) is called a partial cone metric space.

If P is normal, then (X, p) is called normal partial cone metric space. From the definition it is clear that if $p(x, y) = \theta$, then $x = y$. But in general, converse may not be true. Also every cone metric space is partial cone metric space with zero (θ) self distance, but there are partial cone metric spaces which are not cone metric space.

Example 2.1. [13] *Let* $E = \mathbb{R}^2$, $P = \{(x, y) : x, y \geq 0\}$, $X = \mathbb{R}^+$ *and* $p : X \times X \to E$ *is defined by* $p(x, y) = max\{x, y\}(1, \alpha)$ *for all* $x, y \in X$, *where* $\alpha \geq 1$ *is a constant. Then* (X, p) *is a normal partial cone metric space, but it is not a cone metric space. Indeed if* $x = y > 0$, *then we have* $p(x, y) = x(1, \alpha) \neq \theta$.

Example 2.2. [13] *Let* $X = [0, 1]$, $E = C^1_{\mathbb{R}}[0, 1]$ *with norm* $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $P = \{f : f(t) \geq 0$ *for all* $t \in [0, 1]\}$. *Define* $p : X \times X \to E$ *by* $p(x, y) = max\{x, y\}\varphi(t)$, *where* $\varphi(t) = e^t \in E$. *Then* (X, p) *is a non-normal partial cone metric space, but it is not a cone metric space. Indeed if* $x = y > 0$, *then we have* $p(x, y) = x\varphi(t) \neq \theta$.

Let (X, p) be a partial cone metric space, P a solid cone. For $c \in P^0, x \in X$ let $B_p(x, c) = \{y \in X : p(x, y) \ll c + p(x, x)\}$ and $\beta = \{B_p(x, c) : x \in X, c \in P^0\}$. Then $\tau_p = \{U \subset X :$ for all $x \in U$ there exists $B \in \beta$ and $x \in B \subset U\} \cup \emptyset$, is a topology on X. Therefore, every partial cone metric space is a topological space with topology τ_p.

Definition 2.5. [13] Let (X, p) be a partial cone metric space and $\{x_n\}$ be a sequence in X. If for every $c \in P^0$ there is a positive integer n_0 such that, $p(x_n, x) \ll c + p(x, x)$ for all $n > n_0$, then $\{x_n\}$ is said to be convergent and converges to x, and x is the limit of $\{x_n\}$ and we denote this by $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

There should be no confusion between the convergence in (X, p) and in (X, d). When we say that sequence is convergent in X, it means it is convergent in (X, p). If sequence is convergent in (X, d), then we say that "sequence is properly convergent".

Lemma 2.1. [13] *Let* (X, p) *be a partial cone metric space and define* $d : X \times X \to E$ *by* $d(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, *for all* $x, y \in X$. *Then* (X, d) *is a cone metric space.*

We call d the induced cone metric on (X, p). Unless we specify otherwise, whenever we consider a cone metric on X, it will be the induced cone metric.

Definition 2.6. [13] Let E be a real Banach space and P be a solid cone. A sequence $\{a_n\}$ in E is called a c-sequence in E if for every $c \in P^0$ there exists $n_0 \in \mathbb{N}$ such that, $a_n \ll c$ for all $n > n_0$.

Definition 2.7. [13] Let (X,p) be a partial cone metric space, $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be a θ-Cauchy sequence in (X,p) if $\{p(x_n,x_m)\}$ is a c-sequence in E, i.e., if for every $c \in P^0$ there exists $n_0 \in \mathbb{N}$ such that, $p(x_n,x_m) \ll c$ for all $n, m > n_0$. (X,p) is said to be θ-complete if every θ-Cauchy sequence $\{x_n\}$ in (X,p) converges with respect to the topology τ_p to $x \in X$ such that, $p(x,x) = \theta$.

Lemma 2.2. [13] *If (X,p) be a partial cone metric space, $\{x_n\}$ a sequence in X. If $\{x_n\}$ is θ-Cauchy sequence, then it is a proper Cauchy sequence.*

Lemma 2.3. [13] *If the induced cone metric space (X,d) is complete, then the partial cone metric space (X,p) is θ-complete.*

Remark 2. Every closed subset of a θ-complete partial cone metric space is θ-complete.

Lemma 2.4. [13] *If (X,p) is a partial cone metric space, $\{x_n\}$ is a sequence in X and $x_n \to x \in X$ (properly), then $\{x_n\}$ is convergent and x is a limit of $\{x_n\}$. Furthermore, if $p(x,x) = \theta$, then $\{x_n\}$ is θ-Cauchy sequence.*

Definition 2.8. [10] Let X be a nonempty set, $m \in \mathbb{N}$ and let $T\colon X \to X$ be a mapping. Then $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T if:

(a) $A_i, i = 1, 2, \ldots, m$ are nonempty subsets of X;

(b) $T(A_1) \subseteq A_2, T(A_2) \subseteq A_3, \cdots, T(A_{m-1}) \subseteq A_m, T(A_m) \subseteq A_1$.

Next, we define the generalized cyclic contractions in a partial cone metric space.

Definition 2.9. Let (X,p) be a partial cone metric space, $T\colon X \to X$ a mapping and let $X = \bigcup_{i=1}^{m} A_i$ be a cyclic representation of X with respect to T. Then T is called a generalized cyclic contraction if there exists $k \in [0,1)$ such that

$$p(Tx,Ty) \preceq kM(x,y) \quad \text{for all} \quad x \in A_i, y \in A_{i+1}, \tag{2.1}$$

where $A_{m+i} = A_i$ for all $i \in \{1,2,\ldots,m\}$ and

$$M(x,y) \in \left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\right\}. \tag{2.2}$$

Now, we can state our main results.

3. Main Results

First, we prove the following lemma which will be used in the sequel.

Lemma 3.1. *Let (X,p) be a partial cone metric space, $A_i, i = 1,2,\ldots,m, m \in \mathbb{N}$ be nonempty subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T\colon Y \to Y$ be a generalized cyclic contraction. If T has a fixed point $u \in Y$, then it is unique with $p(u,u) = \theta$ and $u \in \bigcap_{i=1}^{m} A_i$.*

Proof. If $u \in Y$ be a fixed point of T, then $T^n u = u$ for all $n \in \mathbb{N}$. Since $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T, by definition we have $u \in \bigcap_{i=1}^{m} A_i$. Now, from (2.1) we have

$$p(u,u) = p(Tu,Tu) \preceq kM(u,u), \tag{3.1}$$

where

$$M(u,u) \in \left\{p(u,u), p(u,Tu), p(u,Tu), \frac{p(u,Tu)+p(u,Tu)}{2}\right\} = \{p(u,u)\}.$$

Therefore, from (3.1) we have $p(u,u) \preceq kp(u,u)$, and by (f) of Remark 1 we obtain $p(u,u) = \theta$. Thus, if $u \in Y$ is a fixed point of T, then $p(u,u) = \theta$. Now suppose that

$v \in Y$ be another fixed point of T, then we have $v \in \bigcap_{i=1}^{m} A_i$ and $p(v,v) = \theta$. Since $u, v \in \bigcap_{i=1}^{m} A_i$ it follows from (2.1) that

$$p(u,v) = p(Tu,Tv) \preceq kM(u,v), \tag{3.2}$$

where

$$M(u,v) \in \left\{ p(u,v), p(u,Tu), p(v,Tv), \frac{p(u,Tv)+p(v,Tu)}{2} \right\} = \{p(u,v), \theta\}.$$

If $M(u,v) = \theta$, then $u = v$ and uniqueness follows. If $M(u,v) = p(u,v)$, then from (3.2) we have $p(u,v) \preceq kp(u,v)$ and again by (f) of Remark 1 we obtain $p(u,v) = \theta$, that is, $u = v$. Thus the fixed point of T is unique. □

Theorem 3.1. *Let (X,p) be a θ-complete partial cone metric space, $A_i, i = 1,2,\dots,m$, $m \in \mathbb{N}$ be nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T\colon Y \to Y$ be a generalized cyclic contraction. Then T has a unique fixed point $u \in Y$. Moreover, $p(u,u) = \theta$, $u \in \bigcap_{i=1}^{m} A_i$ and each Picard sequence $\{x_n\} = \{T^n x_0\}, x_0 \in Y$ converges to u in the topology τ_p.*

Proof. Let $x_0 \in Y$, then $x_0 \in A_i$ for at least one $i \in \{1,2,\dots,m\}$. Let the sequence $\{x_n\}$ be the Picard sequence with initial value x_0, that is, $x_n = T^n x_0$. If $x_k = x_{k-1}$ for any $k \in \mathbb{N}$, then x_k is a fixed point of T and the result follows by Lemma 3.1. Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Since $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T we have $x_1 = Tx_0 \in A_{i+1}, x_2 = Tx_1 \in A_{i+2}, \cdots$. Therefore, $x_n \in A_{i+n}$ for $n \geq 0$, where $A_{m+i} = A_i$ for all $i \in \mathbb{N}$. Now for all $n \in \mathbb{N}$ we obtain from (2.1) that

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \preceq kM(x_{n-1}, x_n), \tag{3.3}$$

where

$$
\begin{aligned}
M(x_{n-1},x_n) &\in \left\{p(x_{n-1},x_n), p(x_{n-1},Tx_{n-1}), p(x_n,Tx_n), \frac{p(x_{n-1},Tx_n)+p(x_n,Tx_{n-1})}{2}\right\} \\
&= \left\{p(x_{n-1},x_n), p(x_n,x_{n+1}), \frac{p(x_{n-1},x_{n+1})+p(x_n,x_n)}{2}\right\}.
\end{aligned}
$$

We consider the following cases:

(a) If $M(x_{n-1},x_n) = p(x_n,x_{n+1})$, then from (3.3) we have $p(x_n,x_{n+1}) \preceq kp(x_n,x_{n+1})$ and $k \in [0,1)$ by (f) of Remark 1 we have $p(x_n,x_{n+1}) = \theta$, that is, $x_n = x_{n+1}$. This contradiction shows that $M(x_{n-1},x_n) \neq p(x_n,x_{n+1})$.

(b) If $M(x_{n-1},x_n) = \dfrac{p(x_{n-1},x_{n+1})+p(x_n,x_n)}{2}$, that is, $M(x_{n-1},x_n) \preceq \dfrac{1}{2}\big[p(x_{n-1},x_n)+p(x_n,x_{n+1})\big]$ so we obtain from (3.3) that $p(x_n,x_{n+1}) \preceq \dfrac{k}{2}\big[p(x_{n-1},x_n)+p(x_n,x_{n+1})\big]$, that is, $p(x_n,x_{n+1}) \preceq \dfrac{k}{2-k}p(x_{n-1},x_n) \preceq kp(x_{n-1},x_n)$.

(c) If $M(x_{n-1},x_n) = p(x_{n-1},x_n)$, then again from (3.3) we obtain $p(x_n,x_{n+1}) \preceq kp(x_{n-1},x_n)$.

Thus, from inequality (3.3) we have only one possibility that $p(x_n,x_{n+1}) \preceq kp(x_{n-1},x_n)$ for all $n \in \mathbb{N}$ and by repetition of this process we obtain

$$
p(x_n,x_{n+1}) \preceq k^n p(x_0,x_1) \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.4}
$$

If $m,n \in \mathbb{N}$ with $m > n$, then it follows from (3.4) that

$$
\begin{aligned}
p(x_n,x_m) &\preceq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \cdots + p(x_{m-1},x_m) \\
&\quad -[p(x_{n+1},x_{n+1}) + \cdots + p(x_{m-1},x_{m-1})] \\
&\preceq k^n p(x_0,x_1) + k^{n+1}p(x_0,x_1) + \cdots + k^{m-1}p(x_0,x_1) \\
&\preceq \frac{k^n}{1-k}p(x_0,x_1).
\end{aligned}
$$

Since $k \in [0,1)$ we have $\frac{k^n}{1-k}p(x_0,x_1) \to \theta$ and by (a) and (d) of Remark 1, for every $c \in E$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $p(x_n,x_m) \ll c$ for all $n > n_0$.

Therefore, $\{p(x_n, x_m)\}$ is a c-sequence, that is $\{x_n\}$ is a θ-Cauchy sequence in Y. Since Y is closed and X is θ-complete, there exists $u \in Y$ such that $\{x_n\}$ converges with respect to τ_p to the point $u \in Y$ such that $p(u, u) = \theta$ and so the sequence $\{p(x_n, u)\}$ is a c-sequence. We shall show that u is the fixed point of T.

Note that the sequence $\{x_n\}$ has an infinite number of terms in A_i, $i \in \{1, 2, \ldots, m\}$. Hence we can construct a subsequence of $\{x_n\}$ in each A_i, $i \in \{1, 2, \ldots, m\}$ which converges to u. Since $\{p(x_n, u)\}$ is a c-sequence and each A_i is closed we obtain $u \in \bigcap_{i=1}^{m} A_i$.

As, $u \in \bigcap_{i=1}^{m} A_i$ using (3.3) we obtain

$$\begin{aligned} p(u, Tu) &\preceq p(u, x_{n+1}) + p(x_{n+1}, Tu) - p(x_{n+1}, x_{n+1}) \\ &\preceq p(u, x_{n+1}) + p(Tx_n, Tu) \\ &\preceq p(u, x_{n+1}) + kM(x_n, u), \qquad (3.5) \end{aligned}$$

where

$$\begin{aligned} M(x_n, u) &\in \left\{ p(x_n, u), p(x_n, Tx_n), p(u, Tu), \frac{p(x_n, Tu) + p(u, Tx_n)}{2} \right\} \\ &= \left\{ p(x_n, u), p(x_n, x_{n+1}), p(u, Tu), \frac{p(x_n, Tu) + p(u, x_{n+1})}{2} \right\}. \end{aligned}$$

Now we consider the following cases:

(a') If $M(x_n, u) = p(x_n, u)$, then from (3.5) we have $p(u, Tu) \preceq p(u, x_{n+1}) + kp(x_n, u)$ and $\{p(x_n, u)\}$ is a c-sequence therefore, for every $c \in E$ with $\theta \ll c$, there exists $n_1 \in \mathbb{N}$ such that $p(x_n, u) \ll c/2k, p(u, x_{n+1}) \ll c/2$ for all $n > n_1$. So, by (a) and (c) of Remark 1 we have $p(u, Tu) = \theta$, that is, $Tu = u$.

(b') If $M(x_n, u) = p(x_n, x_{n+1})$, then since $\{p(x_n, x_m)\}$ is a c-sequence, by a similar process as in the case (a') we obtain $Tu = u$.

(c') If $M(x_n, u) = p(u, Tu)$, then from (3.5) we have $p(u, Tu) \preceq p(u, x_{n+1}) + kp(u, Tu)$, that is, $p(u, Tu) \preceq \frac{1}{1-k} p(u, x_{n+1})$. Again, $\{p(x_n, u)\}$ is a c-sequence therefore, for every $c \in E$ with $\theta \ll c$, there exists $n_2 \in \mathbb{N}$ such that $p(x_{n+1}, u) \ll (1-k)c$ for all $n > n_2$. So, by (a) and (c) of Remark 1 we have $p(u, Tu) = \theta$, that is, $Tu = u$.

(d') If $M(x_n, u) = \frac{p(x_n, Tu) + p(u, x_{n+1})}{2}$, then we have

$$\begin{aligned} M(x_n, u) &\preceq \frac{p(x_n, u) + p(u, Tu) + p(u, x_{n+1}) - p(u, u)}{2} \\ &\preceq \frac{p(x_n, u) + p(u, Tu) + p(u, x_{n+1})}{2}. \end{aligned}$$

Since $\{p(x_n, u)\}$ is a c-sequence therefore, for every $c \in E$ with $\theta \ll c$, there exists $n_3 \in \mathbb{N}$ such that $p(x_{n+1}, u) \ll \frac{(2-k)c}{2(2+k)}$, $p(x_n, u) \ll \frac{(2-k)c}{2k}$ for all $n > n_3$. Again, with similar arguments we obtain $Tu = u$.

Thus, in each case we obtain that $Tu = u$, i.e., u is a fixed point of T. The remaining part of the proof follows from the Lemma 3.1. □

Remark 3. The above theorem generalizes the corresponding fixed point theorems for cyclic contractions in metric and partial spaces. For example, the Corollary 2.5 of Abbas et al. [11] and Theorem 4.1 of Nashine et al. [12], for $j = 1$ are particular cases (when the partial cone metric spaces is replaced by the partial metric space) of the above theorem. Therefore, the result of Ćirić [9] is also deduced by the above theorem.

The following corollary is a generalization and extension of result of Kirk et al. [10] in partial cone metric spaces.

Corollary 3.1. *Let (X, p) be a θ-complete partial cone metric space, $A_i, i = 1, 2, \ldots, m$, $m \in \mathbb{N}$ be nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T\colon Y \to Y$ is* *a cyclic contraction, that is, there exists $k \in [0,1)$ such that*

$$p(Tx,Ty) \preceq kp(x,y) \quad \text{for all} \quad x \in A_i, y \in A_{i+1},$$

where $A_{m+i} = A_i$ for all $i \in \{1,2,\dots,m\}$. Then T has a unique fixed point $u \in Y$. Moreover, $p(u,u) = \theta$, $u \in \bigcap_{i=1}^{m} A_i$ and each Picard sequence $\{x_n\} = \{T^n x_0\}, x_0 \in Y$ converges to u in the topology τ_p.

Example 3.1. *Let $E = \mathbb{R}^2$, the Euclidian plane and $P = \{(x,y) \in \mathbb{R}^2 \colon x,y \geq 0\}$ a cone in E. Let $X = A \cup B$, where $A = \{(x,0) \colon x \in [0,1]\}, B = \{(0,x) \colon x \in [0,1]\}$. Define a mapping $p \colon X \times X \to E$ by*

$$p(a,b) = p(b,a) = \left(\frac{4}{3}, 1\right) \max\{x,y\}, \text{ if } a = (x,0), b = (y,0);$$
$$p(a,b) = p(b,a) = \left(1, \frac{2}{3}\right) \max\{x,y\}, \text{ if } a = (0,x), b = (0,y);$$
$$p(a,b) = p(b,a) = \left(\frac{4}{3}x + y, x + \frac{2}{3}y\right), \text{ if } a = (0,x), b = (y,0).$$

Then, (X,p) is a partial cone metric space. Now the induced cone metric d is given by

$$d(a,b) = d(b,a) = \left(\frac{4}{3}, 1\right) |x-y|, \text{ if } a = (x,0), b = (y,0);$$
$$d(a,b) = d(b,a) = \left(1, \frac{2}{3}\right) |x-y|, \text{ if } a = (0,x), b = (0,y);$$
$$d(a,b) = d(b,a) = \left(\frac{4}{3}x + y, x + \frac{2}{3}y\right), \text{ if } a = (0,x), b = (y,0).$$

Then, since (X,d) is a complete cone metric space, (X,p) is θ-complete. Define $T \colon X \to X$ by

$$T(x,0) = \begin{cases} \frac{1}{2}(0,x), & \text{if } x \in [0,1/4]; \\ (0,x), & \text{if } x \in (1/4,1], \end{cases} \quad T(0,x) = \begin{cases} \frac{1}{2}(x,0), & \text{if } x \in [0,1/2]; \\ (x,0), & \text{if } x \in (1/2,1); \\ (0,0), & \text{if } x = 1. \end{cases}$$

Let $Y = A_1 \cup A_2$, $A_1 = \{(0,x) \colon x \in [0,1/2]\}$, $A_2 = \{(x,0) \colon x \in [0,1/4]\}$. Then, $T(A_1) \subseteq A_2, T(A_2) \subseteq A_1$. Then, $Y = A_1 \cup A_2$ is a cyclic representation of Y with *respect to* T *and* T *is a cyclic contraction with* $k = 3/4$. *Note that, all the conditions of Corollary 3.1 are satisfied and* $(0,0) \in A_1 \cap A_2$ *is the unique fixed point of* T. *Now it is easy to see that* T *is not a Hardy-Rogers type contraction on* X *(see for details* [13]*). Indeed, for* $a = (x,0), b = (0,x)$ *with* $1/2 < x < 1$, *there exist no nonnegative constants* $a_i, i = 1,2,3,4,5$ *such that* $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ *and*

$$p(Ta,Tb) \preceq a_1 p(a,b) + a_2 p(a,Ta) + a_3 p(b,Tb) + a_4 p(a,Tb) + a_5 p(b,Ta).$$

Therefore, the result of Malhotra et al. [13] *are not applicable. Also, one can see that the ordinary metric versions of Theorem 3.1 are not applicable here.*

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