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BAER GAMMA RINGS WITH INVOLUTIONS

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Abstract. The concept of involution in Γ-rings is introduced and with the help of involutions, we obtain some characterizations of Baer Γ-rings.

1. Introduction

As a generalization of rings, the concept of Γ-rings was first introduced by N. Nobusawa [6]. After words Barnes [1] generalized the notion of Nobusawa's Γ-rings and gave a new definition of a Γ-ring. Now a days, Γ-rings means the Γ-rings in the sense of Barnes [1] where other Γ-rings are known as N -rings i.e., gamma rings in the sense of Nobusawa. Many Mathematicians worked on Γ-rings and obtained some fruitful results that are a generalization of many classical ring theories. In the Book " Rings with operators " Kaplansky [3] worked on Baer rings and obtained various results relating to involution and Baer rings. Paul and Sabur [9] worked on Lie and Jordan structures in simple Γ-rings and generalized some results of classical rings into Γ-rings. Paul and Sabur [10] also worked on Baer Gamma rings and obtained some characterizations of this Γ-ring.

In this paper, we introduce the notion of an involution in Γ-rings and generalize

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some results of classical Baer rings into gamma Baer rings with the help of the new concept of an involution. In [10], an example of a Baer gamma ring is given

2. Preliminaries

Definition 2.1. Gamma Ring: Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \to M$ (sending (x, α, y) into $x\alpha y$) such that
$(i)(x+y)\alpha z = x\alpha z + y\alpha z, x(\alpha+\beta)z = x\alpha z + x\beta z, x\alpha(y+z) = x\alpha y + x\alpha z$
$(ii)(x\alpha y)\beta z = x\alpha(y\beta z)$, where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then M is called a Γ-ring in the sense of Barnes [1].

Definition 2.2. Sub Γ-ring: Let M be a Γ-ring. A non-empty subset S of a Γ-ring M is a sub Γ-ring of M if $a, b \in S$, then $a - b \in S$ and $a\gamma b \in S, \forall \gamma \in \Gamma$.

Definition 2.3. Ideal of Γ-rings: A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{c\alpha a : c \in M, \alpha \in \Gamma, a \in A\}(A\Gamma M)$ is contained in A. If A is both a left and a right ideal of M, then we say that A is an ideal or two-sided ideal of M. If A and B are both left (respectively right or two-sided) ideals of M, then $A + B = \{a + b : a \in A, b \in B\}$ is clearly a left (respectively right or two-sided) ideal, called the sum of A and B. We can say every finite sum of left (respectively right or two-sided) ideal of a Γ-ring is also a left (respectively right or two-sided) ideal.

It is clear that the intersection of any number of left (respectively right or two-sided) ideal of M is also a left (respectively right or two-sided) ideal of M. If A is a left ideal of M, B is a right ideal of M and S is any non-empty subset of M, then the set, $A\Gamma S = \{\sum_{i=1}^{n} a_i\gamma s_i : a_i \in A, \gamma \in \Gamma, s_i \in S, n$ is a positive integer$\}$ is a left ideal of M and $S\Gamma B$ is a right ideal of M. $A\Gamma B$ is a two-sided ideal of M. If $a \in M$, then the principal ideal generated by a denoted by $< a >$ is the intersection of all ideals containing a and is the set of all finite sum of elements of the form $na + x\alpha a + a\beta y + u\gamma a\mu v$, where n is an integer, x, y, u, v are elements of M and $\alpha, \beta, \gamma, \mu$ are elements of Γ. This is the smallest ideal generated by a. Let $a \in M$. The smallest left (right) ideal generated by a is called the principal left (right) ideal $< a \mid (\mid a >)$.

Definition 2.4. Unity element of a Γ-ring: Let M be a Γ-ring. M is called a Γ-ring with unity if there exists an element $e \in M$ such that $a\gamma e = e\gamma a = a$ for all $a \in M$ and some $\gamma \in \Gamma$. We shall frequently denote e by 1 and when M is a Γ-ring with unity, we shall often write $1 \in M$. Note that not all Γ-rings have an unity. When a Γ-ring has an unity, then the unity is unique.

Definition 2.5. Nilpotent element: Let M be a Γ-ring. An element x of M is called nilpotent if for some $\gamma \in \Gamma$, there exists a positive integer $n = n(\gamma)$ such that $(x\gamma)^n x = (x\gamma x\gamma ... \gamma x\gamma)x = 0$.

Definition 2.6. Nil ideal: An ideal A of a Γ-ring M is a nil ideal if every element of A is nilpotent that is, for all $x \in A$ and some $\gamma \in \Gamma$, $(x\gamma)^n x = (x\gamma x\gamma ... \gamma x\gamma)x = 0$, where n depends on the particular element x of A.

Definition 2.7. Nilpotent ideal: An ideal A of a Γ-ring M is called nilpotent if $(A\gamma)^n A = (A\gamma A\gamma ... \gamma A\gamma)A = 0$, where $n$is the least positive integer.

Definition 2.8. Annihilator of a subset of a Γ-ring: Let M be a Γ-ring. Let S be a subset of M. Then the left annihilator $l(S)$ of S is defined by $L(S) = \{m \in M : m\gamma S = 0$ for every $\gamma \in \Gamma\}$, whereas the right annihilator $r(S)$ is defined by $R(S) = \{m \in M : S\gamma m = 0$ for every $\gamma \in \Gamma\}$.

Definition 2.9. Idempotent element: Let M be a Γ-ring. An element e of M is called idempotent if $e\gamma e = e \neq 0$ for some $\gamma \in \Gamma$.

Definition 2.10. Centre of a Γ-ring: Let M be Γ-ring. The centre of M, written as Z, is the set of those elements in M that commute with every element in M, that is, $Z = \{m \in M : m\gamma x = x\gamma m$ for all $x \in M$ and $\gamma \in \Gamma\}$.

Definition 2.11. ΓM**-homomorphism:** Let M be a Γ-ring. Let A and B be the left ideals of M. A ΓM-homomorphism is a function $\phi : A \to B$ such that
$(i)\phi(x+y) = \phi(x) + \phi(y)$ for all $x, y \in A$
$(ii)(m\gamma x) = m\gamma(x)$ for all $x \in A, m \in M$ and $\gamma \in \Gamma$.
In case, A and B are right ideals, then (i) and (ii) become
$(i)\prime\ \phi(x+y) = \phi(x) + \phi(y)$ for all $x, y \in A$
$(ii)\prime\ \phi(x\gamma m) = \phi(x)\gamma m$ for all $x \in A, m \in M$ and $\gamma \in \Gamma$.

Definition 2.12. ΓM**-isomorphism:** Let M be a Γ-ring. Let A and B be two left ideals of M. Let $\phi : A \to B$ be a ΓM-homomorphism from A into B. We call ϕ , a ΓM-isomorphism, if ϕ is one-one and onto. We say that A and B are ΓM-isomorphic and we write $A \cong B$.

Definition 2.13. Baer Γ-ring: A Γ-ring M is called a Baer Γ-ring if the right annihilator of every non-empty subset of M is generated by an idempotent element of M.

3. Baer Gamma Rings with Involutions

Definition 3.1. Let M be a Γ-ring. A mapping $I : M \to M$ is called an **involution** if $(i)I(a+b) = I(a) + I(b), (ii)I(ab) = I(b)\alpha I(a)$ and $(iii)I^2(a) = a$, for all $a, b \in M, \alpha \in \Gamma$.

Example 3.1. *Let R be an associative ring with 1 having an involution * . Let $M = M1.2(R)$ and $\Gamma = \{ \begin{pmatrix} n_1.1 \\ n_2.1 \end{pmatrix} : n_1, n_2 \in \boldsymbol{Z} \}$. Then M is a Γ-ring. Define $I : M \to M$ by $I((a,b)) = (a*, b*)$. Then it is clear that I is an involution on M.*

We know that if e is the idempotent elements of a Γ-ring M, then $M\Gamma e$ and $e\Gamma M$ are respectively left ideal and right ideal of M, which is shown in [7].

Theorem 3.1. *Let e and f be idempotents in a Γ-ring M. The following are equivalent (1) $e\Gamma M, f\Gamma M$ are ΓM-isomorphism*
(2) $M\Gamma e, M\Gamma f$ are ΓM-isomorphism
(3) There exist elements $x \in e\Gamma M\Gamma f, y \in f\Gamma M\Gamma e$ with $x\alpha y = e, y\alpha x = f, \alpha \in \Gamma$.

Proof. Since condition (3) is left-right symmetric it will suffice to identify (2) and (3). (3) implies (2). Map $M\Gamma e$ to $M\Gamma f$ by right multiplication by x, $M\Gamma f$ to $M\Gamma e$ by right multiplication by y. The product both ways to clearly the identity. (2) implies (3). Let ϕ be the map from $M\Gamma e$ to $M\Gamma f$ and set EMAFK.BIB.Zel$\phi(e) = x$. Then since ϕ is a ΓM-homomorphism map we have $\phi(a) = \phi(a\alpha e) = a\alpha\phi(e) = a\alpha x$ for any in $a \in M\Gamma e$ and $\alpha \in \Gamma$ i.e., ϕ is a right multiplication by x. In particular $x = \phi(e) = e\alpha x$ and $x \in e\Gamma M\Gamma f$. Similarly the map from $M\Gamma f\Gamma$ to $M\Gamma e$ is right multiplication by an element $y \in f\Gamma M\Gamma e$. Evidently $x\alpha y = e, y\alpha x = f, \alpha \in \Gamma$. □

Definition 3.2. Idempotents e, f in a Γ-ring M are equivalent, written $e \sim f$ if they satisfy (and hence all) of the conditions in theorem 3.3. Note that a Baer Γ-ring is finite if and only whenever $e \sim 1$, then $e = 1$.

Definition 3.3. An element of a Γ-ring M with involution I is called self-adjoint if $I(x) = I$. A projection is a self-adjoint idempotent. A subset S is self-adjoint if $x \in S$ implies $I(x) \in S$. A Baer Γ-ring with involution I is a Γ-ring with involution I such that for any subset $S, R(S) = e\Gamma M$ with e a projection.

By applying the involution we get that in a Baer Γ-ring with involution I the left annihilator of any subset is like wise generated by a projection. In particular, a Baer Γ-ring with involution I is a Baer Γ-ring. The projection of e generating $e\Gamma M$ is unique. For if $e\Gamma M = f\Gamma M$ with e and f projections, we find $e = f\alpha e, \alpha \in$

$\Gamma, f = e\alpha f = I(e\alpha f) = I(f)\alpha I(e) = f\alpha e = I(f\alpha e) = I(e) = e$. Because of this uniqueness, we can call e the right-annihilating projection of the subset S of M. Even more useful is $g = 1 - e$ which we shall call the right projection of S. Then $s\alpha g = s\alpha(1 - e) = s\alpha 1 - s\alpha e = s$ for all $s \in S$ and $\alpha \in \Gamma$ is the smallest such projection. If f is an idempotent in a Baer Γ-ring with involution I and e is its right projection, we readily see that $e \sim f$.

Theorem 3.2. *Let e and f be idempotents in a Baer Γ-ring with involution $I, f \in e\Gamma M\Gamma e$. Let g and h be the right projections of e and f. Then $e - f \sim g - h$.*

Proof. Noting that $g \geq h, e\gamma g = e, g\gamma e = g, f\gamma h = f, h\gamma f = f, \gamma \in \Gamma$, we verify directly that $e - e\gamma h$ and $g - g\gamma f$ implement an equivalence of $e - g$ and $f - h$. □

Definition 3.4. For projection e, f in a Γ-ring with involution I write $e \leq f$ in case$e = e\gamma f, \gamma \in \Gamma$ (which is equivalent to $e = f\gamma e$). One readily sees that this relation makes the projections into a partially ordered set.

Theorem 3.3. *The projections in a Baer Γ-ring with involution I form a complete lattice.*

Proof. Given a family $\{e_i\}$ of projections, let e be their right projection. One readily sees that e is the least upper bound (LUB) of e_i's. Dually, there is a greatest lower bound (GLB). Hence the theorem is proved. □

Definition 3.5. Let M be a Baer Γ-ring with involution I. B is a sub Γ-ring of M. We say that B is a Baer sub Γ-ring with involution I of M if
(1) B is a self adjoint sub Γ-ring (2) If $S \subset B$ and e is the right annihilating projection of $S(in M)$, then $e \in B$.

If B is a Baer sub Γ-ring with involution I, then B is itself obviously a Baer Γ-ring with involution I. Its unity element is the same as that of M (take the annihilator of

0). The lattice of projections in B is a complete sub lattice of that of M. If M is a Baer Γ-ring with involution I and e is a projection in M, the projections of $e\Gamma M\Gamma e$ are the projections of $f \in M$ with $f \leq e$. It follows easily that $e\Gamma M\Gamma e$ is a Baer Γ-ring with involution I and that a family of projections in $e\Gamma M\Gamma e$ has the same LUB whatever computed in $e\Gamma M\Gamma e$ or in M.

Theorem 3.4. *Let M be a Baer Γ-ring with involution I and S be a self-adjoint subset of M. Let T be the commuting Γ-ring of S. Then T is a Baer sub Γ-ring with involution I of M.*

Proof. Since S is self-adjoint, the sub Γ-ring T is also self-adjoint. Given $V \subset T$, write $R(V) = e\Gamma M$ (this is the annihilator in M of course). We must show that e lies in T. Thus given , we have to prove $e\gamma s = s\gamma e, \gamma \in \Gamma$. Given $s \in S$, we have $s\gamma v = v\gamma s$ and $v\gamma e = 0$, then $v\gamma(1-e)\gamma s\gamma e = v\gamma s\gamma e - v\gamma e\gamma s\gamma e = v\gamma s\gamma e - 0 = v\gamma s\gamma e = s\gamma v\gamma e = s\gamma 0 = 0$. Since v is arbitrary in $V, (1-e)\gamma s\gamma e \in e\Gamma M$. Hence $(1-e)\gamma s\gamma e = 0$. Thus $s\gamma e = e\gamma s\gamma e$. Apply involution I, we have $I(sve) = I(e\gamma s\gamma e)$. This implies that $I(e)\gamma I(s) = I(e)\gamma I(s)\gamma I(e)$. So $e\gamma s = e\gamma s\gamma e$. Hence $s\gamma e = e\gamma s$. □

Corollary 3.1. *The center of a Baer Γ-ring with involution I is a Baer sub Γ-ring with involution I.*

Theorem 3.5. *In a Baer Γ-ring M with involution $I.x\alpha I(x) = 0$ implies $x = 0, x \in M, \alpha \in \Gamma$.*

Proof. Let e be the right annihilating projection of x. Then $x\alpha e = 0$. Now $I(x\alpha e) = I(0) = 0$. This implies that $I(e)\alpha I(x) = 0$. So $e\alpha I(x) = 0$. Since $x\alpha I(x) = 0$, we have $I(x) \in e\Gamma M, I(x) = e\alpha I(x) = 0$. Now $x = I^2(x) = I(I(x)) = I(0) = 0$ It follows that a Baer Γ-ring with involution I has no nil left or right ideals. For let A be a nil right ideal in a Baer Γ-ring with involution I. If $x \in A$, then $y = x\alpha I(x) \in A$. If

$(y\alpha)^n y$ is the smallest power of y that is 0, let $z = (y\alpha)^{n-1}y$. Then $z\alpha I(z) = z\alpha z = 0$ whence $z = 0$ by theorem 3.12. Hence $x = 0$. The argument for a nil left ideal is analogous. □

A fortiori, a Baer Γ-ring with involution I has no nilpotent ideals. We turn now to the consideration of equivalence of projection in a Baer Γ-ring with involution I.

Theorem 3.6. *Let M be a Baer Γ-ring with involution I, an element of M such that $x\alpha I(x)$ is a projection of e for $\alpha \in \Gamma$. Then $I(x)\alpha x$ is also a projection of f. We have $x \in e\Gamma M\Gamma f, I(x) \in f\Gamma M\Gamma e$ and thus $e \sim f$.*

Proof. Set $y = e\alpha x - x$.Then

$$\begin{aligned}
y\alpha I(y) &= (e\alpha x - x)\alpha I(e\alpha x - x) \\
&= ((e-1)\alpha x)\alpha I(e\alpha x - x) \\
&= ((e-1)\alpha x)\alpha(I(x)\alpha I(e) - I(x)) \\
&= ((e-1)\alpha x)\alpha(I(x)\alpha e - I(x)) \\
&= (e-1)\alpha x\alpha I(x)\alpha(e-1) \\
&= (e-1)\alpha e\alpha(e-1) \\
&= (e\alpha e - 1\alpha e)\alpha(e-1) \\
&= (e-e)\alpha(e-1) \\
&= 0\alpha(e-1) \\
&= 0\alpha(e-1) \\
&= 0
\end{aligned}$$

By Theorem 3.12, $y = 0$. So $e\alpha x - x = 0$. Thus $e\alpha x = x$. If $f = I(x)\alpha x$ then $I(f) = I(I(x)\alpha x) = I(x)\alpha I^2(x) = I(x)\alpha x = f$ and $f\alpha f = I(x)\alpha x\alpha I(x)\alpha x =$

$I(x)\alpha e\alpha x = I(x)\alpha x = f$. Now $f\alpha I(x) = I(x)\alpha x\alpha I(x) = I(x)\alpha e = I(x)$. Thus $x\alpha I(x)\alpha x = e\alpha x = x, I(x) \in f\Gamma M\Gamma e$. Hence $e \sim f$. □

Definition 3.6. An element x in a Γ-ring M with involution I is called a partial isometry if $x\alpha I(x)$ and $I(x)\alpha x, \alpha \in \Gamma$ are projections.

Definition 3.7. In a Γ-ring M with involution I, projections e, f are called I-equivalent, written $e \sim^I f$, if there exists a partial isometry $x \in e\Gamma M\Gamma f$ with $x\alpha I(x) = e, I(x)\alpha x = f$.

It is easy verified that $\sim^I$ is an equivalence relation and that $e \sim^I f$ implies $e \sim f$. Note that if M is a Baer Γ-ring with involution I the condition $x \in e\Gamma M\Gamma f$ in the definition of I-equivalence is redundant (Theorem 3.13). We now wish to make some comparisons between Baer Γ-rings and Baer Γ-rings with involution I. We begin by exhibiting a condition that can convert a Baer Γ-ring into a Baer Γ-ring with involution I.

Theorem 3.7. *Let M be a Γ-ring with involution I and suppose that for every x in $M, 1 + I(x)\alpha x, \alpha \in \Gamma$ is invertible in M. Then for any idempotent f in M there exists a projection e such that $f\Gamma M = e\Gamma M$.*

Proof. Let $x = I(f) - f$. Then $I(x) = I(I(f) - f) = f - I(f)$. Therefore $I(x)\alpha x = (f - I(f))\alpha(I(f) - f)$. So $1 + I(x)\alpha x = 1 + (f - I(f))\alpha(I(f) - f)$. Since $1 + I(x)\alpha x$ is invertible in $M, 1 + (f - f - I(f)\alpha(I(f) - (f)$ is invertible in M. Take $z = 1 + (f - 1(f))\alpha(I(f) - f)$. Then, z is invertible, say $t = z^{-1}$. Also we have, $z = I(z)$, then $t = I(t)$. Therefore

$$
\begin{aligned}
f\alpha z &= f\alpha(1 - f - I(f) + f\alpha I(f) + I(f)\alpha f) \\
&= f\alpha 1 - f\alpha f - f\alpha I(f) + f\alpha f\alpha I(f) + f\alpha I(f)\alpha f \\
&= f - f - f\alpha I(f) + f\alpha I(f) + f\alpha I(f)\alpha f \\
&= f\alpha I(f)\alpha f
\end{aligned}
$$

Similarly $z\alpha f = f\alpha I(f)\alpha f$. It follows that t commutes with f. We have also seen that t commutes with $I(f)$. Now we choose $e = f\alpha I(f)\alpha t$. Then $I(e) = I(f\alpha I(f)\alpha t) = I(t)\alpha I^2(f)\alpha I(f) = t\alpha f\alpha I(f) = f\alpha t\alpha I(f) = f\alpha I(f)\alpha t = e$. Also

$$
\begin{aligned}
e\alpha e &= f\alpha I(f)\alpha t\alpha f\alpha I(f)\alpha t \\
&= t\alpha f\alpha I(f)\alpha f\alpha I(f)\alpha t \\
&= t\alpha(f\alpha I(f)\alpha f)\alpha I(f)\alpha t \\
&= t\alpha z\alpha f\alpha I(f)\alpha t \\
&= (t\alpha z)\alpha(f\alpha I(f)\alpha t) \\
&= 1\alpha e = e
\end{aligned}
$$

Thus e is a projection. Evidently $f\alpha e = e$ whence $e\Gamma M \subset f\Gamma M$. Again

$$
\begin{aligned}
e\alpha f &= f\alpha I(f)\alpha f\alpha t \\
&= f\alpha z\alpha t \\
&= f\alpha(z\alpha t) \\
&= f\alpha 1 = f
\end{aligned}
$$

Therefore $f\Gamma M \subset e\Gamma M$. Hence $f\Gamma M = e\Gamma M$. □

Corollary 3.2. *Let M be a Baer Γ-ring with an involution I and suppose that $1 + I(x)\alpha x, \alpha \in \Gamma$ is invertible for every x in M. Then M is a Baer Γ-ring with involution I.*

Next we give a condition which identifies the two versions of equivalence.

Theorem 3.8. *Let M be a Γ-ring with involution I. Assume that for any $y \in M$ there exists a self-adjoint $z \in M$ which commutes with everything that commutes with $I(y)\alpha y$ and satisfies $z\alpha z = I(y)\alpha y, \alpha \in \Gamma$. Then equivalent projections in M are I-equivalent.*

Proof. Let the projections e, f be equivalent via $x, y, x \in e\Gamma M\Gamma f, y \in f\Gamma M\Gamma f, x\alpha y = e, y\alpha x = f$, Choose z (relative to y) are permitted by the hypothesis. We have $x\alpha I(x)\alpha I(y)\alpha y = x\alpha I(y\alpha x)\alpha y = x\alpha I(f)\alpha y = x\alpha y = e$ Since e is self-adjoint, $e = I(e)$. Then $e = I(e) = I(x\alpha I(x)\alpha I(y)\alpha y) = I(y)\alpha I^2(y)\alpha I^2(x)\alpha I(x) = I(y)\alpha y\alpha x\alpha I(x)$ Therefore $x\alpha I(x)\alpha I(y)\alpha y) = I(y)\alpha y\alpha x\alpha I(x)$. Thus $x\alpha I(x)$ commutes with $y\alpha I(y)$ and hence also with z. Now we have $I(y)\alpha y\alpha e = I(y)\alpha y$. Then $I(I(y)\alpha y\alpha e) = I(I(y)\alpha y)$. So, $I(e)\alpha I(y)\alpha I^2(y) = I(y)\alpha I^2(y)$. Thus $e\alpha I(y)\alpha y = I(y)\alpha(y)$. Hence $e\alpha I(y)\alpha y = I(y)\alpha(y)\alpha e$. Thus $e\alpha z = z\alpha e$. The element $w = e\alpha z\alpha x \in e\Gamma M\Gamma f$ implements the desired I-equivalence of e and f. Now

$$\begin{aligned} I(w)\alpha w &= I(e\alpha z\alpha x)\alpha(e\alpha z\alpha x) \\ &= I(x)\alpha I(z)\alpha I(e)\alpha e\alpha z\alpha x \\ &= I(x)\alpha z\alpha e\alpha e\alpha z\alpha x \\ &= I(x)\alpha z\alpha e\alpha z\alpha x \\ &= I(x)\alpha z\alpha z\alpha e\alpha x \\ &= I(x)\alpha z\alpha z\alpha x \\ &= I(x)\alpha I(y)\alpha y\alpha x \\ &= I(y\alpha x)\alpha(y\alpha x) \\ &= I(f)\alpha f = f. \end{aligned}$$

Again we have

$$w\alpha I(w) = e\alpha z\alpha x\alpha I(e\alpha z\alpha x)$$
$$= e\alpha z\alpha x\alpha I(x)\alpha I(z)\alpha I(e)$$
$$= e\alpha z\alpha x\alpha I(x)\alpha z\alpha e$$
$$= e\alpha(x\alpha I(x))\alpha Z\alpha z\alpha e$$
$$= e\alpha x\alpha I(x)\alpha Z\alpha z\alpha e$$
$$= x\alpha I(x)\alpha I(y)\alpha y\alpha e$$
$$= x\alpha I(y\alpha x)\alpha y\alpha e$$
$$= x\alpha I(f)\alpha y$$
$$= x\alpha f\alpha y$$
$$= x\alpha y = e.$$

Hence e and f are I-equivalent. □

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