ON $\rho-$CONTRACTION IN G-METRIC SPACE

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ABSTRACT. In this paper, we introduced a new type of a contractive condition defined on G-metric space, namely a $\rho-$contraction, which generalizes the weak contraction. We also proved some fixed point theorems for such a condition in ordered metric spaces. A supporting example of our results is provided in the last part of our paper as well.

1. INTRODUCTION

It is well known that the Banach contraction principle has been improved in different directions in different spaces by mathematicians over the years. Even in the contemporary research, it remains a heavily investigated branch.

In 1997, Alber and Guerre-Delabriere [3] have introduced the concept of weak contraction in Hilbert spaces. Later, Rhoades [19] showed, in 2001, that these results are also valid in complete metric spaces. In 2009, Harjani and Sadarangani [11] carried the work of Rhoades [19] into partially ordered metric spaces. Harjini and Sadarangani [11] also proved fixed point theorems for noncontinuous mappings, nonincreasing mappings and even for non-monotonic mappings.

On the other hand in 2006, Mustafa in collaboration with Sims introduced a new notion of generalized metric space called G-metric space [14], which are called G-metric spaces as generalization of metric space (X,d), to develop and to introduce a new fixed point theory for a variety of mappings in this new setting, also to extend known metric space theorems to a more general setting. After that several fixed point results were proved in these spaces. Some of these works are noted in [1, 2, 15, 16, 17, 18].

The aim of this paper is to introduced a weak condition which resulted in the concept called a $\rho-$contraction.

Definition 1.1. Let $(X, \preceq, G)$ be an ordered G-metric space. A function $\rho : X \times X \times X \to R$ is called a $\rho-$function in X if it satisfies the following conditions:

(i) $\rho(x, y, z) \geq 0$ for every comparable $x, y, z \in X$;

(ii) for any sequence $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ in X such that x_n, y_n and z_n are comparable at each $n \in N$, if $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} \rho(x_n, y_n, z_n) = \rho(x, y, z)$;

(iii) for any sequence $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ in X such that x_n, y_n and z_n are comparable at each $n \in N$, if $\lim_{n\to\infty} \rho(x_n, y_n, z_n) = 0$ then $\lim_{n\to\infty} G(x_n, y_n, z_n) = 0$.

If, in addition, then following condition is also satisfied:

(A) for any sequence $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ in X such that x_n, y_n and z_n are comparable at each $n \in N$, if the limit $\lim_{n\to\infty} G(x_n, y_n, z_n)$ exists, then the limit $\lim_{n\to\infty} \rho(x_n, y_n, z_n)$ also exists,

then ρ is said to be a $\rho-$function of type (A) w.r.to $\preceq$ in X.

Proposition 1.1. *Let $(X, \preceq, G)$ be an ordered G-metric space and $\rho : X \times X \times X \to R$ be a $\rho-$ function w.r.to $\preceq$ in X. If $x, y, z \in X$ are comparable and $\rho(x, y, z) = 0$ then $x = y = z$.*

Proof. Let $x, y, z \in X$ be comparable and $\rho(x, y, z) = 0$. Define $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ to be three constant sequences in X such that $x_n = x$, $y_n = y$ and $z_n = z$ for all $n \in X$. It follows from the definition of a $\rho-$function, since x, y and z are comparable that $G(x, y, z) = 0$. That is $x = y = z$. □

Corollary 1.1. *Let $(X, \preceq, G)$ be a totally ordered G-metric space and $\rho : X \times X \times X \to R$ be a $\rho-$function w.r.to $\preceq$ in X. If $x, y, z \in X$ and $\rho(x, y, z) = 0$ then $x = y = z$.*

Proof. Since X is totally ordered set, any $x, y, z \in X$ are comparable. The rest of the proof is straight forward. □

Example 1.1. *Let $X = R$. Define $G, \rho : X \times X \times X \to R^+$ with $G(x, y, z) = |x - y| + |y - z| + |z - x|$ and $\rho(x, y, z) = 1 + |x - y| + |y - z| + |z - x|$. If X is endowed with a usual ordering $\preceq$, then $(X, \preceq, G)$ is a totally ordered G- metric space with $\rho-$as a $\rho-$function of type (A) w.r.to $\preceq$ in X.*

Note that $\rho(x, y, z) \neq 0$ for all $x, y, z \in X$ even when $x = y = z$. This example show that the converges of Proposition 1.1 and that the Corollary 1.1 are not generally true.

Definition 1.2. Let $(X, \preceq, G)$ be an ordered G-metric space, a mapping $f : X \to X$ is called $\rho-$contraction w.r.to $\preceq$ if there exists a $\rho-$function $\rho : X \times X \times X \to R$ w.r.to $\preceq$ in X such that

(1.1)
$$G(fx, fy, fz) \leq G(x, y, z) - \rho(x, y, z)$$

for any comparable $x, y, z \in X$. Naturally, if there exists a $\rho-$function of type (A) w.r.to $\preceq$ in X such that inequality 1.1 holds for any comparable $x, y, z \in X$, then f is said to be a $\rho-$contraction of type (A) w.r.to $\preceq$.

2. Main results

Theorem 2.1. *Let $(X, \preceq, G)$ be a complete ordered G-metric space and $f : X \to X$ be continuous and nondecreasing $\rho-$ contraction of type (A) w.r.to $\preceq$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then $\{f^n x_0\}_{n=1}^{\infty}$ converges to a fixed point of f in X.*

Proof. For the existence of fixed point, we choose $x_0 \in X$ such that $x_0 \preceq fx_0$. If $x_0 = fx_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. We define a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n = f^n x_0$. Since $x_0 \preceq fx_0$ and f is nondecreasing w.r.to $\preceq$, we obtain

$$x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ....$$

If there exists $n_0 \in N$ such that $\rho(x_{n_0}, x_{n_0+1}, x_{n_0+2}) = G(x_{n_0}, x_{n_0+1}, x_{n_0+2})$, then by the notion of $\rho-$ contractivity, the proof is finished. Therefore, we assume that $\rho(x_n, x_{n+1}, x_{n+2}) < G(x_n, x_{n+1}, x_{n+2})$ for all $n \in N$. Also assume that $\rho(x_n, x_{n+1}, x_{n+2}) \neq 0$ for all $n \in N$. Otherwise we can find $n_0 \in N$ with $x_{n_0} = x_{n_0+1}$, that is $x_{n_0} = fx_{n_0}$, and the proof is finished. Hence, we consider only the case where $0 < \rho(x_n, x_{n+1}, x_{n+2}) < G(x_n, x_{n+1}, x_{n+2})$ for all $n \in N$.

Since $x_n \preceq x_{n+1}$ for all $n \in N$, we have

$$
\begin{aligned}
G(x_n, x_{n+1}, x_{n+2}) &= G(fx_{n-1}, fx_n, fx_{n+1}) \\
&\leq G(x_{n-1}, x_n, x_{n+1}) - \rho(x_{n-1}, x_n, x_{n+1}) \\
&\leq G(x_{n-1}, x_n, x_{n+1})
\end{aligned}
$$

for all $n \in N$. Therefore, we have $\{G(x_{n-1}, x_n, x_{n+1})\}_{n=1}^{\infty}$ nondecreasing. Since $\{G(x_{n-1}, x_n, x_{n+1})\}_{n=1}^{\infty}$ is bounded, there exists $l \geq 0$ such that $\lim_{n\to\infty} G(x_{n-1}, x_n, x_{n+1}) = l$. Thus, there exists $q \geq 0$ such that $\lim_{n\to\infty} \rho(x_{n-1}, x_n, x_{n+1}) = q$.

Assume that $l > 0$. Then, by the ρ–contractivity of f, we have

$$l \leq l - q.$$

Hence, $q = 0$, which implies that $l = 0$, a contradiction. Therefore, we have

$$(2.1) \qquad\qquad \lim_{n\to\infty} G(x_{n-1}, x_n, x_{n+1}) = 0.$$

Now we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Assume the contrary. Then there exists $\epsilon_0 > 0$ for which we can define subsequences $\{x_{m_k}\}_{k=1}^{\infty}$, $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{x_{p_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that n_k is minimal in the sense that $n_k > m_k > p_k > k$ and $G(x_{p_k}, x_{m_k}, x_{n_k}) \geq \epsilon_0$. Therefore, $G(x_{p_k}, x_{m_k}, x_{n_k-1}) < \epsilon_0$. Observe that

$$
\begin{aligned}
\epsilon_0 \;\; & \leq \;\; G(x_{p_k}, x_{m_k}, x_{n_k}) \\[2mm]
& \leq \;\; G(x_{p_k}, x_{p_k-1}, x_{p_k-1}) + G(x_{p_k-1}, x_{m_k}, x_{n_k}) \\[2mm]
& < \;\; \epsilon_0 + G(x_{p_k-1}, x_{m_k}, x_{n_k}).
\end{aligned}
$$

Letting $k \to \infty$, we obtain $\epsilon_0 \leq \lim_{k\to\infty} G(x_{p_k}, x_{m_k}, x_{n_k}) \leq \epsilon_0$ and so

$$(2.2) \qquad\qquad \lim_{k\to\infty} G(x_{p_k}, x_{m_k}, x_{n_k}) = \epsilon_0.$$

Similarly we have

$$(2.3) \qquad\qquad \lim_{k\to\infty} G(x_{p_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon_0.$$

Further we deduce that the limit $\lim_{k\to\infty} \rho(x_{p_k-1}, x_{m_k-1}, x_{n_k-1})$ also exists. Now by the ρ–contractivity, we have

$$
\begin{aligned}
G(x_{p_k}, x_{m_k}, x_{n_k}) \;\; & = \;\; G(fx_{p_k-1}, fx_{m_k-1}, fx_{n_k-1}) \\[2mm]
& \leq \;\; G(x_{p_k-1}, x_{m_k-1}, x_{n_k-1}) - \rho(x_{p_k-1}, x_{m_k-1}, x_{n_k-1}).
\end{aligned}
$$

From 2.2 and 2.3, we may find that

$$(2.4) \qquad\qquad 0 \leq -\lim_{n \to \infty} \rho(x_{p_k - 1}, x_{m_k - 1}, x_{n_k - 1}),$$

which further implies that $\lim_{n \to \infty} \rho(x_{p_k - 1}, x_{m_k - 1}, x_{n_k - 1}) = 0$. Notice that $x_{m_k - 1} \preceq$ $x_{n_k - 1}$ at each $k \in N$. Consequently, we obtain that $\lim_{n \to \infty} G(x_{p_k - 1}, x_{m_k - 1}, x_{n_k - 1}) = 0$, which is a contradiction. So $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, there exists $x^\star$ such that $x_n = f^n x_0 \to x^\star$ as $n \to \infty$. Finally, the continuity of f and $f f^n x_0 = f^{n+1} x_0 \to x^\star$ implies that $f x^\star = x^\star$.

Therefore, $x^\star$ is a fixed point of f. □

Next, we drop the continuity of f in Theorem 2.1 and find out that we can still guarantee a fixed point if we strengthen the condition of a partially ordered set to be a sequentially ordered set.

Theorem 2.2. *Let $(X, \preceq, G)$ be a complete sequentially ordered G-metric space and $f : X \to X$ be nondecreasing $\rho-$contraction of type (A) w.r.to $\preceq$. If there exists $x_0 \in X$ with $x_0 \preceq f x_0$, then $\{f^n x_0\}_{n=1}^{\infty}$ converges to a fixed point of f in X.*

Proof. If we take $x_n = f^n x_0$ in the proof of the Theorem 2.1, then we conclude that $\{x_n\}_{n=1}^{\infty}$ converges to a point $x^\star \in X$.

Next we prove that $x^\star$ is fixed point of f in X. Indeed suppose that $x^\star$ is not a fixed point of f, i.e., $G(x^\star, f x^\star, f x^\star) \neq 0$. Since $x^\star$ is comparable with x_n for all $n \in N$, we have

$$
\begin{aligned}
G(x^\star, f x^\star, f x^\star) &\leq G(x^\star, f x_n, f x_n) + G(f x_n, f x^\star, f x^\star) \\
&\leq G(x^\star, f x_n, f x_n) + G(x_n, x^\star, x^\star) - \rho(x_n, x^\star, x^\star) \\
&\leq G(x^\star, x_{n+1}, x_{n+1}) + G(x_n, x^\star, x^\star) - \rho(x_n, x^\star, x^\star) \\
&\leq G(x^\star, x_{n+1}, x_{n+1}) + G(x_n, x^\star, x^\star)
\end{aligned}
$$

for all $n \in N$. By the definition of convergent sequence, we have for any $\epsilon > 0$, there exists $n \in N$ such that $G(x_n, x_n, x^\star) < \frac{\epsilon}{2}$. Therefore we have

$$G(x^\star, fx^\star, fx^\star) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

As easily seen, $G(x^\star, fx^\star, fx^\star) = 0$, which is contradiction. Hence $x^\star$ is a fixed point of f. □

Corollary 2.1. *Let $(X, \preceq, G)$ be a complete totally ordered G-metric space and $f : X \to X$ be nondecreasing $\rho-$contraction of type (A) w.r.to $\preceq$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then $\{f^n x_0\}_{n=1}^\infty$ converges to a fixed point of f in X.*

Proof. Take $x_n = f^n x_0$ as in proof of Theorem 2.1. Since the total ordering implies the partial ordering, we conclude that $\{x_n\}_{n=1}^\infty$ converges to a fixed point.

Next we show that the fixed point of f is unique. Assume that u, v and w are three distinct fixed points of f, i.e., $G(u, v, w) \neq 0$. Since X is totally ordered, u, v and w are comparable. Thus, we have

$$
\begin{aligned}
G(u, v, w) &= G(fu, fv, fw) \\
&\leq G(u, v, w) - \rho(u, v, w)
\end{aligned}
$$

which is a contradiction. Therefore, $u = v = w$ and the fixed point of f is unique. □

We can still guarantee the uniqueness of the fixed point by weakening the total ordering as stated and proved in the next theorem.

Theorem 2.3. *Let $(X, \preceq, G)$ be a complete partially ordered G-metric space and $f : X \to X$ be continuous and nondecreasing $\rho-$contraction of type (A) w.r.to $\preceq$. Suppose that for each $x, y, z \in X$, there exists $r \in X$ which is comparable to x, y, z. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then $\{f^n x_0\}_{n=1}^\infty$ converges to a fixed point of f in X.*

Proof. Take $x_n = f^n x_0$ as in proof of Theorem 2.1. Since the total ordering implies the partial ordering, we conclude that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point.

Next we show that the fixed point of f is unique. Assume that u, v, w are distinct fixed points of f, i.e., $G(u, v, w) \neq 0$. Since $u, v, w \in X$, there exists $r \in X$ such that r is comparable to u, v, w. We will prove this part by showing that the sequence $\{r_n\}_{n=1}^{\infty}$ given by $r_n = f^n r$ converges to u, v and w. Therefore we have

$$
\begin{aligned}
G(u, f^n r, f^n r) &\leq G(u, f^{n-1}r, f^{n-1}r) - \rho(u, f^{n-1}r, f^{n-1}r) \\
&\leq G(u, f^{n-1}r, f^{n-1}r).
\end{aligned}
$$

(2.5)

If we define a sequence $y_n = G(u, f^{n-1}r, f^{n-1}r)$ and $z_n = \rho(u, f^{n-1}r, f^{n-1}r)$, we may obtain from 2.5 that $\{y_n\}_{n=1}^{\infty}$ is nonincreasing and there exist $l, q \geq 0$ such that $\lim_{n \to \infty} y_n = l$ and $\lim_{n \to \infty} z_n = q$.

Assume that $l > 0$. Then by the $\rho-$contractivity of f, we have

$$ l \leq l - q $$

which contradiction. Hence $\lim_{n \to \infty} y_n = 0$. In the same way, we can show that $\lim_{n \to \infty} G(u, f^{n-1}r, f^{n-1}r) = 0$. That is, $\{r_n\}_{n=1}^{\infty}$ converges to u, v and w. Since the limit of convergent sequence is unique, we conclude that $u = v = w$. Hence, this yields the uniqueness of fixed point. $\qquad \square$

Theorem 2.4. *Let $(X, \preceq, G)$ be a complete sequentially ordered G-metric space and $f : X \to X$ be continuous and nondecreasing $\rho-$contraction of type (A) w.r.to $\preceq$. Suppose that for each $x, y, z \in X$, there exists $r \in X$ which is comparable to x, y, z. If there exists $x_0 \in X$ with $x_0 \preceq f x_0$, then $\{f^n x_0\}_{n=1}^{\infty}$ converges to a fixed point of f in X.*

Proof. Take $x_n = f^n x_0$ as in proof of Theorem 2.1. Since the total ordering implies the partial ordering, we conclude that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point. The rest of the proof is similar to the proof of Theorem 2.3. □

Remark 1. In parallel with the study of Theorems 2.1, 2.2, 2.3 and 2.4, we can also prove in the same way that if the mapping f is nonincreasing, the above theorems still hold. However, we will omit the result for nonincreasing mappings.

Next we drop the monotonically conditions of f and find out that we can still apply our results to confirm the existence and uniqueness of a fixed point of f.

Theorem 2.5. *Let $(X, \preceq, G)$ be a complete partially ordered G-metric space and $f : X \to X$ be continuous $\rho-$contraction of type (A) w.r.to $\preceq$ such that the comparability of $x, y, z \in X$ implies comparability of $fx, fy, fz \in f(X)$. If there exists $x_0 \in X$ such that x_0 and fx_0 are comparable, then $\{f^n x_0\}_{n=1}^{\infty}$ converges to a fixed point of f in X.*

Proof. For the existence of fixed point, we choose $x_0 \in X$ such that x_0 and fx_0 are comparable. If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. We define a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n = f^n x_0$. Since x_0 and fx_0 are comparable, we have x_n and x_{n+1} comparable for all $n \in N$.

If there exists $n_0 \in N$ such that $\rho(x_{n_0}, x_{n_0+1}, x_{n_0+2}) = G(x_{n_0}, x_{n_0+1}, x_{n_0+2})$, then by the notion of $\rho-$contractivity, the proof is finished. Therefore, we assume that $\rho(x_n, x_{n+1}, x_{n+2}) < G(x_n, x_{n+1}, x_{n+2})$ for all $n \in N$. Also assume that $\rho(x_n, x_{n+1}, x_{n+2}) \neq 0$ for all $n \in N$. Otherwise we can find $n_0 \in N$ with $x_{n_0} = x_{n_0+1}$, that is $x_{n_0} = fx_{n_0}$, and the proof is finished. Hence, we consider only the case where $0 < \rho(x_n, x_{n+1}, x_{n+2}) < G(x_n, x_{n+1}, x_{n+2})$ for all $n \in N$.

Since x_n and x_{n+1} are comparable for all $n \in N$, we have

$$
\begin{aligned}
G(x_n, x_{n+1}, x_{n+2}) &= G(fx_{n-1}, fx_n, fx_{n+1}) \\
&\leq G(x_{n-1}, x_n, x_{n+1}) - \rho(x_{n-1}, x_n, x_{n+1}) \\
&\leq G(x_{n-1}, x_n, x_{n+1})
\end{aligned}
$$

for all $n \in N$. Therefore, we have $\{G(x_{n-1}, x_n, x_{n+1})\}_{n=1}^{\infty}$ nonincreasing. Since $\{G(x_{n-1}, x_n, x_{n+1})\}_{n=1}^{\infty}$ is bounded, there exists $l \geq 0$ such that $\lim_{n\to\infty} G(x_{n-1}, x_n, x_{n+1}) = l$. Thus, there exists $q \geq 0$ such that $\lim_{n\to\infty} \rho(x_{n-1}, x_n, x_{n+1}) = q$.

Assume that $l > 0$. Then, by the $\rho-$contractivity of f, we have

$$
l \leq l - q.
$$

Hence, $q = 0$, which implies that $l = 0$, a contradiction. Therefore, we have

$$
(2.6) \qquad \lim_{n\to\infty} G(x_{n-1}, x_n, x_{n+1}) = 0.
$$

Now we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Assume the contrary. Then there exists $\epsilon_0 > 0$ for which we can define subsequences $\{x_{m_k}\}_{k=1}^{\infty}$, $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{x_{p_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that n_k is minimal in the sense that $n_k > m_k > p_k > k$ and $G(x_{p_k}, x_{m_k}, x_{n_k}) \geq \epsilon_0$. Therefore, $G(x_{p_k}, x_{p_k-1}, x_{p_k-1}) < \epsilon_0$. Observe that

$$
\begin{aligned}
\epsilon_0 &\leq G(x_{p_k}, x_{m_k}, x_{n_k}) \\
&\leq G(x_{p_k}, x_{p_k-1}, x_{p_k-1}) + G(x_{p_k-1}, x_{m_k}, x_{n_k}) \\
&< \epsilon_0 + G(x_{p_k-1}, x_{m_k}, x_{n_k}).
\end{aligned}
$$

Letting $k \to \infty$, we obtain $\epsilon_0 \leq \lim_{k\to\infty} G(x_{p_k}, x_{m_k}, x_{n_k}) \leq \epsilon_0$ and so

$$
(2.7) \qquad \lim_{k\to\infty} G(x_{p_k}, x_{m_k}, x_{n_k}) = \epsilon_0.
$$

Similarly we have

(2.8)
$$\lim_{k\to\infty} G(x_{p_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon_0.$$

Further we deduce that the limit $\lim_{k\to\infty} \rho(x_{p_k-1}, x_{m_k-1}, x_{n_k-1})$ also exists. Now by the $\rho-$contractivity, we have

$$
\begin{aligned}
G(x_{p_k}, x_{m_k}, x_{n_k}) &= G(fx_{p_k-1}, fx_{m_k-1}, fx_{n_k-1}) \\
&\leq G(x_{p_k-1}, x_{m_k-1}, x_{n_k-1}) - \rho(x_{p_k-1}, x_{m_k-1}, x_{n_k-1}).
\end{aligned}
$$

From 2.7 and 2.8, we may find that

(2.9)
$$0 \leq - \lim_{n\to\infty} \rho(x_{p_k-1}, x_{m_k-1}, x_{n_k-1}),$$

which further implies that $\lim_{n\to\infty} \rho(x_{p_k-1}, x_{m_k-1}, x_{n_k-1}) = 0$. Notice that $x_{m_k-1} \preceq x_{n_k-1}$ at each $k \in N$. Consequently, we obtain that $\lim_{n\to\infty} G(x_{p_k-1}, x_{m_k-1}, x_{n_k-1}) = 0$, which is a contradiction. So $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, there exists $x^\star$ such that $x_n = f^n x_0 \to x^\star$ as $n \to \infty$. Finally, the continuity of f and $f f^n x_0 = f^{n+1} x_0 \to x^\star$ implies that $fx^\star = x^\star$.

Therefore, $x^\star$ is a fixed point of f. $\qquad\square$

Theorem 2.6. *Let $(X, \preceq, G)$ be a complete sequentially ordered G-metric space and $f : X \to X$ be continuous $\rho-$contraction of type (A) w.r.to $\preceq$ such that the comparability of $x, y, z \in X$ implies comparability of $fx, fy, fz \in f(X)$. If there exists $x_0 \in X$ such that x_0 and fx_0 are comparable, then $\{f^n x_0\}_{n=1}^{\infty}$ converges to a fixed point of f in X.*

Theorem 2.7. *Let $(X, \preceq, G)$ be a complete totally ordered G-metric space and $f : X \to X$ be $\rho-$contraction of type (A) w.r.to $\preceq$ such that the comparability of $x, y, z \in X$ implies comparability of $fx, fy, fz \in f(X)$. If there exists $x_0 \in X$ such that x_0 and fx_0 are comparable, then $\{f^n x_0\}_{n=1}^{\infty}$ converges to a fixed point of f in X.*

Theorem 2.8. *Let $(X, \preceq, G)$ be a complete partially ordered G-metric space and $f :$ $X \to X$ be continuous $\rho-$contraction of type (A) w.r.to $\preceq$ such that the comparability of $x, y, z \in X$ implies comparability of $fx, fy, fz \in f(X)$. If there exists $x_0 \in X$ such that x_0 and fx_0 are comparable, then $\{f^n x_0\}_{n=1}^{\infty}$ converges to a fixed point of f in X.*

Theorem 2.9. *Let $(X, \preceq, G)$ be a complete sequentially ordered G-metric space and $f : X \to X$ be $\rho-$contraction of type (A) w.r.to $\preceq$ such that the comparability of $x, y, z \in X$ implies comparability of $fx, fy, fz \in f(X)$. If there exists $x_0 \in X$ such that x_0 and fx_0 are comparable, then $\{f^n x_0\}_{n=1}^{\infty}$ converges to a fixed point of f in X.*

3. EXAMPLE

We give an example to ensure the applicability of our theorems.

Example 3.1. *Let $X = [0, 1] \times [0, 1]$ and suppose that we write $x = (x_1, x_2)$ and $y = (y_1, y_2)$ for all $x, y \in X$. Define $G, \rho : X \times X \times X \to R$ by*

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z \\ 2\max\{x_1 + y_1 + z_1, x_2 + y_2 + z_2\} & \text{otherwise} \end{cases}$$

and

$$\rho(x, y, z) = \begin{cases} 0, & \text{if } x = y = z \\ \max\{x_1, x_2 + y_2 + z_2\} & \text{otherwise.} \end{cases}$$

Let $\preceq$ be an ordering in X such that for $x, y, z \in X$, $x \preceq y \preceq y$ if and only if $x_1 = y_1 = z_1$ and $x_2 \preceq y_2 \preceq z_2$. Then $(X, \preceq, G)$ is a partially ordered G-metric space with $\rho-$function of type (A) w.r.to $\preceq$ in X.

Now, let f be a self mapping on X defined by $f(x) = f(x_1, x_2) = \left(0, \frac{x_2^2}{2}\right)$ for all $x \in X$. It is obvious that f is continuous and nondecreasing w.r.to $\preceq$.

Let $x, y, z \in X$ be comparable w.r.to $\preceq$. If $x = y = z$, then they clearly satisfy the inequality 1.1. On the other hand, if $x \neq y \neq z$, we have

$$\begin{aligned}
G(fx, fy, fz) &= G(f(x_1, x_2), f(y_1, y_2), f(z_1, z_2)) \\
&= G\left(\left(0, \frac{x_2^2}{2}\right), \left(0, \frac{y_2^2}{2}\right), \left(0, \frac{z_2^2}{2}\right)\right) \\
&= 2\max\{0, \frac{x_2^2}{2} + \frac{y_2^2}{2} + \frac{z_2^2}{2}\} \\
&= x_2^2 + y_2^2 + z_2^2 \\
&\leq x_2 + y_2 + z_2 \\
&\leq \max\{3x_1, x_2 + y_2 + z_2\} \\
&\leq 2\max\{3x_1, x_2 + y_2 + z_2\} - \max\{3x_1, x_2 + y_2 + z_2\} \\
&\leq 2\max\{3x_1, x_2 + y_2 + z_2\} - \max\{x_1, x_2 + y_2 + z_2\} \\
&\leq G(x, y, z) - \rho(x, y, z).
\end{aligned}$$

Therefore the inequality 1.1 is satisfied for every comparable $x, y, z \in X$. So f is a continuous and nondecreasing $\rho-$contraction of type (A) w.r.to $\preceq$. Let $x_0 = (0,0)$, so we have $x_0 \preceq fx_0$. Now applying Theorem 2.1, we conclude that f has a fixed point in X which is the point (0,0).

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