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.

A CLASS OF PRIMARY SUBSEMIMODULES

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Abstract. The partial functions under disjoint-domain sums and functional composition is a so-ring, an algebraic structure possessing a natural partial ordering, an infinitary partial addition and a binary multiplication, subject to a set of axioms. In this paper we introduce the notion of primary subsemimodule with respect to a prime subsemimodule in partial semimodules and singular partial semiring with respect to a partial semimodule.

1. Introduction

The study of $pfn(D, D)$(the set of all partial functions of a set D to itself), $Mfn(D, D)$(the set of all multi functions of a set D to itself) and $Mset(D, D)$(the set of all total functions of a set D to the set of all finite multi sets of D) play an important role in the theory of computer science, and to abstract these structures Manes and Benson[4] introduced the notion of sum ordered partial semirings (so-rings). In

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[5], we have obtained the ideal theory of so-rings. In [6], [7] we characterize prime, semiprime and primary subsemimodules with prime, semiprime and primary partial ideals respectively. In this paper we introduce the singular partial semiring with respect to a partial semimodule and we generalize the results of T. K. Dutta and M. L. Das[2] and we characterize primary subsemimodules wrt a prime subsemimodule with primary partial ideal wrt a prime partial ideal.

2. **Preliminaries**

In this section we collect some important definitions and results for our use in this paper.

Definition 2.1. [8] A partial semiring is a quadruple $(R, \Sigma, \cdot, 1)$, where (R, Σ) is a partial monoid, $(R, \cdot, 1)$ is a monoid with multiplicative operation $\cdot$ and unit 1, and the additive and multiplicative structures obey the following distributive laws. If $\Sigma(x_i : i \in I)$ is defined in R, then for all y in R, $\Sigma(y \cdot x_i : i \in I)$ and $\Sigma(x_i \cdot y : i \in I)$ are defined and
$y \cdot [\Sigma_i x_i] = \Sigma_i (y \cdot x_i)$; $[\Sigma_i x_i] \cdot y = \Sigma_i (x_i \cdot y)$.
In addition to that, if $(R, \cdot, 1)$ is a commutative monoid then the partial semiring $(R, \Sigma, \cdot, 1)$ is called a commutative partial semiring.

Throughout this paper R stands for a commutative partial semiring.

Definition 2.2. [1] Let R be a partial semiring. A subset N of R is said to be a partial ideal of R if the following are satisfied
(I1). if $(x_i : i \in I)$ is summable family in R and $x_i \in N$ for every $i \in I$ then $\Sigma x_i \in N$,
(I2). if $x \in N$ and $r \in R$ then $xr, rx \in N$.

Definition 2.3. [1] A proper partial ideal P of a partial semiring R is said to prime if and only if for any partial ideals A, B of R, $AB \subset P$ implies $A \subset P$ or $B \subset P$.

Spec(R) denotes the set of all prime partial ideals of a partial semiring R. For convenience we denote the set $\{H \in spec(R) \mid I \subset H\}$ by $V(I)$.

Theorem 2.1. [5] *If I is a partial ideal of a commutative partial semiring R then $\bigcap V(I) = \{a \in R \mid a^n \in I$ for some positive integer $n\}$.*

Definition 2.4. [8] Let $(R, \Sigma, \cdot, 1)$ be a partial semiring and $(M, \overline{\Sigma})$ be a partial monoid. Then M is said to be a left partial semimodule over R if there exists a function $* : R \times M \longrightarrow M : (r, x) \mapsto r * x$ which satisfies the following axioms for $x, (x_i : i \in I)$ in M and $r_1, r_2, (r_j : j \in J)$ in R

(i). if $\overline{\Sigma_i} x_i$ exists then $r * (\overline{\Sigma_i} x_i) = \overline{\Sigma_i}(r * x_i)$,

(ii). if $\Sigma_j r_j$ exists then $(\Sigma_j r_j) * x = \overline{\Sigma_j}(r_j * x)$,

(iii). $r_1 * (r_2 * x) = (r_1 \cdot r_2) * x$,

(iv). $1_R * x = x$,

(v). $0_R * x = 0_M$.

Definition 2.5. [6] Let $(M, \overline{\Sigma})$ be a left partial semimodule over a partial semiring R. Then a nonempty subset N of M is said to be a subsemimodule of M if and only if N is closed under $\overline{\Sigma}$ and $*$.

Definition 2.6. [6] Let N be a subsemimodule of a left partial semimodule M over R. Then $(N : M) = \bigcap\{(N : m) \mid m \in M\}$ is called the associated partial ideal of N.

Definition 2.7. [6] Let M be a partial semimodule over R. Then M is said to be multiplication partial semimodule if for all subsemimodules N of M there exists a partial ideal I of R such that $N = IM$.

Definition 2.8. [6] Let M be a multiplication partial semimodule over R and N, K be subsemimodules of M such that $N = IM$ and $K = JM$ for some partial ideals I, J of R. Then the multiplication of N and K is defined as $NK = (IM)(JM) = (IJ)M$.

Definition 2.9. [6] Let M be a multiplication partial semimodule over R and $m_1, m_2 \in M$ such that $Rm_1 = I_1M$ and $Rm_2 = I_2M$ for some partial ideals I_1, I_2 of R. Then the multiplication of m_1 and m_2 is defined as $m_1m_2 = (I_1M)(I_2M) = (I_1I_2)M$.

Definition 2.10. [6] Let M be a partial semimodule over R and N be a proper subsemimodule of M. Then N is said to be prime subsemimodule of M if for any $r \in R$ and $n \in M$, $r * n \in N$ implies $r \in (N : M)$ or $n \in N$.

Theorem 2.2. [6] *Let M be a multiplication partial semimodule over R and N be a subsemimodule of M. Then N is prime subsemimodule of M if and only if $(N : M)$ is a prime partial ideal of R.*

Definition 2.11. [7] A proper partial ideal I of R is said to be primary if for any $a, b \in R$, $ab \in I$ implies $a \in I$ or $b^n \in I$ for some $n \in Z^+$.

Lemma 2.1. [7] *If I is a primary partial ideal of R then $\sqrt{I}$ is a prime partial ideal of R.*

Definition 2.12. [7] A proper subsemimodule N of a partial semimodule M over R is said to be primary if for any $a \in R$, $x \in M$, $a * x \in N$ implies $x \in N$ or $a \in \bigcap V((N : M))$.

Lemma 2.2. [7] *Let M be a multiplication partial semimodule over R and N be a subsemimodule of M. Then $\bigcap V((N : M)) = (\bigcap V(N) : M)$.*

Definition 2.13. [7] Let M be a partial semimodule over R and N be a subsemimodule of M. Then define $(N :_M r) = \{m \in M \mid r * m \in N\}$ for any $r \in R$.

3. **Singular partial semiring wrt a partial semimodule**

Definition 3.1. Let M be a partial semimodule over a partial semiring R. Then for any $r \in R$, define $(0 :_M r) = \{m \in M \mid r * m = 0_M\}$, called the annihilator of $\{r\}$ in M, denoted by $A_M(r)$.

Remark 1. $A_M(r)$ is a subtractive subsemimodule of M.

Definition 3.2. Let N be a nonzero subsemimodule of a partial semimodule M over R. Then N is said to be an essential subsemimodule if N has nonzero intersection with all nonzero subsemimodules of M.

Definition 3.3. Let M be a partial semimodule over R. Then define
$S_M(R) = \{r \in R \mid A_M(r)$ is an essential subsemimodule of $M\}$.
i.e., $S_M(R) = \{r \in R \mid A_M(r) \bigcap N \neq 0_M$ for every nonzero subsemimodule N of $M\}$.

Theorem 3.1. *$S_M(R)$ is a subtractive partial ideal of R.*

Proof. Since $A_M(0_R)$ is an essential subsemimodule of M, $0_R \in S_M(R)$.
Let $(r_i : i \in I)$ be a summable family in R and $r_i \in S_M(R)$, $i \in I$. Then $\Sigma_i r_i$ exists and $A_M(r_i) \bigcap N \neq 0_M$ $\forall$ nonzero subsemimodule N of M, $i \in I$. $\Rightarrow (\bigcap A_M(r_i)) \bigcap N \neq 0_M$ $\forall$ nonzero subsemimodule N of M. $\Rightarrow A_M(\Sigma_i r_i) \bigcap N \neq 0_M$ $\forall$ nonzero subsemimodule N of M and hence $A_M(\Sigma_i r_i) \in S_M(R)$.

Now let $r \in S_M(R)$ and $r' \in R$. Then $A_M(r)$ is an essential subsemimodule of M. Now for any nonzero subsemimodule N of M, $r' * N$ is a nonzero subsemimodule of M. Since $A_M(r)$ is essential, $A_M(r) \bigcap (r' * N) \neq 0_M$. $\Rightarrow \exists\ 0_M \neq r' * n \in A_M(r) \bigcap (r' * N)$ where $0_M \neq n \in N$. $\Rightarrow (rr') * n = r * (r' * n) = 0_M$. $\Rightarrow 0_M \neq n \in A_M(rr') \bigcap N$. $\Rightarrow A_M(rr')$ is an essential subsemimodule of M. $\Rightarrow rr' \in S_M(R)$. Hence $S_M(R)$ is a partial ideal of R.

Let $r, r' \in R$ be such that $r, r+r' \in S_M(R)$. Then $A_M(r)$ and $A_M(r+r')$ are essential subsemimodules of M. Now for any nonzero subsemimodule N of M, $A_M(r) \bigcap A_M(r+r') \bigcap N \neq 0_M$. $\Rightarrow \exists 0_M \neq n \in A_M(r) \bigcap A_M(r+r')$. $\Rightarrow r * n = 0_M$ and $(r+r') * n = 0_M$. $\Rightarrow r' * n = 0_M$. $\Rightarrow 0_M \neq n \in A_M(r') \bigcap N$. $\Rightarrow A_M(r')$ is an essential subsemimodule of M. $\Rightarrow r' \in S_M(R)$. Hence $S_M(R)$ is a subtractive partial ideal of R. □

Definition 3.4. Let M be a partial semimodule over R. Then $S_M(R)$ is called singular partial ideal of R with respect to M.

Theorem 3.2. $S_M(R) = \{x \in R \mid x * N = 0_M$ *for some essential subsemimodule* N *of* $M\}$.

Proof. Take $S = \{x \in R \mid x * N = 0_M$ for some essential subsemimodule N of $M\}$. Let $x \in S_M(R)$. Then $A_M(x)$ is an essential subsemimodule of M. Moreover $x * A_M(x) = 0_M$. $\Rightarrow x \in S$. Now for any $x \in S$, $x * N = 0_M$ for some essential subsemimodule N of M. $\Rightarrow N \subseteq A_M(x)$. Since N is essential, $A_M(x)$ is an essential subsemimodule of M. $\Rightarrow x \in S_M(R)$. Hence $S_M(R) = S$. □

Theorem 3.3. $S_M(R) = \{x \in R \mid x * N = 0_M$ *for some essential subtractive subsemimodule* N *of* $M\}$.

Proof. Take $S' = \{x \in R \mid x * N = 0_M$ for some essential subtractive subsemimodule N of $M\}$. By above theorem, $S' \subseteq S \subseteq S_M(R)$. Now let $x \in S_M(R)$. Then $x * N = 0_M$ for some essential subsemimodule N of M. Let $\bar{N}$ be the subtractive closure of N. i.e., $\bar{N} = \{m \in M \mid \exists\ m' \in N \ni m + m' \in N\}$. Since $N \subseteq \bar{N}$ and N is essential, $\bar{N}$ is an essential subtractive subsemimodule of M. Now let $m \in \bar{N}$. Then $\exists\ m' \in N \ni m + m' \in N$. $\Rightarrow x * m' = 0_M$ and $x * (m + m') = 0_M$. $\Rightarrow x * m = 0_M\ \forall\ m \in \bar{N}$. $\Rightarrow x * \bar{N} = 0_M$. $\Rightarrow x \in S'$. Hence $S_M(R) = S'$. □

Definition 3.5. Let M be a partial semimodule over R. Then R is said to be singular partial semiring with respect to M (in short., singular wrt M) if $S_M(R) = R$.

Theorem 3.4. *If R is a nil partial semiring then R is singular wrt every partial semimodule over R.*

Proof. Let M be a partial semimodule over R, $a \in R$ and N be a nonzero subsemimodule of M. Then $\exists$ a positive integer m such that $a^m = 0_R$. $\Rightarrow$ for any $0_M \neq n \in N$, $0_M \neq Rn \subseteq N$. $\Rightarrow a^m Rn = 0_M$. Let n be the least positive integer such that $a^n Rn = 0_M$ and $a^{n-1}Rn \neq 0_M$. $\Rightarrow a^{n-1}Rn \subseteq A_M(a)$. $\Rightarrow$ $0_M \neq a^{n-1}Rn \subseteq A_M(a) \bigcap N$. $\Rightarrow A_M(a)$ is an essential subsemimodule of M. $\Rightarrow$ $a \in S_M(R)$. Hence $S_M(R) = R$. Therefore R is singular wrt any partial semimodule M. □

Remark 2. If I is a partial ideal of a partial semiring R then $S_M(I) = I \bigcap S_M(R)$.

Remark 3. If K is a partial ideal of a partial semiring R then $KS_M(R) \subseteq S_M(K)$.

Proof. Let $x \in KS_M(R)$. Then $x = \Sigma_i x_i y_i$ where $x_i \in K$, $y_i \in S_M(R)$, $i \in I$. $\Rightarrow$ $x_i y_i \in K$ and $A_M(y_i)$ is an essential subsemimodule of M, $i \in I$. Now for any $i \in I$, $A_M(y_i) \subseteq A_M(x_i y_i)$. $\Rightarrow A_M(x_i y_i)$ is an essential subsemimodule of M, $i \in I$. $\Rightarrow$ $x_i y_i \in S_M(K)$, $i \in I$. $\Rightarrow x = \Sigma_i x_i y_i \in S_M(K)$. Hence the remark. □

Definition 3.6. Let M, M' be partial semimodules over R. Then a surjective homomorphism $\gamma : M \to M'$ is said to be semiisomorphism if $\ker\gamma = 0$.

Theorem 3.5. *If $\gamma : M \to M'$ is a semiisomorphism and*
$S_M(R) = R$ then $S_{M'}(R) = R$.

Proof. Suppose if $S_{M'}(R) \subset R$. Then $\exists\ 0 \neq r \in R \ni r \notin S_{M'}(R)$. Since $r \notin S_{M'}(R)$, $\exists$ a nonzero subsemimodule N' of $M' \ni A_{M'}(r) \bigcap N' = 0$. $\Rightarrow 0 \neq n' \in N' \subseteq M'$.

Since γ is onto, $\exists\ 0 \neq n \in M \ni \gamma(n) = n' \in N'$. Take $N = \{a \in M \mid \gamma(a) \in N'\}$. Then N is a nonzero subsemimodule of M. $\Rightarrow A_M(r) \bigcap N \neq 0$. $\Rightarrow \exists\ 0 \neq b \in N \ni r * b = 0$. $\Rightarrow r * \gamma(b) = \gamma(r * b) = 0$. $\Rightarrow \gamma(b) \in A_{M'}(r) \bigcap N' = 0$. $\Rightarrow b \in \ker\gamma = 0$. $\Rightarrow b = 0$, a contradiction. Hence $S_{M'}(R) = R$. □

Theorem 3.6. *If $\gamma : M \to M'$ is a semiisomorphism and $S_{M'}(R) = 0$ then $S_M(R) = 0$.*

Proof. Suppose if $S_M(R) \neq 0$. Then $\exists\ 0 \neq r \in R \ni r \in S_M(R)$. $\Rightarrow 0 \neq r \in R$ and $A_M(r)$ is an essential subsemimodule of M. Since $S_{M'}(R) = 0$, $r \notin S_{M'}(R)$. $\Rightarrow \exists$ a nonzero subsemimodule N' of $M' \ni A_{M'}(r) \bigcap N' = 0$. Since $N' \neq 0$, $\exists\ 0 \neq n' \in N' \subseteq M'$. Since γ is onto, $\exists\ 0 \neq n \in M \ni \gamma(n) = n' \in N'$. Take $N = \{a \in M \mid \gamma(a) \in N'\}$. Then N is a nonzero subsemimodule of M. $\Rightarrow A_M(r) \bigcap N \neq 0$. $\Rightarrow \exists\ 0 \neq b \in N \ni r * b = 0$. $\Rightarrow \gamma(b) \in A_{M'}(r) \bigcap N' = 0$. $\Rightarrow b \in \ker\gamma = 0$, a contradiction. Hence $S_M(R) = 0$. □

Theorem 3.7. *If $\gamma : M \to M'$ is a semiisomorphism and $S_M(R) = 0$ then $S_{M'}(R) = 0$.*

Proof. Suppose if $S_{M'}(R) \neq 0$. Then $\exists\ 0 \neq r \in R \ni r \in S_{M'}(R)$. $\Rightarrow A_{M'}(r)$ is an essential subsemimodule of M'. Since $S_M(R) = 0$, $r \notin S_M(R)$. $\Rightarrow \exists$ a nonzero subsemimodule N of $M \ni A_M(r) \bigcap N = 0$. Since $N \neq 0$, $\exists\ 0 \neq n \in N$. If $\gamma(n) = 0$ then $n \in \ker\gamma = 0$, a contradiction. Hence $\gamma(n) \neq 0$. $\Rightarrow N' = \gamma(N)$ is a nonzero subsemimodule of M'. $\Rightarrow A_{M'}(r) \bigcap N' \neq 0$. $\Rightarrow \exists\ 0 \neq x \in N' \ni r * x = 0$. $\Rightarrow \exists\ 0 \neq a \in N \ni \gamma(a) = x$ and $r * \gamma(a) = 0$. $\Rightarrow \gamma(r * a) = 0$. $\Rightarrow r * a \in \ker\gamma = 0$. $\Rightarrow a \in A_M(r) \bigcap N = 0$, a contradiction. Hence $S_{M'}(R) = 0$. □

Definition 3.7. A prime subsemimodule N of M is said to be associated prime of R if $N = (0 :_M r) = A_M(r)$ for some $0 \neq r \in R$.

Denote the set of all associated primes of R by $AP_M(R)$ or simply $AP(R)$.

Remark 4. If $N \in AP(R)$ then N is a subtractive subsemimodule of M.

Lemma 3.1. *Any maximal element of* $\{A_M(r) \mid 0 \neq r \in R\}$ *is a prime subsemimodule of* M*.*

Proof. Let $A_M(x) = (0 :_M x)$ be a maximal element of $\{A_M(r) \mid 0 \neq r \in R\}$. Let $r \in R$ and $m \in M \ni r * m \in A_M(x)$ and $r \notin (A_M(x) : M)$. $\Rightarrow x * (r * m) = 0$ and $rM \nsubseteq A_M(x)$. $\Rightarrow (xr) * m = 0$ and $x * (rM) \neq 0$. $\Rightarrow (rx) * m = 0$ and $rx \neq 0$. $\Rightarrow$ $m \in A_M(rx)$. Clearly $A_M(x) \subseteq A_M(rx)$. Since $A_M(x)$ is maximal, $A_M(x) = A_M(rx)$. $\Rightarrow m \in A_M(x)$. Hence $A_M(x)$ is a prime subsemimodule of M. □

Theorem 3.8. *If* M *is Noetherian then* $AP(R) \neq 0$ *if and only if* $R \neq 0$*.*

Proof. Suppose $R \neq 0$. Then $\exists\ 0 \neq s \in R$. Take $\mathbb{A} = \{A_M(r) \mid 0 \neq r \in R\}$. clearly $A_M(s) \in \mathbb{A}$. $\Rightarrow \mathbb{A}$ is a nonempty family of subsemimodules of M. Since M is Noetherian, $\mathbb{A}$ has a maximal element. Let it be $A_M(x)$ for some $0 \neq x \in R$. Then by above lemma, $A_M(x) \in AP(R)$ and hence $AP(R) \neq 0$.

Conversely suppose $AP(R) \neq 0$. $\Rightarrow \exists$ a prime subsemimodule $N = A_M(x)$ for some $0 \neq x \in R$. Hence $R \neq 0$. □

4. **Primary subsemimodules wrt a prime subsemimodule**

Definition 4.1. Let I be a partial ideal and P be a prime partial ideals of R. Then I is said to be primary with respect to P (in short., primary wrt P) if for any $a, b \in R$ $\ni ab \in I$ and $b \notin P$ then $a^n \in I$ for some $n \in Z^+$.

Theorem 4.1. *If* I *is a primary partial ideal of* R *then* I *is primary wrt* $\bigcap V(I)$*.*

Proof. Suppose I is primary. Then by lemma 3.6 of [7], $\bigcap V(I)$ is prime partial ideal of R. Let $a, b \in R \ni ab \in I$ and $b \notin \bigcap V(I)$. Then $ab \in I$ and $b^n \notin I$ for all $n \geq 1$. Since I is primary, $a \in I$. Hence I is primary wrt $\bigcap V(I)$. □

Theorem 4.2. *Let I be a partial ideal and P be a prime partial ideal of R. Then I is primary wrt P if and only if for any partial ideals A, B of R, $AB \subseteq I$ and $B \nsubseteq P$ implies $a^n \in I$ for some $n \in \mathbb{Z}^+$, $\forall\ a \in A$.*

Proof. Suppose I is primary wrt P and let A, B be partial ideals of $R \ni AB \subseteq I$ and $B \nsubseteq P$. Then $\exists\ b \in B \ni b \notin P$. Now for any $a \in A$, $ab \in AB \subseteq I$ and $b \notin P$. $\Rightarrow$ $a^n \in I$ for some $n \in \mathbb{Z}^+$.

Conversely suppose for any A, B of R, $AB \subseteq I$ and $B \nsubseteq P$ implies $a^n \in I$ for some $n \in \mathbb{Z}^+$, $\forall\ a \in A$. Let $a, b \in R \ni ab \in I$ and $b \notin P$. Now $(aR)(bR) = (ab)R \subseteq I$ and $bR \nsubseteq P$. $\Rightarrow$ $(ar)^n \in I$ for some $n \in \mathbb{Z}^+$, $\forall\ ar \in aR$. $\Rightarrow$ $a^n \in I$ for some $n \in \mathbb{Z}^+$. Hence I is primary wrt P. □

Definition 4.2. Let N be a proper subsemimodule and P be a prime subsemimodule of a partial semimodule M. Then N is said to be primary wrt P if for any $a \in R$, $m \in M$, $a * m \in N$ and $m \notin P$ implies $a^n \in (N : M)$ for some $n \in \mathbb{Z}^+$.

Remark 5. The intersection of any two primary subsemimodules wrt P of a partial semimodule M is primary wrt P.

Proof. Let N_1, N_2 be two primary subsemimodules wrt P of M and let $a \in R$, $m \in M$ $\ni a * m \in N_1 \bigcap N_2$ and $m \notin P$. Then $a^n \in (N_1 : M)$ and $a^k \in (N_2 : M)$ for some $n, k \in \mathbb{Z}^+$. Take $l = \max\{n, k\}$. Then $a^l \in (N_1 : M) \bigcap (N_2 : M) = (N_1 \bigcap N_2 : M)$ for some $l \in \mathbb{Z}^+$. Hence $N_1 \bigcap N_2$ is primary wrt P. □

Theorem 4.3. *Let M be a partial semimodule over R, N be a proper subsemimodule and P be a prime subsemimodule of M. If N is primary wrt P then its associated partial ideal $(N : M)$ is a primary partial ideal wrt $(P : M)$ of R.*

Proof. Suppose N is primary wrt P of M. Since P is prime subsemimodule of M, $(P : M)$ is a prime partial ideal of R. Let $a, b \in R \ni ab \in (N : M)$ and $b \notin (P : M)$. Then $a(bM) = (ab)M \subseteq N$ and $bM \nsubseteq P$. $\Rightarrow a^n \in (N : M)$ for some $n \in \mathbb{Z}^+$. Hence $(N : M)$ is a primary partial ideal wrt $(P : M)$ of R. □

The following is an example of a partial semimodule M over R in which the converse of above theorem is not true in general.

Example 4.1. *Let R be the partial semiring $\mathbb{N}$ with finite support addition and usual multiplication. Then $M = \mathbb{N} \times \mathbb{N}$ is a left partial semimodule over R by the scalar multiplication $*(x, (a, b)) \mapsto (xa, xb)$. Then $K = 0 \times 4\mathbb{N}$ is a subsemimodule of M and $P = \mathbb{N} \times 3\mathbb{N}$ is a prime subsemimodule of M. Here $(K : M) = 0$ and $(P : M) = \{r \in \mathbb{N} \mid r * (\mathbb{N} \times \mathbb{N}) \subseteq \mathbb{N} \times 3\mathbb{N}\} = 3\mathbb{N}$ which is a prime partial ideal of R. Clearly $(K : M)$ is primary wrt $(P : M)$. Since $2 * (0, 2) \in K$, $(0, 2) \notin P$ and $2^n \notin (K : M)\ \forall\ n \in Z^+$, K is not primary subsemimodule wrt P of M.*

Now we show that the converse of above theorem is true for the multiplication partial semimodules.

Theorem 4.4. *Let M be a multiplication partial semimodule over R, N be a subsemimodule and P be a prime subsemimodule of M. Then N is primary wrt P if and only if $(N : M)$ is primary wrt $(P : M)$.*

Proof. By above theorem, we get the necessary part. for sufficient part, suppose $(N : M)$ is primary wrt $(P : M)$. Let $a \in R$, $m \in M \ni a * m \in N$ and $m \notin P$.

Since M is multiplication partial semimodule, $\exists$ a partial ideal I of $R \ni Rm = IM$. $\Rightarrow (aI)M = a(IM) = a * (Rm) \subseteq N$ and $IM = Rm \nsubseteq P$. $\Rightarrow aI \subseteq (N : M)$ and $I \nsubseteq (P : M) \Rightarrow a^n \in (N : M)$ for some $n \in \mathbb{Z}^+$. Hence N is primary wrt P. □

Theorem 4.5. *Let M be a multiplication partial semimodule over R, N be a subsemimodule and P be a prime subsemimodule of M. Then the following conditions are equivalent:*

(1). N is primary wrt P,

(2). for any subsemimodules U, V of M, $UV \subseteq N$ and $V \nsubseteq P$ implies $u^n \in N$ for some

$n \in \mathbb{Z}^+$, $\forall\ u \in U$,

(3). for any $m_1, m_2 \in M$, $m_1m_2 \in N$ and $m_2 \notin P$ implies $m_1^n \in N$ for some $n \in \mathbb{Z}^+$.

Proof. (1)$\Rightarrow$(2): Suppose N is primary wrt P and let U, V be subsemimodules of M $\ni UV \subseteq N$ and $V \nsubseteq P$. Since M is multiplication partial semimodule, $\exists$ partial ideal I, J of $R \ni U = IM$ and $V = JM$. $\Rightarrow UV = (IJ)M \subseteq N$ and $V = JM \nsubseteq P$. $\Rightarrow$ $IJ \subseteq (N : M)$ and $J \nsubseteq (P : M)$. Since $(N : M)$ is primary wrt $(P : M)$, $i^n \in (N : M)$ for some $n \in \mathbb{Z}^+$, $\forall\ i \in I$. $\Rightarrow u^n = (i * m)^n \in (iM)^n = (iM)^n = i^nM \subseteq N$ for some $n \in \mathbb{Z}^+$, $\forall\ u \in U = IM$.

(2)$\Rightarrow$(3): Suppose for any subsemimodules U, V of M, $UV \subseteq N$ and $V \nsubseteq P$ implies $u^n \in N$ for some $n \in \mathbb{Z}^+$, $\forall\ u \in U$. Let $m_1, m_2 \in M \ni m_1m_2 \in N$ and $m_2 \notin P$. Since M is multiplication semimodule, $\exists$ partial ideals I, J of $R \ni Rm_1 = IM$ and $Rm_2 = JM$. $\Rightarrow m_1m_2 = (Rm_1)(Rm_2) = (IJ)M \subseteq N$ and $Rm_2 = JM \nsubseteq P$. $\Rightarrow$ $(r * m_1)^n \in N$ for some $n \in \mathbb{Z}^+$, $\forall\ r * m_1 \in Rm_1$ and hence $m_1^n \in N$ for some $n \in \mathbb{Z}^+$.

(3)$\Rightarrow$(1): Suppose for any $m_1, m_2 \in M$, $m_1m_2 \in N$ and $m_2 \notin P$ implies $m_1^n \in N$ for some $n \in \mathbb{Z}^+$. We prove $(N : M)$ is primary wrt $(P : M)$. Let I, J be partial ideals of

$R \ni IJ \subseteq (N : M)$, $i^n \notin (N : M)$, $\forall\, n \in \mathbb{Z}^+$, for some $i \in I$ and $J \nsubseteq (P : M)$. Then $(IJ)M \subseteq N$, $i^n M \nsubseteq N$ and $JM \nsubseteq P$. $\Rightarrow \exists\, j \in J$, $m_1, m_2 \in M \ni i^n * m_1 \in IM \setminus N$ and $j * m_2 \in JM \setminus P$. Now $(i * m_1)(j * m_2) \in (IM)(JM) = (IJ)M \subseteq N$ and $j * m_2 \notin P$. $\Rightarrow (i * m_1)^l \in N$ for some $l \in \mathbb{Z}^+$. $\Rightarrow i^l * m_1 = (i * m_1)^l \in N$, a contradiction. Hence $(N : M)$ is primary wrt $(P : M)$. $\Rightarrow N$ is primary wrt P. □

Theorem 4.6. *If N is primary subsemimodule of a multiplication partial semimodule M then N is primary wrt $\bigcap V(N)$.*

Proof. Since N is primary, $(N : M)$ is primary partial ideal of R (by theorem 4.3). $\Rightarrow (N : M)$ is primary wrt $\bigcap V((N : M))$ (by theorem 3.2). $\Rightarrow (N : M)$ is primary wrt $(\bigcap V(N) : M)$ (by lemma 1.16). $\Rightarrow N$ is primary $\bigcap V(N)$ (by theorem 4.4). □

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