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SEMICLASSICAL PSEUDODIFFERENTIAL OPERATORS WITH OPERATOR SYMBOL

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Abstract. This work is a generalization to the symbol operator case of the classical h-pseudodifferentials operators. We are interested to the properties of composition, symbolic calculus, and the L^2-continuity of these operators type.

1. Introduction

The main motivation of the h-pseudodifferential calculus is to get an algebraic correspondence between the classical observables and the quantum observables (one calls it a quantization of the classical observables). In particular, this would permit us to localize (within the limits allowed by the uncertainty principle) both in position and momentum variables any quantum state ψ, take a smooth cutoff function $\chi(x,\xi) \in C_0^{\infty}(\mathbb{R}^{2n})$ (the space of smooth compactly supported functions and χ is close to the characteristic function of some compact subset of $\mathbb{R}^{2n}$). Then its associated quantum observable $\chi(x, hD_x)\psi$ will have the effect of (essentially) cutting off the Cartesian product $Supp\psi \times Supp\widehat{\psi}$ outside $Supp\psi$ (here $Supp$ stands for the support). Another important feature of this calculus will consist in inverting the elliptic operators. If $a(x,\xi)$ is a classical observable that never vanishes (and therefore is invertible in multiplicative algebra of smooth functions), one would like to be able to invert also

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its quantization $a(x, hD_x)$. This procedure will be possible when a satisfies a little bit more that it is an invertible element of a special kind of subalgebra of $C^\infty(\mathbb{R}^n)$, called spaces of symbols. We refer the reader to the books of [12, 9, 16].

The idea of the elaboration of this work is inspired works Martinez, Klein-Martinez-Seiler-Wang [8], Martinez-Messirdi [11] and Messirdi-Senoussaoui [13, 14] on the study of the spectrum of operators of the type

$$P = -h^2\Delta_x - \Delta_y + V(x, y) = -h^2\Delta_x + V(x) \text{ on } L^2\left(\mathbb{R}^n_x \times \mathbb{R}^p_y\right),$$

where $V(x, y)$ is the potential and is $h = \frac{1}{\sqrt{M}} \to 0$ (is proportional to the inverse of the square-root of the nuclear mass), and the construction of asymptotic expansions in powers of $\sqrt{h}$ for eigenvalues and associated eigenfunctions of P of the types:

$$\sum_{j\geq 0} \alpha_j h^{j/2} \text{ and } e^{-\psi(x)/h}\left(\sum_{j\geq 0} a_j(x, y) h^{j/2}\right),$$

where $\psi(x)$ is the Agmon distance between x and the potential well.

We generalize to the case operator the well know scalar h-pseudodifferential calculus developed by many authors ([2, 4, 5, 6, 7, 10, 15, 16, 17, 19]). We introduce the notion of symbol operator which satisfy estimates of special kind. We try to specify a little bit more the way in which the symbols may depend on the semiclassical parameter h. We define the notion $\sim$(the so-called asymptotic equivalence of symbols), which will be used used in the study of the semiclassical expansion of the spectrum.

We are interested in the composition of two h-pseudodifferentials operators with operator symbol. We see that it is always possible, and that one can construct one to one correspondences between h-pseudodifferential operators with operator symbol and symbols depending on $2n$ variables.

For the h-pseudodifferential operator with operator symbol, an interesting question is under which conditions on a these operators are bounded on L^2. The last part of this paper provides a rather complete answer to this question.

We note the work of Balazard-Konlein [1] and Senoussaoui [18] to study these operators type. This work presents a new approach for this study.

2. Spaces of symbols

Definition 2.1. An order function g is a $C^\infty\left(\mathbb{R}^d;\mathbb{R}_+^*\right)$- function satisfying:

$$|\partial_x^\alpha g(x)| \leq C_\alpha g\left(x\right),\ \forall\alpha\in\mathbb{N}^d,\forall x\in\mathbb{R}^d$$

or

$$\partial_x^\alpha g = \mathcal{O}\left(g\right),\ \forall\alpha\in\mathbb{N}^d.$$

The simplest examples is given by $\langle x\rangle = \left(1+|x|^2\right)^{\frac{m}{2}}$, where m is a natural number. Other examples are $e^{f(x)}$ where f is smooth and bounded together with all its derivatives.

Proposition 2.1. *If g is an order function on $\mathbb{R}^d$, then so is the function $\frac{1}{g}$.*

Proof. Indeed, one has to show that for any $\alpha\in\mathbb{N}^d$, $\partial^\alpha\left(\frac{1}{g}\right) = \mathcal{O}\left(\frac{1}{g}\right)$. Setting $\widetilde{g} = \frac{1}{g}$ and using the Leibniz formula to differentiate the identity $g\widetilde{g} = 1$ α times, the required estimate is easily obtained by induction on $|\alpha|$. □

Definition 2.2. A function $a = a\left(x;h\right)$ defined on $\mathbb{R}^d\times\left]0,h_0\right]\longrightarrow\mathcal{L}\left(H,K\right)$ with operator valued, for some $h_0 > 0$ and H,K are Hilbert spaces, is said to be in $S_g^d\left(H,K\right)$ if a depends smoothly on x and for any $\alpha\in\mathbb{N}^d$ one has

$$\left\|\partial_x^\alpha a\left(x;h\right)\right\|_{\mathcal{L}(H,K)} = \mathcal{O}\left(g\left(x\right)\right)$$

uniformly with respect to $(x;h)\in\mathbb{R}^d\times\left]0,h_0\right]$

In particular, $S_1^d\left(H,K\right)$ is the set of $C^\infty\left(\mathbb{R}^d;\mathcal{L}\left(H,K\right)\right)$ parameterized by some $h\in\left]0,h_0\right]$ that are uniformly bounded together with all their derivatives.

- If $V = V(x) \in S_1^n(H,K)$, then the operator $\xi^2 + V(x)$ is in $S_g^{2n}(H,K)$ where $g(\xi) = \langle \xi \rangle^2$.
- Any $\chi \in C_0^\infty\left(\mathbb{R}^d; \mathcal{L}(H,K)\right)$ (the space of compactly supported C^∞ functions on $\mathbb{R}^d$ with operator valued) is in $S_1^d(H,K)$.

By proposition 2.1and the Leibniz formulas we have the equivalence

$$a \in S_g^d(H,K) \Longleftrightarrow \frac{1}{g} a \in S_1^d(H,K). \tag{2.1}$$

We endow $S_g^d(H,K)$ with the topology associated with the family of seminorms $N_\alpha(a) = \sup \frac{1}{g} \|\partial_x^\alpha a\|_{\mathcal{L}(H,K)}$, and it can be verified easily that this makes $S_g^d(H,K)$ a Fréchet space (topological vector space where the topology is defined by a family of seminorms). The algebraic properties of the space $S_g^d(H,K)$ are the following.

Proposition 2.2. *Let g_1 and g_2 be two order functions on $\mathbb{R}^d$, and let $a \in S_{g_1}^d(H,L)$, $b \in S_{g_2}^d(L,K)$ where H, L, K are Hilbert spaces. Then $g_1 g_2$ is also an order function and $ba \in S_{g_1 g_2}^d(H,K)$.*

Proof. This is an obvious consequence of the Leibniz formula. □

Definition 2.3. A symbol $a \in S_g^d(H,K)$ is said to be elliptic if there exists a positive constant C_0 such that

$$\|a(x;h)\|_{\mathcal{L}(H,K)} \geq \frac{1}{C_0} g(x)$$

uniformly with respect to $(x;h) \in \mathbb{R}^d \times]0,h_0]$.

Then we have the following proposition:

Proposition 2.3. *If $a \in S_g^d(H,K)$ is elliptic, then $a^{-1} \in S_{\frac{1}{g}}^d(K,H)$.*

Proof. Set $b = a^{-1}$. Then the result is obtained by differentiating iteratively the relation $ba = I_{\mathcal{L}(H)}$ and by using the Leibniz formula. □

2.1. **Semiclassical expansions of symbols.** Throughout this section g denotes an arbitrary order function on $\mathbb{R}^d$, and H, K are Hilbert spaces.

Definition 2.4. Let $a \in S_g^d(H,K)$ and let $(a_j)_{j\in\mathbb{N}}$ be a sequence of symbols of $S_g^d(H,K)$. Then we say that a is asymptotically equivalent to the formal sum $\sum_{j=0}^{\infty} h^j a_j$ in $S_g^d(H,K)$, and we write

$$a \sim \sum_{j=0}^{\infty} h^j a_j$$

if for any $N \in \mathbb{N}$ and for any $\alpha \in \mathbb{N}^d$ there exist $h_{N,\alpha} > 0$ and $C_{N,\alpha} > 0$ such that

$$\left\| \partial^\alpha \left(a - \sum_{j=0}^{N} h^j a_j \right) \right\|_{\mathcal{L}(H,K)} \leq C_{N,\alpha} h^N g$$

uniformly on $\mathbb{R}^d \times]0, h_{N,\alpha}]$.

In other words, for any $N > 0$ the symbol a can be approximated by $\sum_{j=0}^{N} h^j a_j$ up to a symbol that vanishes together with all its derivatives as h^N when goes to zero. In practice, the existence of $h_{N,\alpha}$ will not be explicitly written, being referred to as h small enough at the end of an estimate.

In particular case where all the a_j's are identically zero, we write

$$a = \mathcal{O}(h^\infty) \text{ in } S_g^d(H,K) \text{ if } a \sim 0 \text{ in } S_g^d(H,K).$$

An important and surprising feature is that although a series of the type $\sum_{j=0}^{\infty} h^j a_j$ has no reason to be convergent, one can always find a symbol that is, asymptotically equivalent to it. The following proposition gives an answer:

Proposition 2.4. *Let $(a_j)_{j\in\mathbb{N}}$ be an arbitrary sequence of symbols of $S_g^d(H,K)$. Then there exists $a \in S_g^d(H,K)$ such that $a \sim \sum_{j=0}^{\infty} h^j a_j$ in $S_g^d(H,K)$. Moreover, a is unique up to $\mathcal{O}(h^\infty)$ in $S_g^d(H,K)$. Such a symbol a is called resummation of the formal symbol $\sum_{j=0}^{\infty} h^j a_j$.*

Proof. First of all, dividing everything by g and using (2.1), we can assume without loss of generality that $g \equiv 1$.

Since the unicity up to $\mathcal{O}(h^\infty)$ is obvious, we concentrate on the existence of a. Then let $\chi \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp}\chi \subset [-2,2]$, $\chi = 1$ on $[-1,1]$.

We have the following lemma:

Lemma 2.1. *There exists a decreasing sequence of positive numbers $(\varepsilon_j)_{j\in\mathbb{N}}$ converging to zero, such that for any $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \le j$, one has*

$$\sup_{x\in\mathbb{R}^d} \left\| \left(1-\chi\left(\frac{\varepsilon_j}{h}\right)\right) \partial^\alpha a_j(x;h) \right\|_{\mathcal{L}(H,K)} \le h^{-1}$$

for h small enough.

Proof. Setting

$$C_j = \sup_{|\alpha|\le j, x\in\mathbb{R}^d} \|\partial^\alpha a_j(x;h)\|_{\mathcal{L}(H,K)}$$

and using the fact that $1-\chi\left(\frac{\varepsilon_j}{h}\right)$ is non zero only for $h \le \varepsilon_j$, we have

$$h \sup_{x\in\mathbb{R}^d} \left\| \left(1-\chi\left(\frac{\varepsilon_j}{h}\right)\right) \partial^\alpha a_j(x;h) \right\|_{\mathcal{L}(H,K)} \le C_j\varepsilon_j \le 1$$

if one has chosen the decreasing sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ in such a way that $\varepsilon_j \le \frac{1}{C_j}$ (for all $j \ge 0$ one take, e.g., $\varepsilon_j = \min\left\{(k+C_k)^{-1}; k \le j\right\}$). □

We then set

$$a(x;h) = \sum_{j\ge 0} h^j \left(1-\chi\left(\frac{\varepsilon_j}{h}\right)\right) a_j(x;h),$$

where actually, the sum contains only a finite number (depending on $h > 0$ fixed) of nonzero terms (since $\varepsilon_j < h$ if j becomes large). Thus a is a smooth function of $x \in \mathbb{R}^d$, and for any $\alpha \in \mathbb{N}^d$ one has

$$\|\partial^\alpha a(x;h)\|_{\mathcal{L}(H,K)} \le \sum_{j\le|\alpha|} h^j \|\partial^\alpha a_j(x;h)\|_{\mathcal{L}(H,K)} + \sum_{j>|\alpha|} h^j \left\| \left(1-\chi\left(\frac{\varepsilon_j}{h}\right)\right) \partial^\alpha a_j(x;h) \right\|_{\mathcal{L}(H,K)}$$

and therefore, using Lemma 2.1,

$$\|\partial^\alpha a(x;h)\|_{\mathcal{L}(H,K)} \leq C_\alpha + \sum_{j>|\alpha|} h^j \leq C'_\alpha,$$

where C_α, C'_α are positive constants.

Thus $a \in S^d_1(H,K)$, and for any $\alpha \in \mathbb{N}^d$ and $N \geq |\alpha|$ one has

$$\left\|\partial^\alpha \left(a - \sum_{j=0}^{N} h^j a_j\right)\right\|_{\mathcal{L}(H,K)} \leq \sum_{j=0}^{N} h^j \left\|\chi\left(\frac{\varepsilon_j}{h}\right)\partial^\alpha a_j\right\|_{\mathcal{L}(H,K)} + \sum_{j\geq N+1} h^j \left\|\left(1-\chi\left(\frac{\varepsilon_j}{h}\right)\right)\partial^\alpha a_j\right\|_{\mathcal{L}(H,K)}.$$

Using again Lemma 2.1, we get

$$\left\|\partial^\alpha \left(a - \sum_{j=0}^{N} h^j a_j\right)\right\|_{\mathcal{L}(H,K)} \leq \sum_{j=0}^{N} h^{N+j}\varepsilon_j^{-N} \left|\left(\frac{\varepsilon_j}{h}\right)^N \chi\left(\frac{\varepsilon_j}{h}\right)\right| C_{j,\alpha} + \sum_{j\geq N+1} h^{j-1},$$

where the $C_{j,\alpha}$'s are positive constants. Since the function $\mathbb{R} \ni t \longmapsto t^N\chi(t)$ is bounded, we deduce easily from the estimate above that there exists a constant C_N such that for any $h > 0$ sufficiently small,

$$\left\|\partial^\alpha \left(a - \sum_{j=0}^{N} h^j a_j\right)\right\|_{\mathcal{L}(H,K)} \leq C_N h^N.$$

□

3. h-Pseudodifferential operators with operator symbol

Definition 3.1. For $a \in S^{3n}_{\langle\xi\rangle^m}(H,K)$ and for $u \in C_0^\infty(\mathbb{R}^n, H)$ (the space of smooth compactly supported functions with Hilbert valued), we define the h-pseudodifferential operator with operator symbol by

$$Op_h(a)\,u(x;h) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\xi} a(x,y,\xi)\,u(y)\,dyd\xi. \tag{3.1}$$

Proposition 3.1. *For all $a \in S^{3n}_{\langle\xi\rangle^m}(H,K)$ and for any $\nu \in \mathbb{R}$ the operator $h^{-\nu}Op_h(a) : C_0^\infty(\mathbb{R}^n, H) \longrightarrow C^\infty(\mathbb{R}^n, K)$ is linear continuous.*

Proof. In general the integral (3.1) is not absolutely convergent, so we use the technique of the oscillatory integral developed by Hörmander see [7, 5, 16, 19] □

Example 3.1. **i):** *The scalar case:*

$$a(x,y,\xi)=\sum_{|\alpha|\le m} b_\alpha(x)\,\xi^\alpha$$

with $b_\alpha \in S_1^{3n}(\mathbb{R},\mathbb{C})$, *we get*

$$Op_h(a)=\sum_{|\alpha|\le m} b_\alpha(x)\,(hD_x)^\alpha,\ \ D_x=-i\partial_x$$

ii): *Inverse of* $1-h^2\Delta_x+V(x)$, $V(x)\in\mathcal{L}(H,K)$: *taking* $a(x,y,\xi)=\left(1+\xi^2+V(x)\right)^{-1}$, *we get an operator that satisfies*

$$\left(1-h^2\Delta_x+V(x)\right)\circ Op_h\left(\left(1+\xi^2+V(x)\right)^{-1}\right)=I_{\mathcal{L}(K)}\ \text{on}\ C_0^\infty(\mathbb{R}^n,K)$$

Theorem 3.1. *For all* $a\in S^{3n}_{\langle\xi\rangle^m}(H,K)$, $Op_h(a)$ *can be extended in a unique way to a linear continuous operator* $\mathcal{S}(\mathbb{R}^n,H)\longrightarrow\mathcal{S}(\mathbb{R}^n,K)$ *(the Schwarz space with Hilbert valued). And by duality* $Op_h(a)$ *can be extended in a unique way to a linear continuous operator* $\mathcal{S}'(\mathbb{R}^n,K)\longrightarrow\mathcal{S}'(\mathbb{R}^n,H)$

Proof. For any $\alpha,\beta\in\mathbb{N}^n$, writing

$$x^\beta\partial_x^\alpha I_k u(x)=\left(\int_{|x-y|\le\frac{1}{2}|x|}+\int_{|x-y|\ge\frac{1}{2}|x|}\right)x^\beta\partial_x^\alpha\left[e^{\frac{i}{h}(x-y)\xi}\left({}^tL\right)^k(au)\right]dyd\xi \tag{3.2}$$

with $L=\dfrac{1}{1+\xi^2}\left(1-h\xi D_y\right)$, we see that for $k>m+n+|\alpha|$ the first integral is $\mathcal{O}(1)$, because for any $\gamma>0$,

$$x^\beta\langle\xi\rangle^{m+|\alpha|-k}\langle y\rangle^{-\gamma}=\mathcal{O}\left(\langle\xi\rangle^{m+|\alpha|-k}\langle y\rangle^{|\beta|-\gamma}\right)$$

uniformly on $\left\{|x-y|\le\frac{1}{2}|x|\right\}$, and is therefore integrable with respect to (y,ξ) on $\mathbb{R}^{2n}$ if $\gamma>|\beta|+n$.

On the other hand, setting

$$L'=\frac{1}{1+|x-y|^2}\left(1+h(x-y)D_\xi\right),$$

we see by integrating by parts with respect to ξ that for any $N \in \mathbb{N}$, the second integral can be rewritten as sum of terms of the type

$$C_{\alpha',\alpha''} \int_{|x-y| \geq \frac{1}{2}|x|} x^{\beta} e^{\frac{i}{h}(x-y)\xi} \left({}^{t}L'\right)^{N} \left[\xi^{\alpha'} \partial_x^{\alpha''} \left({}^{t}L\right)^{k} (au)\right] dy d\xi$$

(with $\alpha' + \alpha'' = \alpha$ and $C_{\alpha',\alpha''}$ are constant) and is therefore $\mathcal{O}(1)$ if we take $N \geq |\beta|$.

As a consequence, $Op_h(a)\, u \in \mathcal{S}(\mathbb{R}^n, K)$, and moreover, the previous consideration actually show that $\left\|x^{\beta} \partial_x^{\alpha} Op_h(a)\, u(x)\right\|_K$ can be estimated by a finite number of seminorms of u in $\mathcal{S}(\mathbb{R}^n, H)$. □

3.1. **Composition.** Thanks to Theorem 3.1, there is no theoretical problem in defining the composition of two h-pseudodifferential operators with operator symbol. The problem is only to know whether this composed operator is again itself a h-pseudodifferential operator.

Let H, K and L three Hilbert spaces and let $m, m' \in \mathbb{R}$.

Theorem 3.2 (theorem of composition)**.** *For all* $a \in S^{3n}_{\langle\xi\rangle^m}(K, L)$ *and* $b \in S^{3n}_{\langle\xi\rangle^{m'}}(H, K)$ *there exists* $c \in S^{3n}_{\langle\xi\rangle^{m+m'}}(H, L)$ *such that*

$$Op_h(a) \circ Op_h(b) = Op_h(c).$$

Moreover, a possible choice for c *is given by the oscillatory integral*

$$c(x, y, \xi) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(x-z)(\eta-\xi)} a(x, z, \eta)\, b(z, y, \xi)\, dz d\eta := a\#b(x, y, \xi),$$

which satisfies

$$a\#b(x, y, \xi) \sim \sum_{|\alpha| \geq 0} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_z^{\alpha} \partial_\eta^{\alpha} \left(a(x, z, \eta)\, b(z, y, \xi)\right) \Big|_{\substack{z=x \\ \eta=\xi}} \text{ in } S^{3n}_{\langle\xi\rangle^{m+m'}}(H, L).$$

Proof. Making integrations by parts and using the same decomposition as in (3.2), we see that for $u \in C_0^{\infty}(\mathbb{R}^n, H)$ we have

$$Op_h(a)\, u(z) = \lim_{\substack{\varepsilon \to 0^+ \\ \delta \to 0^+}} \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h - \varepsilon\langle\xi\rangle - \delta\langle z\rangle}(x-y)\xi} b(z, y, \xi)\, u(y)\, dy d\xi, \tag{3.3}$$

where lim takes place for the topology of $\mathcal{S}(\mathbb{R}^n, K)$. As a consequence, the continuity of $Op_h(a)$: $\mathcal{S}(\mathbb{R}^n, K) \to \mathcal{S}(\mathbb{R}^n, L)$ gives

$$(2\pi h)^{2n} Op_h(a) \circ Op_h(b) u(x) =$$
$$\lim_{\substack{\varepsilon\to 0^+ \\ \delta\to 0^+}} \int e^{\frac{i}{h}(x-z)\eta} a(x,z,\eta) \left(\int e^{\frac{i}{h-\varepsilon\langle\xi\rangle-\delta\langle z\rangle}(x-y)\xi} b(z,y,\xi) u(y) dyd\xi \right) dzd\eta$$

which by a similar argument can be rewritten as

$$(2\pi h)^{2n} Op_h(a) \circ Op_h(b) u(x) =$$
$$\lim_{\substack{\varepsilon\to 0^+ \\ \delta\to 0^+}} \int e^{\frac{i}{h}(x-z)\eta + \frac{i}{h-\varepsilon\langle\xi\rangle-\delta\langle z\rangle-\delta\langle\eta\rangle}(x-y)\xi} a(x,z,\eta) b(z,y,\xi) u(y) dyd\xi dzd\eta,$$

and therefore

$$Op_h(a) \circ Op_h(b) u(x) = \lim_{\substack{\varepsilon\to 0^+ \\ \delta\to 0^+}} \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h-\varepsilon\langle\xi\rangle}(x-y)\xi} c_\delta(x,y,\xi) u(y) dyd\xi \tag{3.4}$$

with

$$c_\delta(x,y,\xi) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h-\delta\langle z\rangle-\delta\langle\eta\rangle}(x-z)(\eta-\xi)} a(x,z,\eta) b(z,y,\xi) dzd\eta.$$

As a consequence, by the dominated convergence theorem it is enough to prove that $c_\delta = \mathcal{O}\left(\langle\xi\rangle^{m+m'}\right)$ (with respect of the norm) uniformly with respect to δ, that for all $(x,y,\xi) \in \mathbb{R}^{3n}$, $c_\delta(x,y,\xi)$ has a limit $c_0(x,y,\xi)$ as $\delta \to 0^+$ (so that the first limit $\delta \to 0^+$ can be taken in (3.4), leading to a convergent integral), and that $c_0 \in S^{3n}_{\langle\xi\rangle^{m+m'}}(H, L)$ (so the second limit $\varepsilon \to 0^+$ can be taken in (3.4), leading to an oscillatory integral).

Set

$$L_1 = \left(1 + \frac{|\eta-\xi|^2}{h^2} + \frac{|x-z|^2}{h^2}\right)^{-1} \left(1 - \frac{(\eta-\xi)}{h} D_z + \frac{(x-z)}{h} D_\eta\right)$$

and let $\chi_1 \in C_0^\infty(\mathbb{R})$, $\chi_1(s) = 1$ for $|s| \leq 1$, $\chi_1(s) = 0$ for $|s| \geq 2$. For $x, y \in \mathbb{R}^n$, we set $\chi(x,y) = \chi_1(|x-y|)$. Then for any $k \geq |m| + 2n + 1$ one has

$$\begin{aligned} c_\delta(x,y,\xi) &= \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(x-z)(\eta-\xi)} \left({}^tL_1\right)^k \left(e^{-\delta\langle z\rangle - \delta\langle\eta\rangle} a(x,z,\eta)\, b(z,y,\xi)\right) dz d\eta \\ &= d_\delta(x,y,\xi) + e_\delta(x,y,\xi) + f_\delta(x,y,\xi), \end{aligned}$$

where

$$(2\pi h)^n d_\delta(x,y,\xi) =$$

$$\begin{aligned} &\int e^{\frac{i}{h}(x-z)(\eta-\xi)} \left({}^tL_1\right)^k \left((1-\chi(\xi,\eta))\, e^{-\delta\langle z\rangle - \delta\langle\eta\rangle} a(x,z,\eta)\, b(z,y,\xi)\right) dz d\eta \\ &= \int \mathcal{O}\left(\frac{\langle\eta\rangle^m \langle\xi\rangle^{m'}}{\left(1 + h^{-1}|\eta-\xi| + h^{-1}|x-z|\right)^k}\right) dz d\eta \\ &= \int \mathcal{O}\left(\frac{\langle\eta\rangle^m \langle\xi\rangle^{m'}}{\left(1 + \frac{1+|\eta-\xi|}{2h}\right)^{k-n-\frac{1}{2}}}\right) d\eta, \end{aligned}$$

and thus in the case $m \geq 0$,

$$\begin{aligned} (2\pi h)^n d_\delta(x,y,\xi) &= \int \mathcal{O}\left(h^{k-n-\frac{1}{2}} \frac{\left(\langle\xi\rangle + \langle\eta-\xi\rangle\right)^m \langle\xi\rangle^{m'}}{\langle\eta-\xi\rangle^{k-n-\frac{1}{2}}}\right) d\eta \\ &= \mathcal{O}\left(h^{k-n-\frac{1}{2}} \langle\xi\rangle^{m+m'}\right). \end{aligned}$$

In the case $m < 0$, one splits the integral into the two regions $\left\{|\eta| \geq \frac{\langle\xi\rangle}{2}\right\}$ and $\left\{|\eta| \leq \frac{\langle\xi\rangle}{2}\right\}$. In the first region one has $\langle\eta\rangle^m = \mathcal{O}\left(\langle\xi\rangle^m\right)$, and therefore one gets the same estimate as before. In the second region one has $\langle\eta-\xi\rangle \geq \frac{\langle\xi\rangle}{C}$ for some positive constant C, and therefore the corresponding integral can be estimated by $\mathcal{O}\left(h^{k-n-\frac{1}{2}} \langle\xi\rangle^{m'-(k-2n-1)}\right)$.

Similarly,

$$(2\pi h)^n e_\delta(x,y,\xi) =$$
$$\int e^{\frac{i}{h}(x-z)(\eta-\xi)} \left({}^tL_1\right)^k \left[\chi(\xi,\eta)(1-\chi(x,z)) e^{-\delta\langle z\rangle-\delta\langle\eta\rangle} a(x,z,\eta) b(z,y,\xi)\right] dzd\eta$$
$$= \mathcal{O}\left(h^{k-n-\frac{1}{2}} \langle\xi\rangle^{m+m'}\right)$$

for all $k \geq |m| + 2n + 1$, and uniformly with respect to $(x,y,\xi) \in \mathbb{R}^{3n}$ and $\delta > 0$. Actually, the same argument also given that for any $\alpha \in \mathbb{N}^{3n}$,

$$\|\partial^\alpha d_\alpha(x,y,\xi)\|_{\mathcal{L}(H,L)} + \|\partial^\alpha e_\alpha(x,y,\xi)\|_{\mathcal{L}(H,L)} = \mathcal{O}\left(h^\infty \langle\xi\rangle^{m+m'}\right) \tag{3.5}$$

uniformly with respect to $(x,y,\xi) \in \mathbb{R}^{3n}$ and $\delta > 0$.

So it remains to study the last term f_δ, which, by integrations by parts, can be written as

$$f_\delta(x,y,\xi) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(x-z)(\eta-\xi)} \chi(\xi,\eta)\chi(x,z) e^{-\delta\langle z\rangle-\delta\langle\eta\rangle} a(x,z,\eta) b(z,y,\xi) dzd\eta.$$

Making the change of variables

$$\begin{cases} z' = z - x \\ \eta' = \eta - \xi \end{cases}$$

we get

$$f_\delta(x,y,\xi) = \frac{1}{(2\pi h)^n} \int e^{-\frac{i}{h}z'\eta'} u^\delta_{x,y,\xi}(z',\eta') dz'd\eta'$$

with

$$u^\delta_{x,y,\xi}(z',\eta') =$$
$$\chi(\xi,\eta'+\xi)\chi(x,z'+x) e^{-\delta\langle z'+x\rangle-\delta\langle\eta'+\xi\rangle} a(x,z'+x,\eta'+\xi) b(z'+x,y,\xi) \in C_0^\infty\left(\mathbb{R}^{2n},\mathcal{L}(H,L)\right).$$

Then we can apply the stationary phase theorem ([5, 7, 16]) to this integral and we obtain for all $N \geq 1$,

$$f_\delta(x,y,\xi) = \sum_{|\alpha|\leq N-1} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_z^\alpha \partial_\eta^\beta u^\delta_{x,y,\xi}(z,\eta)\Big|_{\substack{z=0\\ \eta=0}} + S_N \tag{3.6}$$

with

$$\begin{aligned}
\|S_N\|_{\mathcal{L}(H,L)} &\leq \frac{Ch^N}{N!}\sum_{|\alpha+\beta|\leq 2n+1}\left\|\partial_z^\alpha\partial_\eta^\beta\left(\partial_z\partial_\eta\right)^N u_{x,y,\xi}^\delta\right\|_{L^1(\mathbb{R}^{2n},\mathcal{L}(H,L))}\\
&= \mathcal{O}\left(h^N\int\limits_{|\eta-\xi|\leq 2,|x-z|\leq 2}\langle\eta\rangle^m\langle\xi\rangle^{m'}dzd\eta\right)\\
&= \mathcal{O}\left(h^N\langle\xi\rangle^{m+m'}\right)
\end{aligned}$$

uniformly. Doing the same procedure for $\partial^\gamma f_\delta$, we get, in particular,

$$\left\|\partial^\gamma f_\delta\left(x,y,\xi\right)\right\|_{\mathcal{L}(H,L)}=\mathcal{O}\left(\langle\xi\rangle^{m+m'}\right) \tag{3.7}$$

uniformly with respect to $\delta>0$ and $(x,y,\xi)\in\mathbb{R}^{3n}$.

Moreover, since for $k\geq m+2n+1$ one has

$$\left\|\left({}^tL_1\right)^k\left[e^{-\delta\langle z\rangle-\delta\langle\eta\rangle}a\left(x,z,\eta\right)b\left(z,y,\xi\right)\right]\right\|_{L^1(\mathbb{R}^{2n},\mathcal{L}(H,L))}=\mathcal{O}_{x,y,\xi}\left(1\right)$$

uniformly with respect to δ, we get by the dominated convergence theorem

$$c_\delta\left(x,y,\xi\right)\longrightarrow c_0\left(x,y,\xi\right)\ \text{ as }\delta\longrightarrow 0^+,$$

where

$$c_0\left(x,y,\xi\right)=\frac{1}{\left(2\pi h\right)^n}\int e^{\frac{i}{h}(x-z)(\eta-\xi)}\left({}^tL_1\right)^k\left[a\left(x,z,\eta\right)b\left(z,y,\xi\right)\right]dzd\eta.$$

Since the estimates (3.5) and (3.7) are uniform with respect to δ, we also have

$$c_0\in S^{3n}_{\langle\xi\rangle^{m+m'}}\left(H,L\right),$$

and finally, we deduce from (3.4) that

$$\begin{aligned}
Op_h\left(a\right)\circ Op_h\left(b\right)u\left(x\right) &= \lim_{\varepsilon\to 0^+}\frac{1}{\left(2\pi h\right)^n}\int e^{\frac{i}{h-\varepsilon\langle\xi\rangle}(x-y)\xi}c_0\left(x,y,\xi\right)u\left(y\right)dyd\xi\\
&= \frac{1}{\left(2\pi h\right)^n}\int e^{\frac{i}{h}(x-y)\xi}c_0\left(x,y,\xi\right)u\left(y\right)dyd\xi,
\end{aligned}$$

where the last integral has to be interpreted as an oscillatory one. Taking also the limit $\delta \longrightarrow 0^+$ into (3.6), we obtain the semiclassical asymptotic expansion of $c_0(x, y, \xi)$. □

Proposition 3.2. *Let $m \in \mathbb{R}$ and let $a \in S^{3n}_{\langle\xi\rangle^m}(H, K)$ be an elliptic symbol in the sense of the definition 2.3. Then there exists $b \in S^{3n}_{\langle\xi\rangle^{-m}}(K, H)$ such that*

$$\begin{cases} Op_h(a) \circ Op_h(b) = I_{\mathcal{L}(K)} + Op_h(r), \\ Op_h(b) \circ Op_h(a) = I_{\mathcal{L}(H)} + Op_h(r'), \end{cases}$$

with $r = \mathcal{O}(h^\infty)$ in $S^{3n}_1(K, K)$ and, $r' = \mathcal{O}(h^\infty)$ in $S^{3n}_1(H, H)$.

Proof. By proposition 2.3, we know that $a^{-1} \in S^{3n}_{\langle\xi\rangle^{-m}}(K, H)$. Then, setting $b = a^{-1}$ and using the expansion of $a\#b$ given in the theorem, it is possible to define $b_j \in S^{3n}_{\langle\xi\rangle^{-m}}(K, H)$ recursively, in such a way that if $b \sim \sum h^j b_j$, then

$$a\#b = 1 + \mathcal{O}(h^\infty) \text{ in } S^{3n}_1(K, K) \text{ and } b\#a = 1 + \mathcal{O}(h^\infty) \text{ in } S^{3n}_1(H, H).$$

and by theorem 3.2, this implies

$$\begin{cases} Op_h(a) \circ Op_h(b) = I_{\mathcal{L}(K)} + Op_h(r), \\ Op_h(b) \circ Op_h(a) = I_{\mathcal{L}(H)} + Op_h(r'), \end{cases}$$

$r = \mathcal{O}(h^\infty)$ in $S^{3n}_1(K, K)$ and, $r' = \mathcal{O}(h^\infty)$ in $S^{3n}_1(H, H)$. □

4. Symbolic calculus-change of quantization

If $x \in \mathbb{R}^n$ denotes the position, these functions depend on $2n$ variables only. Then it could seem more convenient to work with h-pseudodifferential operators with symbol operator of the form $a = a(x, \xi)$ depending on $2n$ variables.

Noting that for $a \in S^{2n}_{\langle\xi\rangle^m}(H, K)$ and for $t \in [0, 1]$ we have $a((1-t)x + ty, \xi) \in S^{3n}_{\langle\xi\rangle^m}(H, K)$, we set

$$Op^t_h(a) := Op_h(a((1-t)x + ty, \xi)).$$

The values $t = 0, t = \frac{1}{2}$ and $t = 1$ play a particular role, and they are respectively called:

$t = 0$:standard quantization or left quantization
$t = \frac{1}{2}$:Weyl quantization denoted by Op_h^w
$t = 1$:right quantization.

Remark 1. The Weyl quantization is particulary useful in quantum mechanics because it has the property that when a is a symmetric operator, then $Op_h^w(a)$ is symetric with respect to the $L^2(\mathbb{R}^n, H)$-scalar product.

Theorem 4.1. *Let $b = b(x, y, \xi) \in S^{3n}_{\langle\xi\rangle^m}(H, K)$ and $t \in [0,1]$. Then there exists a unique $b_t(x, \xi) \in S^{2n}_{\langle\xi\rangle^m}(H, K)$ such that*

$$Op_h(b) = Op_h^t(b_t)$$

Moreover, b_t is given by the oscillatory integral

$$b_t(x, \xi) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\xi' - \xi)\theta} b(x + t\theta, x - (1-t)\theta, \xi')\, d\xi' d\theta \tag{4.1}$$

and satisfies

$$b_t(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|} h^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_\theta^\alpha b(x + t\theta, x - (1-t)\theta, \xi)\,|_{\theta=0} \quad \text{in } S^{2n}_{\langle\xi\rangle^m}(H, K).$$

Here b_t is called the t-symbol of $Op_h(a)$, and is denoted by $b_t = \sigma_t(Op_h(a))$.

Proof. We try to find $b_t \in S^{2n}_{\langle\xi\rangle^m}(H, K)$ such that

$$\int e^{\frac{i}{h}(x-y)\xi} b(x, y, \xi)\, d\xi = \int e^{\frac{i}{h}(x-y)\xi} b_t((1-t)x + ty, \xi)\, d\xi.$$

Setting

$$\begin{cases} \theta = x - y \\ z = (1-t) + ty = x - t\theta, \end{cases}$$

or, equivalently,

$$\begin{cases} x = z + t\theta \\ y = z - (1-t)\,\theta, \end{cases}$$

we are led to

$$\int e^{\frac{i}{h}\theta\xi} b\left(z + t\theta, z - (1-t)\,\theta, \xi\right) d\xi = \int e^{\frac{i}{h}\theta\xi} b_t\left(z, \xi\right) d\xi,$$

where the right-hand side has to be interpreted as an oscillatory integral. In particular, by integrations by parts we see that it defines an element of $S_1^{2n}\left(H, K\right)$ with respect to the variables $\left(z, \theta\right)$. Since, moreover, the right-hand side is proportional to the inverse h-Fourier transforms of $\xi \longmapsto b_t\left(z, \xi\right)$, we obtain necessarily:

$$b_t\left(z, \zeta\right) = \frac{1}{\left(2\pi h\right)^n} \int e^{\frac{i}{h}(\xi-\zeta)\theta} b\left(z + t\theta, z - (1-t)\,\theta, \xi\right) d\xi d\theta, \tag{4.2}$$

where now the right-hand side to be interpreted as a Fourier transform with respect to θ. Introducing again a cutoff function of the type $\chi\left(\theta, \xi - \zeta\right)$ and making integrations by parts as in the proof of the theorem of composition, we see in the same way that $b_t S^{2n}_{\langle\zeta\rangle^m}\left(H, K\right)$, and the stationary phase theorem ([16, 5, 7]) also gives

$$b_t\left(z, \zeta\right) \sum_{\alpha\in\mathbb{N}^n} \frac{\left(-1\right)^{|\alpha|} h^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_\theta^\alpha b\left(x + t\theta, x - (1-t)\,\theta, \xi\right)\Big|_{\substack{\xi=\zeta \\ \theta=0}}.$$

Finally, the unicity of b_t is a consequence of (4.2). □

Example 4.1. *If* $V\left(x\right) = -\Delta_y + V\left(x, y\right) \in S_1^n\left(H^2\left(\mathbb{R}_y^n\right), L^2\left(\mathbb{R}_y^n\right)\right)$, *then for all* $t \in [0, 1]$ *one has* $\sigma_t\left(-h^2\Delta_x + V\left(x\right)\right) = \xi^2 + V\left(x\right)$ *which therefore does not depend on* t.

Let H, K, L three Hilbert spaces.

Theorem 4.2 (Symbolic calculus)**.** *Let* $a\left(x, \xi\right) \in S^{2n}_{\langle\xi\rangle^m}\left(K, L\right)$, *and* $b\left(x, \xi\right) \in S^{2n}_{\langle\xi\rangle^{m'}}\left(H, K\right)$. *Then for all* $t \in [0, 1]$ *there exists an unique* $c_t \in S^{2n}_{\langle\xi\rangle^{m+m'}}\left(H, L\right)$ *such that*

$$Op_h^t\left(a\right) \circ Op_h^t\left(b\right) = Op_h^t\left(c_t\right).$$

Moreover, c_t is given by

(4.3)
$$c_t(x,\xi;h) = e^{ih\left(D_\eta D_v - D_u D_\xi\right)}\left[a\left((1-t)x+tu,\eta\right)b\left(tx+(1-t)v,\xi\right)\right]\Big|_{\substack{u=v=x\\ \eta=\xi}} := a\#^t b,$$

and it satisfies

$$c_t(x,\xi;h) \sim \sum_{k\geq 0}\frac{h^k}{i^k k!}\left(D_\eta D_v - D_u D_\xi\right)^k\left[a\left((1-t)x+tu,\eta\right)b\left(tx+(1-t)v,\xi\right)\right]\Big|_{\substack{u=v=x\\ \eta=\xi}}$$

in $S^{2n}_{\langle\xi\rangle^{m+m'}}(H,L)$.

Proof. By the theorem of composition, one has

$$Op_h^t(a)\circ Op_h^t(b) = Op_h(c)$$

with

$$\begin{aligned} c(x,y,\xi) &= \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}(x-z)(\eta-\xi)}a\left((1-t)x+tz,\eta\right)b\left((1-t)z+ty,\xi\right)dzd\eta \\ &\sim \sum_{\alpha\in\mathbb{N}^n}\frac{(-1)^{|\alpha|}h^{|\alpha|}}{i^{|\alpha|}\alpha!}\partial_z^\alpha\partial_\eta^\alpha\left(a\left((1-t)x+tz,\eta\right)b\left((1-t)z+ty,\xi\right)\right)\Big|_{\substack{z=x\\ \eta=\xi}} \end{aligned} \tag{4.4}$$

in $S^{2n}_{\langle\xi\rangle^{m+m'}}(H,L)$. Moreover, by the previous theorem,

$$Op_h(c) = Op_h^t(c_t)$$

with

$$\begin{aligned} c_t(x,\xi) &= \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}(\xi'-\xi)\theta}c\left(x+t\theta,x-(1-t)\theta,\xi'\right)d\xi' d\theta \\ &\sim \sum_{\alpha\in\mathbb{N}^n}\frac{(-1)^{|\alpha|}h^{|\alpha|}}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha\partial_\theta^\alpha\left(c\left(x+t\theta,x-(1-t)\theta,\xi\right)\right)|_{\theta=0} \end{aligned} \tag{4.5}$$

and therefore,

$$\begin{aligned} c_t(x,\xi) &= \frac{1}{(2\pi h)^{2n}}\int e^{\frac{i}{h}(\xi'-\xi)\theta+(x+t\theta-z)(\eta-\xi')}a\left((1-t)(x+t\theta)+tz,\eta\right) \\ &\quad\times b\left((1-t)z+t\left(x-(1-t)\theta\right),\xi'\right)dzd\eta d\xi' d\theta. \end{aligned}$$

Then we make the change of variables

$$(z,\theta)\longmapsto(u,v)=(z+(1-t)\theta,z-t\theta),$$

which has determinant 1 and gives

$$c_t(x,\xi)=\frac{1}{(2\pi h)^{2n}}\int e^{\frac{i}{h}[x(\eta-\xi)+u(\xi'-\xi)-v(\eta-\xi)]}a((1-t)x+tu,\eta)\times b(tx+(1-t)v,\xi')\,dzd\eta d\xi'd\theta. \tag{4.6}$$

On the other hand, by the Fourier-inverse formula we have

$$e^{ih\left[D_\eta D_v-D_uD_\xi\right]}\left[a((1-t)x+tu,\eta)\,b(tx+(1-t)v,\xi)\right]\Big|_{\substack{u=v=x\\ \eta=\xi}}=$$
$$\frac{1}{(2\pi h)^{2n}}\int e^{\frac{i}{h}[(\xi-\eta)(v-x)+(\xi-\xi')(x-u)]}a((1-t)x+tu,\eta)\,b(tx+(1-t)v,\xi')\,dudvd\eta d\xi'.$$

Comparing this last formula with (4.6) we get immediately

$$c_t(x,\xi)=e^{ih\left[D_\eta D_v-D_uD_\xi\right]}\left[a((1-t)x+tu,\eta)\,b(tx+(1-t)v,\xi)\right]\Big|_{\substack{u=v=x\\ \eta=\xi}}.$$

Then the asymptotic expansion of c_t can be obtained either by again introducing cutoff function and using the stationary phase theorem. □

Particular cases

- For $t=0: Op_h^0(a)\circ Op_h^0(b)=Op_h^0\left(c^l\right)$, with

$$c^l(x,\xi)=e^{ihD_\eta D_y}a(x,\eta)\,b(y,\xi)\Big|_{\substack{y=x\\ \eta=\xi}}:=a\#b\sim\sum_{\alpha\in\mathbb{N}^n}\frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a(x,\xi)\,\partial_x^\alpha b(x,\xi). \tag{4.7}$$

- For $t=\frac{1}{2}: Op_h^w(a)\circ Op_h^w(b)=Op_h^w(c^w)$, with

$$\begin{aligned}c^w(x,\xi) &= e^{ih\left[D_\eta D_x-D_yD_\xi\right]}a(y,\eta)\,b(x,\xi)\Big|_{\substack{y=x\\ \eta=\xi}}:=^w a\#b\\ &\sim \sum_{\alpha\in\mathbb{N}^n}\frac{(-1)^{|\alpha|}h^{|\alpha+\beta|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}\left(\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right)\left(\partial_\xi^\alpha\partial_x^\beta b(x,\xi)\right).\end{aligned} \tag{4.8}$$

5. L^2-Boundedness

Until now, we have made our h-pseudodifferential operators with operator symbol act on $\mathcal{S}(\mathbb{R}^n, H)$ and $\mathcal{S}'(\mathbb{R}^n, H)$. However, for applications in quantum mechanic (where the physical states are described by functions in L^2), it is useful to know how these h-pseudodifferential operators transform $L^2(\mathbb{R}^n, H)$. The following result provides a rather complete answer to this problem.

Theorem 5.1 (Calderón-Vaillancourt)**.** *Let $a \in S_1^{3n}(H, K)$ where H, K are two Hilbert spaces. Then $A = Op_h(a)$ is continuous on $L^2(\mathbb{R}^n, H)$, and*

$$\|Op_h(a)\|_{\mathcal{L}(L^2(\mathbb{R}^n,H),L^2(\mathbb{R}^n,K))} \le C_n \left(\sum_{|\alpha| \le M_n} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^{3n},\mathcal{L}(H,K))} \right),$$

where the positive constants C_n depend only on n.

Before giving the proof of the theorem of Calderón-Vaillancourt, let's give three lemmas:

Lemma 5.1. *For all $d \in \mathbb{R}^d$ there exists $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ such that, if we write $\chi_\mu(z) = \chi_0(z - \mu)$ (where $\mu \in \mathbb{Z}^d$), one has*

$$\sum_{\mu \in \mathbb{Z}^d} \chi_\mu = 1 \text{ on } \mathbb{R}^d.$$

Proof. Let $K = \left\{ z \in \mathbb{R}^d;\ |z_j| \le \frac{1}{2} \text{ for } j = 1, ..., d \right\}$. Then K is compact, and therefore there exists $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi \ge 0$ and $\varphi = 1$ on K. Set

$$\psi(z) = \sum_{\mu \in \mathbb{Z}^d} \varphi(z - \mu),$$

we have

$$\forall \nu \in \mathbb{Z}^d,\ \psi(z + \nu) = \psi(z)$$

and by construction,

$$\forall z \in \mathbb{R}^d,\ \psi(z) \ge 1.$$

Then $\chi_0 := \dfrac{\varphi}{\psi}$ solves the problem. $\square$

Lemma 5.2 (Cotlar-Stein Lemma)**.** *Let $\mathcal{H}$ be a Hilbert space, $(A_\mu)_{\mu\in\mathbb{Z}^d}$ a family of bounded operators on $\mathcal{H}$, and $\omega : \mathbb{Z}^d \longrightarrow \mathbb{R}_+$ an application satisfying*

$$\forall \mu,\nu \in \mathbb{Z}^d, \ \ \|A_\mu A_\nu^*\| + \left\|A_\mu^* A_\nu\right\| \leq \omega\left(\mu-\nu\right)$$

and

$$C_0 := \sum_{\mu\in\mathbb{Z}^d} \sqrt{\omega\left(\mu\right)} < +\infty.$$

Then for all $M \geq 0$, one has,

$$\left\|\sum_{|\mu|\leq M} A_\mu\right\| \leq C_0.$$

Proof. See [16, Page 83]. $\square$

Lemma 5.3. *If $Au\left(x\right) = \int K\left(x,y\right)u\left(y\right)dy$ with $K \in C^0\left(\mathbb{R}_x^n \times \mathbb{R}_y^n, \mathcal{L}\left(H,K\right)\right)$ (H, K are Hilbert spaces), then*

$$\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n,H),L^2(\mathbb{R}^n,K))} \leq \left(\sup_x \int \|K\left(x,y\right)\|_{\mathcal{L}(H,K)}\, dy\right)^{1/2} \left(\sup_y \int \|K\left(x,y\right)\|_{\mathcal{L}(H,K)}\, dx\right)^{1/2},$$

where $\mathcal{L}\left(L^2\left(\mathbb{R}^n,H\right), L^2\left(\mathbb{R}^n,K\right)\right)$ denotes the space of bounded linear operators from $L^2\left(\mathbb{R}^n,H\right)$ into $L^2\left(\mathbb{R}^n,K\right)$.

Proof. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|Au\left(x\right)\|_K^2 &\leq \left(\int \|K\left(x,y\right)\|_{\mathcal{L}(H,K)}^{1/2} \|K\left(x,y\right)\|_{\mathcal{L}(H,K)}^{1/2} \|u\left(y\right)\|_H\, dy\right)^2 \\ & \int \|K\left(x,y\right)\|_{\mathcal{L}(H,K)}\, dy. \int \|K\left(x,y\right)\|_{\mathcal{L}(H,K)} \|u\left(y\right)\|_H^2\, dy, \end{aligned}$$

and therefore

$$\begin{aligned} \|Au\|_{L^2(\mathbb{R}^n,K)}^2 &\leq \int \left(\int \|K\left(x,y\right)\|_{\mathcal{L}(H,K)}\, dy. \int \|K\left(x,y\right)\|_{\mathcal{L}(H,K)} \|u\left(y\right)\|_H^2\, dy\right) dx \\ &\leq \left(\sup_x \int \|K\left(x,y\right)\|_{\mathcal{L}(H,K)}\, dy\right) \left(\sup_y \int \|K\left(x,y\right)\|_{\mathcal{L}(H,K)}\, dx\right) \end{aligned}$$

□

Proof of Calderón-Vaillancourt theorem . By theorem 4.1, there exists $b \in b_t(x,\xi) \in S_1^{2n}(H,K)$ such that $A = Op_h^w(b)$. Moreover, using the operator

$$L := \left(1+\theta^2+(\xi-\xi')^2\right)^{-1}\left(1+h\theta D_{\xi'}+h(\xi'-\xi)D_\theta\right)$$

to make integrations by parts in the integral expression of b, we see that for any $\alpha \in \mathbb{N}^{2n}$ the quantity $\|\partial^\alpha b\|_{L^\infty(H,K)}$ can be estimated by a finite number of derivatives of a. As a consequence, it is enough to prove the theorem for $A = Op_h^w(a)$ with $a \in S_1^{2n}(H,K)$. Moreover, by the change of variables $\xi \longmapsto h\xi$ we see that

$$Op_h^w(a(x,\xi)) = Op_1^w a(x,h\xi),$$

and for all $\alpha,\beta \in \mathbb{N}^n$ we have

$$\partial_x^\alpha\partial_\xi^\beta(a(x,h\xi)) = h^{|\beta|}\left(\partial_x^\alpha\partial_\xi^\beta a\right)(x,h\xi) = \mathcal{O}\left(h^{|\beta|}\left\|\partial_x^\alpha\partial_\xi^\beta a\right\|_{L^\infty(H,K)}\right).$$

Therefore, it is indeed enough to prove the result for $h=1$, that is, for the operator $A = Op_1^w(a)$.

Now we apply lemma 5.1 with $d=2n$, and for $\mu \in \mathbb{Z}^{2n}$ we set

$$a_\mu = a\chi_\mu.$$

Since $|\partial^\alpha\chi_\mu(z)| = |(\partial^\alpha\chi_\mu)(z-\mu)| \le \sup|\partial^\alpha\chi_\mu|$ for all $\alpha \in \mathbb{N}^{2n}$, we obtain by Leibniz formula

$$\partial^\alpha a_\mu = \mathcal{O}\left(\sup_{\beta\le\alpha}\left\|\partial^\beta a\right\|_{L^\infty(H,K)}\right) \text{ uniformly with respect to } \mu \in \mathbb{Z}^{2n}. \tag{5.1}$$

We set

$$A_\mu = Op_1^w(a_\mu),$$

so that for any $u \in C_0^\infty(\mathbb{R}^n,H)$, one has

$$Au = \sum_\mu A_\mu u. \tag{5.2}$$

For μ,ν, we have

$$A_\mu A_\nu^* u(x) = \int K_{\mu,\nu}(x,y)\, u(y)\, dy$$

with

$$K_{\mu,\nu}(x,y) = \frac{1}{(2\pi)^{2n}} \int e^{i(x\xi - y\eta - z\xi + z\eta)} a_\mu\left(\frac{x+z}{2},\xi\right)\left({}^t a_\nu\right)\left(\frac{y+z}{2},\eta\right) dz d\eta d\xi,$$

where ${}^t a_\nu$ is the transpose of the operator a_ν. Since a_μ and a_ν are smooth and compactly supported, we see that $K_{\mu,\nu} \in C^\infty\left(\mathbb{R}^{2n}, K\right)$, and we set

$$L = \frac{1}{1+|x-z|^2+|y-z|^2+|\xi-\eta|^2}\left(1+(x-z)D_\xi - (y-z)D_\eta - (\xi-\eta)D_z\right),$$

which satisfies

$$L\left(e^{i(x\xi - y\eta - z\xi + z\eta)}\right) = e^{i(x\xi - y\eta - z\xi + z\eta)}.$$

Using L to integrate by parts, we get for any $N \geq 0$,

$$K_{\mu,\nu}(x,y) = \frac{1}{(2\pi)^{2n}} \int e^{i(x\xi - y\eta - z\xi + z\eta)} \left({}^t L\right)^N a_\mu\left(\frac{x+z}{2},\xi\right)\left({}^t a_\nu\right)\left(\frac{y+z}{2},\eta\right) dz d\eta d\xi.$$

Moreover, when $|\mu-\nu|$ is large enough, we have on $\mathrm{Supp}(a_\mu(t,\tau)\left({}^t a_\nu\right)(s,\sigma))$,

$$\frac{1}{C}|\mu-\nu| \leq |t-s| + |\tau-\sigma| \leq C|\mu-\nu|,$$

where $C>0$ is a constant. Setting $\mu = (\mu_1,\mu_2)$ and $\nu = (\nu_1,\nu_2)$ in $\mathbb{Z}^{2n}$, we deduce from this and (5.1) that

$$\int \|K_{\mu,\nu}(x,y)\|_K\, dy = \int_{\mathcal{D}_{\mu,\nu}} \mathcal{O}\left(\frac{\sup\limits_{|\alpha|\leq N}\left\|\partial^\beta a\right\|^2_{L^\infty(H,K)}}{\left(1+|x-z|^2+|y-z|^2+|\xi-\eta|^2\right)^N}\right) dy dz d\eta d\xi,$$

where $\mathcal{O}$ is uniform with respect to μ, ν and a, and where

$$\mathcal{D}_{\mu,\nu} = \left\{\frac{1}{C}|\mu-\nu| \leq |x-y| + |\xi-\eta| \leq C|\mu-\nu|,\ \ |\xi-\mu_2| \leq C', |\eta-\mu_2| \leq C'\right\}$$

with C' depends only on n.

Since $|x-z|+|y-z|\geq |x-y|$, this gives

$$\int \|K_{\mu,\nu}(x,y)\|_K\, dy = \int_{\mathcal{D}_{\mu,\nu}} \mathcal{O}\left(\frac{(1+|\mu-\nu|)^{2n+2-N} \sup\limits_{|\alpha|\leq N} \left\|\partial^\beta a\right\|^2_{L^\infty(H,K)}}{(1+|x-z|)^{n+1}(1+|x-y|)^{n+1}} \right) dydz,$$

and therefore, for all $N\geq 0$,

$$\sup_{x\in\mathbb{R}^n} \int \|K_{\mu,\nu}(x,y)\|_K\, dy = \mathcal{O}\left((1+|\mu-\nu|)^{2n+2-N} \sup_{|\alpha|\leq N} \left\|\partial^\beta a\right\|^2_{L^\infty(H,K)} \right). \tag{5.3}$$

In the same way, we see that

$$\sup_{y\in\mathbb{R}^n} \int \|K_{\mu,\nu}(x,y)\|_K\, dx = \mathcal{O}\left((1+|\mu-\nu|)^{2n+2-N} \sup_{|\alpha|\leq N} \left\|\partial^\beta a\right\|^2_{L^\infty(H,K)} \right). \tag{5.4}$$

We deduce from the lemma 5.3 and (5.3)-(5.4) that for all $N\geq 0$,

$$\|A_\mu A_\nu^*\|_{\mathcal{L}(L^2(\mathbb{R}^n,K))} = \mathcal{O}\left((1+|\mu-\nu|)^{2n+2-N} \sup_{|\alpha|\leq N} \left\|\partial^\beta a\right\|^2_{L^\infty(H,K)} \right)$$

uniformly with respect to μ,ν, and exactly in the same way one can prove

$$\left\|A_\mu^* A_\nu\right\|_{\mathcal{L}(L^2(\mathbb{R}^n,H))} = \mathcal{O}\left((1+|\mu-\nu|)^{2n+2-N} \sup_{|\alpha|\leq N} \left\|\partial^\beta a\right\|^2_{L^\infty(H,K)} \right).$$

Then we choose $N=4n+3$, so that we can apply the Cotlar-Stein lemma with $d=2n$ and

$$\omega(\mu) = C(1+|\mu|)^{-2n-1} \sup_{|\alpha|\leq N} \left\|\partial^\beta a\right\|^2_{L^\infty(H,K)},$$

where $C>0$ depends only on n. We obtain that for any $M\geq 0$ and for any $u\in C_0^\infty(\mathbb{R}^n,H)$,

$$\left\| \sum_{|\mu|\leq M} A_\mu u \right\|_{L^2(\mathbb{R}^n,K)} \leq C_0 \|u\|_{L^2(\mathbb{R}^n,H)}$$

with

$$C_0 = \sqrt{C} \sum_\mu (1+|\mu|)^{-n-1/2} \sup_{|\alpha|\leq N} \left\|\partial^\beta a\right\|_{L^\infty(H,K)}.$$

Taking the limit $M\longrightarrow +\infty$, we get by (5.2),

$$\|Au\|_{L^2(\mathbb{R}^n,K)} \leq C_0 \|u\|_{L^2(\mathbb{R}^n,H)}$$

□

6. Conclusion

Using proposition 3.2 and theorem 5.1, we get that if $a \in S^{3n}_{\langle\xi\rangle^m}(H,K)$ is elliptic, there exists $b \in S^{3n}_{\langle\xi\rangle^{-m}}(H,K)$ such that

$$\begin{cases} Op_h(a) \circ Op_h(b) = I_{\mathcal{L}(K)} + R_1, \\ Op_h(b) \circ Op_h(a) = I_{\mathcal{L}(H)} + R_2, \end{cases}$$

with

$$\|R_1\|_{\mathcal{L}(K)} + \|R_1\|_{\mathcal{L}(H)} = \mathcal{O}(h^\infty).$$

In particular, if $m = 0$ and h small enough, $Op_h(a) = Op_h$ is invertible:$L^2(\mathbb{R}^n, H) \to L^2(\mathbb{R}^n, K)$, with inverse given by

$$\begin{aligned} (Op_h(a))^{-1} &= Op_h(b)\left[\sum_{k=0}^{\infty}(-1)^k R_1^k\right] = \left[\sum_{k=0}^{\infty}(-1)^k R_1^k\right] Op_h(b) \\ &= Op_h(b) + \mathcal{O}(h^\infty). \end{aligned}$$

7. Example

This theory can be applied to study the spectrum and resonances for the Hamiltonian in the approximation of Born-Oppenheimer [20] of type:

$$P = -h^2\Delta_x - \Delta_y + V(x,y) \text{ on } L^2\left(\mathbb{R}^n_x \times \mathbb{R}^p_y\right).$$

We can consider the operator $P = Op_h(a)$, where

$$a(x,\xi) = \xi^2 - \Delta_y + V(x,y) \in S^{2n}_{\langle\xi\rangle^2}(H,K),\ H = H^2\left(\mathbb{R}^p_y\right) \text{ and } K = L^2\left(\mathbb{R}^p_y\right).$$

If the symbol $a(x,\xi)$ is invertible, than we can construct the inverse of P. We refer to [8, 11, 13, 14].

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