Jordan Journal of Mathematics and Statistics (JJMS) 7(1), 2014, pp.15 - 28

COMPACTIFICATIONS AND F-SPECTRAL SPACES

CEREN SULTAN ELMALI [(1)] AND TAMER UGUR [(2)]

Abstract. If X is T_3, it is showed that the Fan-Gottesman compactification of X can be embedded into compactification (X^*, k) of X obtaining by a combined approach of nets and open filters. By F-spectral, we mean a topological space X such that the Fan-Gottesman compactification of X is a spectral space. We give necessary and sufficient conditions on X in order to get F-spectral.

1. Introduction

The first section of this paper contains some preliminaries about net, filters and a process of obtaining a compactification (X^*, k) of an arbitrary topological space X. In 2005, Hueytzen J. Wu and Wan-Hong Wu described a process of obtaining a compactification of an arbitrary topological space by a combined approach of nets and open filters. Besides they showed the relation among Wallman, Stone-Cech and (X^*, k) compactification under some conditions [12].

In the second section of our paper contains some information about Wallman and Fan-Gottesman compactification. In 1938, Henry Wallman introduced compactification of T_1 spaces having a normal base [6],[9] which is also called Wallman compactification [10]. In 1952, Ky Fan ve Noel Gottesman constructed a compactification, also called Fan-Gottesman compactification, for a regular space with a normal base

2000 *Mathematics Subject Classification.* 201054D35, 54D70.

Key words and phrases. net, open filter, ultrafilter, normal base, Fan-Gottesman compactification, Wallman compactification, H-spectral space, F-spectral space.

Received: Sept.12 , 2012 Accepted : Nov. 23 , 2013 .

[6]. Their method is similar to Wallman compactification. In [5] it is investigated relation between Fan-Gottesman and Wallman compactifications and showed that Fan-Gottesman compactification of some interesting and specific spaces such as normal A_2 and T_4 is Wallman-type compactification. At the end of this section, if studied space is T_3, we show that the Fan-Gottesman compactification of X can be embedded into the compactification (X^*, k). Also we examined the relation between Wallman and Fan-Gottesman compactification via net and filters.

In the third section of this paper contains some preliminaries about T_0 compactification and spectral spaces. In 1993 Herrlich has constructed [7] with any T_0-space X, a minimal compactification $\beta_w X$ called the T_0-compactification of X. For T_1 space, the extension $\beta_w X$ coincides with the Wallman compactification γX of X. In 2004 Karim Belaid, Othman Echi and Riyadh Gargouri [1] have characterized topological spaces X such that one point compactification of X is a spectral space. In 2006 Karim Belaid [2], gave some properties of H-spectral space which he means a topological space X such that its T_0-compactification is spectral. Also he gave necessary and sufficient condition on the T_1-space X in order to get its Wallman compactification spectral. At the end of this section, we define F-spectral spaces and investigate necessary and sufficient condition in order that Fan -Gottesman compactification of T_3-space is spectral.

2. Nets, filters and (X^*, k) compactification

Let A be a family of continuous functions on a topological space X. A net (x_λ) in X will be called an $A-net$, if $(f(x_\lambda))$ converges for each f in A. Then X is compact if

(1) $f(X)$ is contained in a compact subset C_f for each f in A, and
(2) Every $A-net$ has a cluster point in X

Let X be any arbitrary topological space, $C^*(X) = \{f_\alpha : \alpha \in \Lambda\}$ the family of all bounded real-valued continuous functions on X. For a $C^*(X) - net$ (x_i), let

$$\mathcal{F}_{(x_i)} = \{U : U \text{ is open in } X \text{ and } (x_i) \text{ residually in } U\}$$

It is clear that $\mathcal{F}_{(x_i)}$ is an open filter, and for any $f_\alpha \in C^*(X)$, any $\varepsilon \succ 0$, $f_\alpha^{-1}((\delta_\alpha - \varepsilon, \delta_\alpha + \varepsilon)) \in \mathcal{F}_{(x_i)}$, where $\delta_\alpha = \lim(f_\alpha(x_i))$. It is called $\mathcal{F}_{(x_i)}$ the *open filter* on X *induced by* (x_i).

Definition 2.1. If F is a filter on X, let $\Lambda_{\mathcal{F}} = \{(x, F) : x \in F \subset \mathcal{F}\}$. Then $\Lambda_{\mathcal{F}}$ is directed by the relation $(x_1, F_1) \leq (x_2, F_2)$ if $F_2 \subset F_1$, so the map $P : \Lambda_{\mathcal{F}} \to X$ defined by $P(x, F) = x$ is a net in X. It is called the net based on $\mathcal{F}$.

Lemma 2.1. *A filter $\mathcal{F}$ converges to x in X if the net based on $\mathcal{F}$ converges to x.*

Corollary 2.1. *Let Q be an open filter on X, (x_i) is the net based on Q, and*

$$I = \{U : U \textit{ is open in } X \textit{ and } (x_i) \textit{ is in } U\}$$

Then $I = Q$.

For each $C^*(X) - net$ (x_i) in X, let $\left(w_n^{(x_i)}\right)$ be the net based on the open filter $\mathcal{F}_{(x_i)}$ induced by (x_i). It is clear by Definition 2.1., Lemma 2.1. and corollary 2.1. that:

(1) $\left(w_n^{(x_i)}\right)$ is uniquely determined by $\mathcal{F}_{(x_i)}$ and $\mathcal{F}_{(x_i)} = \mathcal{F}_{(x_j)}$, if $\left(w_n^{(x_i)}\right) = \left(w_n^{(x_j)}\right)$

(2) $\mathcal{F}_{(x_i)} = \mathcal{F}_{w_n^{(x_i)}} = \left\{G : G \text{ is open in } X \text{ and } \left(w_n^{(x_i)}\right) \text{ is residually in } G\right\}$

(3) $\left(w_n^{(x_i)}\right)$ is a $C^*(X) - net$ and $\lim\left(f_\alpha\left(w_n^{(x_i)}\right)\right) = \lim(f_\alpha(x_i))$ for all f_α in $C^*(X)$

(4) The following are equivalent:

a: $\left(w_n^{(x_i)}\right)$ converges to x,

b: (x_i) converges to x

c: $\mathcal{F}_{(x_i)}$ converges to x.

In order to avoid the confusion between $\left(w_n^{(x_i)}\right)$ as a net in X and $\left(w_n^{(x_i)}\right)$ as a point in a set, we will use $\left(w_n^{(x_i)}\right)^*$ to represent $\left(w_n^{(x_i)}\right)$ when it regarded as a point in a set just as in [12] .

Let $Y = \left\{\left(w_n^{(x_i)}\right)^* : (x_i) \text{ is a } C^*(X) - \textit{net} \text{ that does not converge in } X\right\}$ and it is noted that $\left(w_n^{(x_i)}\right)$ is the net based on $F_{(x_i)}$. $X^* = X \cup Y$, the disjoint union of X and Y. For each open set $U \subset X$, define $U^* \subset X^*$ to be the set

$$U^* = U \cup \left\{\left(w_n^{(x_i)}\right)^* : \left(w_n^{(x_i)}\right)^* \in Y \text{ and } \left(w_n^{(x_i)}\right) \text{ is residually in } U\right\}$$

It is clear that if $U \subset V$, then $U^* \subset V^*$. It is seen that $\beta = \{U^* : U \text{ is open in } X\}$ is a base for a topology on X^*.

Let $k : X \to X^*$ be defined by $k(x) = x$. Then k is a continuous function from X into X^*. Moreover $k(X)$ is dense in X^* and (X^*, k) is compactification of X.

Let us cite [11],[12] for detailed information about this section.

3. **Wallman and Fan Gottesman compactification**

The Wallman compactification is defined in [11] as follows.

Let X be a T_2 space and γX be the collection of all closed ultrafilters on X. For each closed set $D \subset X$, define $D \quad \subset \gamma X$ to be the set $D \quad = \{F \in \gamma X : D \in F\}$. Let $\zeta = \left\{D \quad : D \text{ is closed subset of } X\right\}$ be the base for the closed sets of the topology on γX, and let $h : X \to \gamma X$ be defined by $h(x) = F_x$, the closed ultrafilter converging to x in X. Then $(\gamma X, h)$ is the Wallman compactification of X.

Now we investigate how Wallman compactification is obtained via normal base.

Let β is a class of closed sets in X. If it satisfies following three conditions, β is called *normal base*.

1) β is closed under finite intersection and unions.

2) If x is not contained in the closed set A, there is a set $B \in \beta$

such that $x \in B \subset X - A$

3) If $A_1 \cap A_2 = \emptyset$, for $\forall\ A_1, A_2 \in \beta$, there exist sets $A_{m,} A_n \in \beta$ such that $A_1 \subset X - A_n, A_2 \subset X - A_m, A_n \cup A_m = X$

Let X be a T_1 space having a normal base and β be a normal base in X. It is considered K space whose element is denoted by letter as $a', b', ..$ consist of finite number of F_i in X such that

$$F_1 \cap F_2 \cap F_3 \cap ... \cap F_n \neq \emptyset$$

and maximal with respect to above property. Let $\tau(F) = \{a' \in K : F \in a'\}$. It is defined topology of K with a family of sets $\delta = \{\tau(F) : F \in \beta\}$ a base of closed set. K is a compact space and compactification of X. This compactification is called Wallman compactification [6],[9],[10]. In order to avoid the confusion it is denoted by γX.

There is very little difference between Fan-Gottesman and Wallman compactification, β forming Wallman compactification is a normal base for closed sets but β forming Fan-Gottesman compactification is a normal base for open sets. It shall not be forgotten that both of these satisfy conditions of normal base.

It is considered that X is a regular space having a base for open set β which satisfies above three properties of normal base. But Ky Fan and Noel Gottesman used for any $A \in \beta$ and any open set G of X such that $cl_x A \subset G$, there exist a $B \in \beta$ such that $cl_x A \subset B \subset cl_x B \subset G$, where closure of A in X will be denoted $cl_X A$, instead of if $A_1 \cap A_2 = \emptyset$, for $\forall\ A_1, A_2 \in \beta$, there exist sets $A_{m,} A_n \in \beta$ such that $A_1 \subset X - A_n, A_2 \subset X - A_m, A_n \cup A_m = X$.

A *chain family on* β is a non-empty family of sets of β such that

$$cl_X A_1 \cap cl_X A_2 \cap cl_X A_3 \cap ... \cap cl_X A_n \neq \emptyset$$

for any finite number of sets A_i of the family. Every chain family on β is contained in at least one maximal chain family on β by Zorn's lemma. Maximal chain families on β will be denoted by letters as $a^*, b^*, ...$ and also the set of all maximal chain families on β will be denoted by FX. FX is a compact, hausdorff spaces and compactification of regular spaces X. This compactification is called Fan-Gottesman compactification [6].

We know the relation between Wallman and Fan-Gottesman compactifications of some specific spaces from [5]. Therefore, we can obtain the Fan-Gottesman compactification by defining the base via nets and filters like the Wallman compactification.

Definition 3.1. Let X be a T_3 space and κX the subcollection of all open ultrafilters on X. For each open set $O \subset X$, define $O^\bullet \subset \kappa X$ to be the set

$$O^\bullet = \left\{ \hat{G} \in \kappa X : O \subset cl_X O \subset V,\ V \text{ is open in } X \text{ and } V \in \hat{G} \right\}$$

Let Φ is the $\{O^\bullet : O \text{ is open subset of } X\}$ set.It is clear that Φ is the base for open sets of topology on κX. κX is a compact space and the Fan-Gottesman compactifications of X. In order to avoid the confusion it is denoted by κX.

On the other hand, for each closed set $D \subset X$, we define $D^\bullet \subset \kappa X$ by

$$D^\bullet = \left\{ \hat{G} \in \kappa X : G \subseteq D \text{ for some } G \text{ in } \hat{G} \right\}$$

The following properties of κX are useful

(1) If $U \subset X$ is open, then $\kappa X - U^\bullet = (\kappa X - U)^\bullet$
(2) If $D \subset X$ is closed, then $\kappa X - D^\bullet = (\kappa X - D)^\bullet$
(3) If U_1 and U_2 are open in X, then $(U_1 \cap U_2)^\bullet = U_1^\bullet \cap U_2^\bullet$ and $(U_1 \cup U_2)^\bullet = U_1^\bullet \cup U_2^\bullet$

Theorem 3.1. *The Fan-Gottesman compactification κX of X can be embedded into the compactification (X^*, k) of X, if X is T_3.*

Proof. We must define a map from κX to (X^*, k) and show that the map is an embeding.

Firstly, let (X^*, k) be compactification of X defined as section1.
Let $\varphi : \kappa X \to (X^*, k)$ be defined by setting that $\varphi\left(\hat{G}_x\right) = x$, if $\hat{G}_x$ is the open ultrafilters converging to x in X; $\varphi\left(\hat{G}\right) = \left(w_n^{(\hat{G})}\right)^*$ and $\left(w_n^{(\hat{G})}\right)$ is the net based on $\hat{G}$, moreover $\left(w_n^{(\hat{G})}\right)$ is the ultranet in X, if $\hat{G}$ is open ultrafilter that does not converge in X. That is;

$$\varphi = \begin{cases} x & \text{, if } \hat{G}_x \text{ is the open ultrafilters converging to } x \text{ in } X \\ \left(w_n^{(\hat{G})}\right)^* & \text{, if } \hat{G} \text{ is open ultrafilter that does not converge in } X \end{cases}$$

From conclusion of Lemma 2.1., $\left(w_n^{(\hat{G})}\right)$ is a $C^*(X) - net$ that does not converge in X. Since $\left(w_n^{(\hat{G})}\right)$ is the net based on $\hat{G}$ thus by corollary 2.1. , the open filter $\hat{G}_{w_n^{(\hat{G})}}$ induced by $\left(w_n^{(\hat{G})}\right)$ is exactly $\hat{G}$. Hence $\left(w_n^{(\hat{G})}\right)$ is in Y defined as section 2.Since X is a T_3 , X is a Hausdorff then for $\forall\ x \neq y$ there exist open neighborhoods U_x of x and U_y of y such that $U_x \cap U_y = \emptyset$. G_x converging to x and G_y converging to y imply that $U_x \supset A$ for some $A \in G_x$ and $U_y \supset B$ for some $B \in G_y$. If $G_x = G_y$ then A and B are both in G_x and $A \cap B \neq \emptyset$. Hence $U_x \cap U_y \supset A \cap B \neq \emptyset$. This contradicts the fact that $U_x \cap U_y = \emptyset$. So $G_x = G_y$ implying $x = y$. Therefore both $\hat{G}$ and $\left(w_n^{(\hat{G})}\right)$ are uniquely determined by a given open ultrafilter $\hat{G}$ that does not converge in X. Thus φ is well-defined.

Secondly, we show that φ is a injective map.

1) If $\hat{G}_x$ and $\hat{G}_y$ are two open ultra filters converging to x and y, respectively, and $\hat{G}_x \neq \hat{G}_y$. Then $\varphi\left(\hat{G}_x\right) = x$ and $\varphi\left(\hat{G}_y\right) = y$. Then, there exist $U_0 \in \hat{G}_x$ and $V_0 \in \hat{G}_y$ such that $U_0 \cap V_0 = \emptyset$. Since $\hat{G}_x$ converges to x and $\hat{G}_y$ converges to y, so $x \in U$ for all $U \in \hat{G}_x$ and $y \in V$ for all $V \in \hat{G}_y$. Thus $U_0 \cap V_0 = \emptyset$ implies that $x \neq y$.

2) If $\hat{G}_1, \hat{G}_2$ are two open ultra filters that don't converge in X and $\hat{G}_1 \neq \hat{G}_2$, then $\varphi\left(\hat{G}_1\right) = \left(w_n^{(\hat{G}_1)}\right)^*$ and $\varphi\left(\hat{G}_2\right) = \left(w_n^{(\hat{G}_2)}\right)^*$. Since $\hat{G}_1, \hat{G}_2$ are two different open ultra filters, the nets $\left(w_i^{(\hat{G}_1)}\right)$ and $\left(w_n^{(\hat{G}_2)}\right)$ based on $\hat{G}_1$ and $\hat{G}_2$, respectively, are different. That is $\left(w_i^{(\hat{G}_1)}\right) \neq \left(w_n^{(\hat{G}_2)}\right)$. Hence $\left(w_i^{(\hat{G}_1)}\right)^* \neq \left(w_n^{(\hat{G}_2)}\right)^*$ in Y.

3) If $\hat{G}_x$ is an open ultra filters converging to x in X and $\hat{G}$ is a open ultra filters that does not converge in X, then $\hat{G}_x \neq \hat{G}$. Since $\varphi\left(\hat{G}_x\right) = x \in X$, $\varphi\left(\hat{G}\right) = \left(w_i^{(\hat{G})}\right)^* \in Y$ and $X \cap Y = \emptyset$, so $\varphi\left(\hat{G}_x\right) \neq \varphi\left(\hat{G}\right)$. Therefore, φ is one to one.

Thirdly, φ and φ^{-1} are continuous.Let U^* be open set in β defined as section 2; i.e., $U^* = U \cup \left\{\left(w_n^{(x_i)}\right)^* : \left(w_n^{(x_i)}\right)^* \in Y \text{ and } \left(w_n^{(x_i)}\right) \text{ is residually in } U\right\}$ then $\varphi^{-1}(U^*) = \left\{\hat{G}_x : x \in U\right\} \cup \left\{\hat{G} : \left(w_n^{(\hat{G})}\right) \text{ is residually in } U\right\}$. If $\hat{G}_x$ converges to x in U, then there is an open set $H \in \hat{G}_x$ such that $H \subset U$. This implies that $(X - U) \notin \hat{G}_x$; i.e., $\hat{G}_x \in \kappa X - (X - U)^\bullet$.If $\left(w_n^{(\hat{G})}\right)$ is eventually in U, since $\left(w_n^{(\hat{G})}\right)$ is the net based on $\hat{G}$, the corollary 2.1. implies that U is in $\hat{G}$, thus $\hat{G}$ is eventually U;i.e., there exists an G in $\hat{G}$ such that $G \subset U$. This implies again that $(X - U) \notin \hat{G}_x$ and therefore $\hat{G}_x \in \kappa X - (X - U)^\bullet$. Thus $\varphi^{-1}(U^*) \subset \kappa X - (X - U)^\bullet$. For $\kappa X - (X - U)^\bullet \subset \varphi^{-1}(U^*)$, let $\hat{G}$ be a open ultrafilter in $\kappa X - (X - U)^\bullet$, then $(X - U) \notin \hat{G}$. This implies that there exists an $G_0 \in \hat{G}$ such that $G_0 \cap (X - U) = \emptyset$; i.e., $G_0 \subset U$. Hence,

a) If $\hat{G}$ converges to a point x in X; i.e., $\hat{G} = \hat{G}_x$. Then x is in G for all G in $\hat{G}_x$ and thus $x \in G_0 \subset U$. This implies that $\hat{G} = \hat{G}_x$ is in $\varphi^{-1}(U^*)$

b) If $\hat{G}$ does not converge in $X, G_0 \subset U$ implies that $\hat{G}$ is eventually in U, i.e., $U \in \hat{G}$. So, the net $\left(w_n^{(\hat{G})}\right)$ based on $\hat{G}$ is eventually in U; i.e., $\hat{G}$ is in $\varphi^{-1}(U^*)$. Thus $\varphi^{-1}(U^*) = \kappa X - (X - U)^\bullet$ is open in κX. Hence φ is continuous.Since $\varphi^{-1}(U^* \cap \varphi(\kappa X)) = \varphi^{-1}(U^*) \cap \varphi^{-1}(\varphi(\kappa X)) = (\kappa X - (X - U)^\bullet) \cap \kappa X = \kappa X - (X - U)^\bullet$, thus $\varphi(\kappa X - (X - U)^\bullet) = U^* \cap \varphi(\kappa X)$ is an open in $\varphi(\kappa X)$ for any open set $\kappa X-(X-U)^{\bullet}$ in κX. Hence, φ^{-1} is continuous on $\varphi(\kappa X)$. Therefore, φ is an embedding of κX into X^{*} $\square$

Theorem 3.2. *The Wallman compactification $(\gamma X, h)$ of X can be embedded into the Fan-Gottesman compactification of X, if X is T_3.*

Proof. It is defined a map from γX into κX to proof the theorem. It is considered base defined by closed ultrafilter as a normal base. Let $\vartheta:\gamma X\rightarrow\kappa X$ be defined by setting that $\vartheta(F_x)=\hat{G}_x$ such that x contained in $\hat{G}_x$, if F_x is the closed ultrafilter converging to x in X. $\vartheta(F)=\left(w_n^{(F)}\right)^{*}$, $\left(w_n^{(F)}\right)$is the net based on open filter $\hat{G}$, if F is the closed ultrafilter that does not converging in X.

$$\vartheta=\begin{cases}\hat{G}_x & \text{, if } F_x \text{ is the closed ultrafilter converging to } x \text{ in } X\\ \left(w_n^{(F)}\right)^{*} & \text{, if } F \text{ is the closed ultrafilter that does not converging in } X\end{cases}$$

It must be shown that ϑ is an embedding between γX and κX.If F_x and F_y are two closed ultra filters converging to x and y, respectively, and $F_x\neq F_y$. Then $\vartheta(F_x)=\hat{G}_x$ and $\vartheta(F_y)=\hat{G}_y$. Then $\hat{G}_x\neq\hat{G}_y$. If F_1,F_2 are two closed ultra filters that don't converge in X and $F_1\neq F_2$, then $\vartheta(F_1)=\left(w_n^{(F_1)}\right)^{*}$ and $\vartheta(F_2)=\left(w_n^{(F_2)}\right)^{*}$. Since F_1,F_2 are two different open ultra filters, the nets $\left(w_i^{(F_1)}\right)^{*}$ and $\left(w_n^{(F_2)}\right)^{*}$ based on F_1 and F_2 , respectively, are different. Then $\left(w_i^{(F_1)}\right)\neq\left(w_n^{(F_2)}\right)$. Hence $\left(w_i^{(F_1)}\right)^{*}\neq\left(w_n^{(F_2)}\right)^{*}$in Y. If F_x is a closed ultrafilters converging to x in X and F is a closed ultrafilters that does not converge in X, then $F_x\neq F$. Since $\varphi(F_x)=\hat{G}_x$, x contained in $\hat{G}_x$, $\varphi(F)=\left(w_i^{(F)}\right)^{*}$, so $\varphi(F_x)\neq\varphi(F)$. Therefore, ϑ is one to one.

Let $U^{\bullet}$ be open set in β; i.e.,

$$U^{\bullet}=\left\{\hat{G}\in\kappa X:U\subset cl_XU\subset V,\ V\text{ is open in }X\text{ and }V\in\hat{G}\right\}$$

then $\vartheta^{-1}(U^{\bullet}) = \{F_x : x \in U\} \cup \left\{F : \left(w_n^{(F)}\right) \text{ is evetually in } U\right\}$. If F_x converges to x in U then, there is an F in F_x such that $F \subset U$. If $\left(w_n^{(F)}\right)$ is eventually in U, since $\left(w_n^F\right)$ is the net based on open filter $\hat{G}$ induced by F, F is eventually in U. Hence, if F_x converges to x in X, it is clearly seen that $F_x = F$. If F_x does not converges to x in X, then $U \in \hat{G}$. So, the net $\left(w_i^{(F)}\right)$ based on G is eventually in U,F is in $\vartheta^{-1}(U^{\bullet})$. Thus $\vartheta^{-1}(U^{\bullet}) = \gamma X - (X - U)$ is an open in γX. Hence ϑ is continuous. Since $\vartheta^{-1}(U^{\bullet} \cap \vartheta(\gamma X)) = \vartheta^{-1}(U^{\bullet}) \cap \vartheta^{-1}(\vartheta(\gamma X)) = \left(\gamma X - (X - U)\right) \cap \gamma X = \gamma X - (X - U)$ $\vartheta\left(\gamma X - (X - U)\right) = U^{\bullet} \cap \vartheta(\gamma X)$ is open in $\vartheta(\gamma X)$ for any open set $\gamma X - (X - U)$ in γX. Hence, ϑ^{-1} is continuous on $\vartheta(\gamma X)$. Therefore ϑ is an embedding of γX into κX. □

4. $\mathbf{T_0}$-compactification and H-spectral space

Let R be a commutative ring with identity. *Spectrum or prime spectrum* of R, denoted $Spec(R)$, is the set of prime ideals of R. The topology on $Spec(R)$ defined by closed set $Z(I) = \{C \in Spec(R) : I \subseteq C\}$ for ideals I of R is called Zariski topology on $Spec(R)$.

By definition, the closure in the Zariski topology of the singleton set $\{P\}$ in $Spec(R)$ consist of all prime ideals of R contain P. In particular, a point P in $Spec(R)$ is closed in the Zariski topology if and only if the prime ideal P is not contained in any other prime ideals of R, i.e., if and only if P is a maximal ideal [3].

A topological space is called spectral if it is homeomorphic to the prime spectrum or a ring equipped with Zariski topology. M. Hochster [8] has characterized spectral spaces as follows:

A space X is spectral if and only if the following axioms hold:

(1) Every nonempty irreducible closed subset of X is the closure of a unique point (that is, sober)

(2) X is compact;

(3) The compact open sets form a basis of X;

(4) The family of compact open sets of X is closed under finite intersections.

H. Herrlich has introduced the following construction [7]
Let X be a T_0-space. Consider the set $\Gamma(X)$ of all filters F on X that satisfy the following two conditions:

(1) F does not converge in X.

(2) Every finite open cover of X contains some member of F

Let $\Omega(X)$ is the set of minimal elements of $\Gamma(X)$ and define:

a: $X_w^* = X \cup \Omega(X)$.

b: $A_w^* = A \cup \{F : F \in \Omega(X) \text{ and } A \in F\}$

$\beta_w = \{A_w^* : A \text{ open in } X\}$ is a base for a topology τ_w^* on X_w^*. (X_w^*, τ_w^*) is compact and called T_0-compactification of X and denoted by $\beta_w X$.
Also, the following properties hold:

(1) If $\beta_w X$ is sober, then X is sober.

(2) If $\beta_w X$ is spectral, then X is sober.

(3) If $\beta_w X$ is normal, then X is normal

(4) If X is normal, then for each distinct elements H and G of $\beta_w X$ there exist two disjoint open sets U and V of X such that $H \in U_w^*$ and $G \in V_w^*$.

(5) If X is normal sober space, then $\beta_w X$ is sober.

Definition 4.1. A subset N of a space X is called nearly closed in X, if there exist a finite subset δ_x of δ and neighborhood V_x of x with $(V_x \cap N) \subseteq \bigcup_{\delta_{x_i} \in \delta_x} \delta_{x_i}$ for every open cover δ of N and every point x of X

The specialization order of a topological space X is defined by $x \leq y$ if and only if $y \in \overline{\{x\}}$. We denoted by $(x \uparrow) = \{y \in X : x \leq y\}$ and $(\downarrow x) = \{y \in X : y \leq x\}$.

Proposition 4.1. *Let X be a T_0-space such that $(\downarrow x) \cap (\downarrow F) = \emptyset$ for each $x \in X$ and each $F \notin (x \uparrow) \cap \Omega(X)$. If X is H-spectral space, then the following properties hold:*

(1) If C is compact open set of $\beta_w X$, then $C \cap X$ is nearly closed set of X.
(2) The nearly closed and open sets form a basis of X.
(3) If U, V are two open sets such that $U \cup V = X$. Then there exists an open nearly closed set N of X such that $N \subseteq U$ and $N \cup V = X$

Remark 1. If X is T_1-space, then $(\downarrow x) \cap (\downarrow F) = \emptyset$ for each $x \in X$ and each $F \notin (x \uparrow)$.

Let us cite [2, 6] for detailed information about this topic

Karim Belaid at al. [1] have characterized A-spectral spaces (that is; one point compactificaiton of X is spectral space) and he give some properties of H-spectral spaces (that is; T_0-compactification of X is spectral space) and defined W-spectral spaces (that is; Wallman compactificaiton of X is spectral space) and characterized of W-spectral spaces [2].

Definition 4.2. Let X be a T_3 space. If its Fan-Gottesman compactification is spectral, it is called F-spectral space [4].

Theorem 4.1. *Let X be a T_3 space. Then X is an F-spectral if and only if there exists a clopen set U such that $G \subseteq U$ and $H \cap U = \emptyset$ for each disjoint open set G and H of X.*

Proof. ($\Rightarrow$)If $G \cap H = \emptyset$, then $(X - G) \cup (X - H) = X$. By 4.1. Proposition and Remark, there is an open nearly closed set K such that $K \subseteq (X - G)$ and $K \cup (X - G) = X$. Therefore $G \subseteq (X - K)$ and $G \cap (X - K) = \emptyset$. On the other hand κX and X are Hausdorff, we get that $(X - K)$ is clopen.

($\Leftarrow$)Let $\gamma = \{U^* : U \text{ clopen set of } X\}$. Let V be an open set of X and $x \in V^*$. If $x \in V$, then $\{x\}$ is closed. Because X be a T_3 space, X be a T_1 and regular. Hence there exists a clopen set U such that $\{x\} \subseteq U \subseteq V$. Thus U^* is clopen neighborhood of x such that $U^* \subseteq V^* x = \Re \in V^* \cap \Omega(X)$, where $\Omega(X)$ is the set of minimal elements of all filters on X. For $\wp \in \kappa X - U^*$, there exist $G \in \Re$ and $H \in \wp$ such that $G \cap H = \emptyset$. Thus there exists a clopen set $U_\wp$ of X such that $G \subseteq U$ and $G \in (X - U_\wp)$. Hence $\{(X - U_\wp)^* : \wp \in \kappa X - V^*\}$ is an open cover of $\kappa X - V^*$. Since $\kappa X - V^*$ is compact, there is a finite collection $\{(X - U_\wp)^* : \wp \in I\}$ such that $\kappa X - V^* = \bigcup \left\{\kappa X - U_\wp^* : \wp \in I\right\}$. Let $U_\Re = \bigcap \{U_\wp : \wp \in I\}$. It is immediate that $U_\Re^*$ is a clopen neighborhood of $\Re$ such that $U_\Re^* \subset V^*$. Therefore, γ is bases of κX. Since each element of γ is clopen, γ is basis of compact sets closed under finite intersection. Every nonempty irreducible close subset of κX is closure of unique point (that is sober). Thus κX is spectral. $\square$

Conclusion 4.1. *Let X be a T_3 space. If X is an F-spectral, then X is a W-spectral.*

Proof. Since X is a T_3 space. The Wallman compactification $(\gamma X, h)$ of X can be embedded into the Fan-Gottesman compactification of X from Theorem 2.1. On the other hand, for each disjoint open set G and H of X, there exists a clopen set U such that $G \subseteq U$ and $H \cap U = \emptyset$, since X is an F-spectral. Then X is a W-spectral from definition of relative topology and 2.4 Theorem in [2]. $\square$

References

[1] K. Belaid, O. Echi, R. Gargouri, A-spectral space, Topology and Its Application 138, (2004) 315-322

[2] K. Belaid, H-spectral space, Topolgy and Its Application 153, (2006) 3019-3023

[3] D. Dummit, R. M. Foote, Abstract Algebra, John Wiley & Sons Inc., USA (2004)

[4] C. Elmalı and T. Uğur,Some Results Concerning Compactifications,Submaximal Spaces, Spectral Spaces and Primitive Words,Far East Journal of Mathematical Sciences 42, (2010) 93-107

[5] C. Elmalı and T. Uğur, Fan-Gottesman Compactification of Some Specific Spaces is Wallman –Type Compactification. Chaos, Solitons & Fractals 42 (2009)17-19

[6] K. Fan and N. Gottesman, On compactification of Freudenthal and Wallman. Indag. Math. 13, (1952) 184–192

[7] H. Herrlich, Compact T_0-spaces and T_0-compactification, Appl. Categ. Structres 1, (1993) 111-132

[8] M. Hochster, Prime ideal structure in commutative ring, Trans. Amer. Math. Soc. 142, (1969) 43-60

[9] O. Njastad, On Wallman-type Compactification. Math. Zeitschr. 91, No:4 (1966) 267–276

[10] H. Wallman, Lattices and Topological Spaces. Annals of Mathematics 39, (1938) 112–126

[11] S. Willard, General Topology, Dover Publications Inc., Mineola, NewYork (1970)

[12] H. Wu and W. Wu, A net and open-filter process of compactification and Stone-Čech, Wallman compactification. Topology and its Applications 153, (2006) 1291-1301

(1) Department of Mathematics, Faculty of Science, Erzurum Technic University, Erzurum, 25100, Turkey.

E-mail address: ceren.elmali@erzurum.edu.tr

(2) Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey.

E-mail address: tugur@atauni.edu.tr