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FINITE LATTICE IMPLICATION ALGEBRAS

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Abstract. In this paper, by considering a finite lattice implication algebra L and $A \subseteq L$, the set of all co-atoms of L, we prove that L is equal to the filter generated by A, that is $L = [A)$. We give a correspondence theorem between the non-trivial minimal filters and co-atoms of L. We prove that if $A = \{a_1, a_2, \cdots, a_n\}$, then $L \cong [a_1) \times [a_2) \times \ldots \times [a_n)$. Finally, we give a characterization of finite lattice implication algebras. In particular, we show that there exists only one lattice implication algebra of prime order.

1. Introduction

In the field of many-valued logic, lattice-valued logic plays an important role in two aspects: One is that it extends the chain-type truth-value field of some well-known presented logic [1, 2] to some relatively general lattices. The other is that the incompletely comparable property of truth value characterized by a general lattice can more efficiently reflect the uncertainty of people's thinking, judging and decision. Hence, lattice-valued logic is becoming a research field which strongly influences the development of algebraic logic, computer science and artificial intelligence technology. In order to establish a logic system with truth value in a relatively general lattice, in 1993, Xu[6] established the lattice implication algebra by combining lattice and

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implication algebra, and investigated many useful structures. Lattice implication algebra provides the foundation in order to establish the corresponding logic system from the algebraic viewpoint. For the general development of lattice implication algebras, the filter theory plays an important role. In this paper, we consider a finite lattice implication algebra L and $A \subseteq L$, the set of all co-atoms of L, then prove that L is equal to the filter generated by A, that is $L = [A)$. Also, we prove that if $A = \{a_1, a_2, \cdots, a_n\}$, then $L \cong [a_1) \times [a_2) \times \cdots \times [a_n)$. Finally, we give a characterization of finite lattice implication algebras.

2. Preliminaries

Definition 2.1. Let $(L, \leq)$ be a lattice. Then;

(i) L is called *bounded*, if there is a greatest element 1 and least element 0 of L,

(ii) If L is bounded, $x \in L$ is called a *co-atom*, if $x < 1$ and there is no $y \in L$ such that $x < y < 1$,

(iii) a unary operation "$'$" on L is called *order reversing involution* if for any $x, y \in L$, $x \leq y$ implies $y' \leq x'$ and $(x')' = x$

Definition 2.2. [6] By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution "$'$" and a binary operation "$\rightarrow$" satisfying the following axioms: for all $x, y, z \in L$;

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(I2) $x \rightarrow x = 1$,

(I3) $x \rightarrow y = y' \rightarrow x'$,

(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,

(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,

(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$, In a lattice implication algebra, we can define a partial ordering "$\leq$" by $x \leq y$ if and only if $x \rightarrow y = 1$. Moreover, $x \prec y$ means that $x < y$ and there exist no $z \in L$ such that $x < z < y$.

Definition 2.3. [8] A subset F of a lattice implication algebra L is called a *filter* of L if it satisfies

(F1) $1 \in F$,

(F2) $x \rightarrow y \in F$ and $x \in F$ imply $y \in F$, for any $x, y \in F$.

The filter $\{1\} \neq F \subseteq L$ is called *minimal* if and only if there is no filter $G \subseteq L$ such that $\{1\} \subset G \subset F$.

It is easy to prove that, for any filter F of a lattice implication algebra L, if $x \leq y$ and $x \in F$, then $y \in F$.

Theorem 2.1. [6] *Let L be a lattice implication algebra and $\varnothing \neq A \subseteq L$. Then*

$$[A) = \{x \in L | \exists\ a_1, \ldots, a_n \in L,\ n \in \mathbb{N}\ s.t.\ a_1 \rightarrow (\cdots \rightarrow (a_n \rightarrow x) \cdots) = 1\}$$

and for any $a \in L$,

$$[a) = \{x \in L |\ \exists\ n \in \mathbb{N},\ s.t.\ a^n \rightarrow x = 1\}$$

where, $a^n \rightarrow x = \underbrace{a \rightarrow (a \rightarrow (\cdots \rightarrow (a}_{n\ time} \rightarrow x) \cdots))$ and $a^0 \rightarrow x = x$

Proposition 2.1. [6, 9] *Let L be a lattice implication algebra. Then the following hold: for any $x, y, z \in L$;*

(P1) $x \rightarrow 1 = 1$,

(P2) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,

(P3) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$.

(P4) $x' = x \rightarrow 0$,

(P5) $x \vee y = (x \rightarrow y) \rightarrow y$,

(P6) $0 \to x = 1$,

(P7) $1 \to x = x$,

(P8) $x \to y \geq x'$ *and* $x \to y \geq y$,

(P9) $(x \to y) \vee (y \to x) = 1$,

(P10) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,

(P11) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,

(P12) $(x \to z) \to (y \to z) = y \to (x \vee z) = (z \to x) \to (y \to x)$,

(P13) $(z \to x) \to (z \to y) = (x \wedge z) \to y = (x \to z) \to (x \to y)$,

(P14) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 2.4. [9] Let L_1 and L_2 be lattice implication algebras, $f : L_1 \longrightarrow L_2$ a mapping from L_1 to L_2. Then

(i) f is called an *implication homomorphism*, if for all $x, y \in L_1$;

$$f(x \to y) = f(x) \to f(y)$$

(ii) If f is an implication homomorphism and satisfies

$$f(x \vee y) = f(x) \vee f(y)\ ,\ f(x \wedge y) = f(x) \wedge f(y)\ ,\ f(x') = (f(x))'$$

for any $x, y \in L_1$, then f is called a *lattice implication homomorphism.*

Theorem 2.2. [9] *Let L_1 and L_2 be lattice implication algebras and $f : L_1 \longrightarrow L_2$ a mapping. Then f is a lattice implication homomorphism if and only if f is an implication homomorphism and $f(0) = 0$.*

Theorem 2.3. [9] *Let L_1, L_2,..., L_n be lattice implication algebras and $L = L_1 \times L_2 \times \cdots \times L_n$. Then $(L, \vee, \wedge, \to, ', 0, 1)$ is a lattice implication algebra, where binary operations $\vee, \wedge, \to$ and unary operation $'$ on L are defined as follows:*

$$(x_1, x_2, \cdots, x_n) \vee (y_1, y_2, \cdots, y_n) = (x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n)$$

$$(x_1, x_2, \cdots, x_n) \wedge (y_1, y_2, \cdots, y_n) = (x_1 \wedge y_1, x_2 \wedge y_2, \cdots, x_n \wedge y_n)$$

$$(x_1, x_2, \cdots, x_n) \rightarrow (y_1, y_2, \cdots, y_n) = (x_1 \rightarrow y_1, x_2 \rightarrow y_2, \cdots, x_n \rightarrow y_n)$$

$$(x_1, x_2, \cdots, x_n)' = (x_1', x_2', \cdots, x_n')$$

Theorem 2.4. [9] *Let $L_1, L_2, \cdots, L_n$ be lattice implication algebras. Then any filter of $L_1 \times L_2 \times \ldots \times L_n$ is as $F_1 \times F_2 \times \cdots \times F_n$ such that F_i is a filter of L_i, for any $1 \leq i \leq n$.*

Note 2.1. *From now on, in this paper we let always L be a finite lattice implication algebra, unless otherwise is stated.*

3. Correspondence Theorem for minimal filters

Lemma 3.1. *Let $x, y, z \in L$. Then the following hold:*
(i) If $x \prec y$, then $y \rightarrow x$ is a co-atom,
(ii) If $x \vee y = 1$, then $x \rightarrow y = y$ and $y \rightarrow x = x$,
(iii) If $x, y \prec z$ and $x \neq y$, then $z \rightarrow x \neq z \rightarrow y$.

Proof. (i) Let $x, y \in L$ and $x \prec y$. Then there exists $z \in L$ such that $y \rightarrow x < z < 1$, by contrary. Then, by (P3), $z \rightarrow x \leq (y \rightarrow x) \rightarrow x$. $x \leq y$, so by (P5), $(y \rightarrow x) \rightarrow x = x \vee y = y$. Moreover, by (P8), $x \leq z \rightarrow x \leq (y \rightarrow x) \rightarrow x = y$.
Now, since $x \prec y$, $z \rightarrow x = x$ or $z \rightarrow x = y$. If $z \rightarrow x = x$, then $(z \rightarrow x) \rightarrow x = x \rightarrow x = 1$. Since $x \leq y \rightarrow x < z$, we have

$$z = x \vee z = (z \rightarrow x) \rightarrow x = 1$$

which is a contradiction. If $z \rightarrow x = y$, then by (I1)

$$z \rightarrow (y \rightarrow x) = y \rightarrow (z \rightarrow x) = y \rightarrow y = 1$$

and so $z \leq y \rightarrow x < z$, which is impossible. Therefore, $y \rightarrow x$ is a co-atom.
(ii) By (P5), $1 = x \vee y = (x \rightarrow y) \rightarrow y$ and so $x \rightarrow y \leq y$. Moreover, by (P8),

$y \le x \to y$ and so by (I4), $x \to y = y$. Similarly, we can show that $y \to x = x$.
(iii) Let $x, y \prec z$ and $x \ne y$ but $z \to x = z \to y$, by contrary. Since $x \le x \vee y \le z$ and $x \prec z$, so $x \vee y = x$ or $x \vee y = z$. If $x \vee y = x$, then $y \le x \prec z$ and $y \ne x$, so $y \not\prec z$, which is impossible. If $x \vee y = z$ then by (P10),

$$z \to x = (z \to x) \vee (z \to y) = z \to (x \vee y) = z \to z = 1$$

and so $z \le x$, which is impossible. Therefore, $z \to x \ne z \to y$. □

Theorem 3.1. *(i) L is a chain if and only if any filters of L is equal to $\{1\}$ or L,*
(ii) If $a \in L$ is the only co-atom of L, then $L = [a)$.

Proof. (i) ($\Rightarrow$) Let L be a chain of order n, F be a filter of L and $F \ne \{1\}$. Then there exists a co-atom $a \in L$ such that for any $1 \ne x \in L$, $x \le a$. Hence $x \vee a = a$ and so $(a \to x) \to x = a$. $(a \to x) \to x < 1$, since $a < 1$, and so $a \to x \nleq x$. Since L is a chain,

$$(*) \qquad a \to x > x$$

Now, we claim that $0 \in [a)$. Let $0 \notin [a)$, by contrary. Then for any $i \in \mathbb{N}$, $a^i \to 0 \ne 1$ and so by (*),

$$a^i \to 0 = a \to (a^{i-1} \to 0) > a^{i-1} \to 0$$

Hence,

$$a^n \to 0 > a^{n-1} \to 0 > \ldots > a^2 \to 0 > a \to 0 > 0$$

Now, since $B = \{a^n \to 0, a^{n-1} \to 0, \ldots, a^2 \to 0, a \to 0, 0\} \subseteq L$, so $n + 1 = |B| \le |L| = n$, which is a contradiction. Thus $0 \in [a)$ and so $[a) = L$. Now, let $x \in F$. L is a chain, so $x \le a$ and $a \in F$, since F is filter. Hence, $L = [a) \subseteq F \subseteq L$ and so $F = L$.
($\Leftarrow$) Let the only filters of L be $\{1\}$ and L and there exist $x, y \in L$ such that $y \nleq x$. Hence $y \to x \ne 1$ and so by hypothesis, $\{1\} \subset [y \to x) = L$. Now, since

$x \to y \in L = [y \to x)$, so there exists $m \in \mathbb{N}$ such that $(y \to x)^m \to (x \to y) = 1$. By (P9), $(x \to y) \vee (y \to x) = 1$, so by Lemma 3.1(ii), $(y \to x) \to (x \to y) = x \to y$ and so

$$\begin{aligned}
1 &= (y \to x)^m \to (x \to y) \\
&= (y \to x)^{m-1} \to ((y \to x) \to (x \to y)) \\
&= (y \to x)^{m-1} \to (x \to y) \\
&\vdots \\
&= (y \to x)^2 \to (x \to y) \\
&= (y \to x) \to ((y \to x) \to (x \to y)) \\
&= (y \to x) \to (x \to y) \\
&= x \to y
\end{aligned}$$

Hence $x \leq y$. Therefore, L is a chain.

(ii) Let $a \in L$ be the only co-atom of L. First we show that L is a chain. For this let $x, y \in L$ such that $x \nleq y$ and $y \nleq x$, by contrary. Then $x \to y \neq 1$ and $y \to x \neq 1$. a is the only co-atom of L, so $x \to y \leq a$ and $y \to x \leq a$ and so by (P9),

$$1 = (x \to y) \vee (y \to x) \leq a < 1$$

which is impossible. Hence $x \leq y$ or $y \leq x$ and so L is a chain. Now, by (i), $L = [a)$, since $\{1\} \neq [a) \subseteq L$. □

Theorem 3.2. *(i) Let $a \in L$ be a co-atom. Then the filter $F = [a)$ is a non-trivial chain minimal filter,*

(ii) Any non-trivial minimal filter of L is the form $F = [a)$, such that $a \in L$ is a co-atom.

Proof. (i) Let $a \in L$ be a co-atom. Obviously, $[a)$ is non-trivial. Now, suppose that $F = [a)$ is not a chain. Since L is finite, so there exist $x, y \in F$ such that $x \nleq y$, $y \nleq x$, $x \prec x \vee y$ and $y \prec x \vee y$. Then by Lemma 3.1(i), $(x \vee y) \to x$ and $(x \vee y) \to y$ are co-atoms of L. Moreover, by Lemma 3.1(iii), $(x \vee y) \to x \neq (x \vee y) \to y$ and this implies that at least one of them is not equal to a. W.O.L.G, let $(x \vee y) \to x = b$ and $b \neq a$. Since by (P8), $b \geq x$ and $x \in F$, so $b \in F$ and by Theorem 2.1, there exists $n \in \mathbb{N}$, such that $a^n \to b = 1$. Now, since a and b are co-atoms, so $a \vee b = 1$ and by Lemma 3.1(ii), $a \to b = b$. Hence $(a \to (a \to b)) = a \to b = b$. By a similar way, we can prove that $1 = a^n \to b = a \to (\ldots \to (a \to (a \to b))\ldots) = b$, which is a contradiction. Therefore, $[a)$ is a chain.

Now, we show that $F = [a)$ is a minimal filter of L. Let E be a non-trivial filter of L such that $E \subseteq F$. Then there exists $1 \neq x \in E \cap F$. We have $x \in F = [a)$, so there exists smallest $n \in \mathbb{N}$ such that $a^n \to x = 1$ and so $a \leq a^{n-1} \to x$. Since a is a co-atom, so $a^{n-1} \to x = 1$ or $a^{n-1} \to x = a$. n is the smallest, so $a^{n-1} \to x \neq 1$ and $a^{n-1} \to x = a$. But, by (P8) we can prove that

$$x \leq a \to x \leq a \to (a \to x) \leq \ldots \leq (a \to (\ldots \to (a \to (a \to x))\ldots)) = a^{n-1} \to x = a$$

and so $x \leq a$. Now since $x \in E$ and E is a filter we have $a \in E$ and so $E \subseteq F = [a) \subseteq E$. Hence $F = E$. Therefore, $F = [a)$ is a minimal filter of L.

(ii) Let F be a non-trivial minimal filter of L. Then $\{1\} \neq F$. Now, let $1 \neq x \in F$. There exists a co-atom $1 \neq a \in L$ such that $x \leq a$, since L is finite and so $a \in F$. Then $[a) \subseteq F$. Since F is minimal and $[a)$ is a filter, so $F = [a)$. □

Theorem 3.3 (Correspondence Theorem)**.** *There exists a bijection map between the set of all non-trivial minimal filters and the set of all co-atoms of L.*

Proof. Let $\mathcal{F} = \{F : F$ is a non-trivial minimal filter of $L\}$, $\mathcal{C} = \{a : a$ is a co-atom of $L\}$ and $\varphi : \mathcal{C} \to \mathcal{F}$ be defined by $\varphi(a) = [a)$. It is clear that φ is well-defined.

Now, we show that φ is one-to-one and onto. Let $\varphi(a) = \varphi(b)$, for $a, b \in \mathcal{C}$. Then $[a) = [b)$. $a \in [a)$, so $a \in [b)$ and so there exists smallest $n \in \mathbb{N}$ such that $a^n \to b = 1$. Similar to the proof of Theorem 3.2(i), we can get that $a \leq b$. Now, since $b \in [b)$, so $b \in [a)$ and similarly we get that $b \leq a$ and so $a = b$. Hence φ is one-to-one. Now, let $F \in \mathcal{F}$. Then by Theorem 3.2(ii), there exists a co-atom $a \in L$ such that $F = [a)$. Now, φ is onto, since $a \in \mathcal{C}$ and $\varphi(a) = F$. □

4. Characterization of finite lattice implication algebras

Theorem 4.1. *Let A be the set of all co-atoms of L. Then $[A) = L$.*

Proof. It is clear that $[A) \subseteq L$. Now, let $1 \neq x \in L$. Since L is finite, so there exist $m \in \mathbb{N}$ and $b_1, b_2, \ldots, b_m \in L$ such that $x \prec b_1 \prec b_2 \prec \ldots \prec b_m \prec 1$. Then by Lemma 3.1(i),

$$b_m = 1 \to b_m, b_m \to b_{m-1}, \ldots, b_2 \to b_1, b_1 \to x$$

are co-atoms of L. Moreover,

$$\begin{aligned}
& b_m \to ((b_m \to b_{m-1}) \to \cdots ((b_3 \to b_2) \to ((b_2 \to b_1) \to ((b_1 \to x) \to x))) \cdots) \\
= \;& b_m \to ((b_m \to b_{m-1}) \to \cdots ((b_3 \to b_2) \to ((b_2 \to b_1) \to (b_1 \vee x))) \cdots) \quad , \quad \text{by (P5)} \\
= \;& b_m \to ((b_m \to b_{m-1}) \to \cdots ((b_3 \to b_2) \to ((b_2 \to b_1) \to b_1)) \cdots) \quad , \quad \text{since } x \leq b_1 \\
= \;& b_m \to ((b_m \to b_{m-1}) \to \cdots ((b_3 \to b_2) \to (b_2 \vee b_1)) \cdots) \quad , \quad \text{by (P5)} \\
= \;& b_m \to ((b_m \to b_{m-1}) \to \cdots ((b_3 \to b_2) \to b_2) \cdots) \quad , \quad \text{since } b_1 \leq b_2 \\
\vdots \;& \\
= \;& b_m \to ((b_m \to b_{m-1}) \to b_{m-1}) \\
= \;& b_m \to (b_m \vee b_{m-1}) \quad , \quad \text{by (P5)} \\
= \;& b_m \to b_m \quad , \quad \text{since } b_{m-1} < b_m \\
= \;& 1
\end{aligned}$$

Now, since $\{b_m, b_m \to b_{m-1}, \ldots, b_2 \to b_1, b_1 \to x\} \subseteq A$, we get that $x \in [A)$ and so $L \subseteq [A)$. Therefore, $L = [A)$. □

Lemma 4.1. *Let F be a filter of L. Then $(F, \wedge, \vee, \to, 1, 0_F = \bigwedge_{x \in F} x)$ is a lattice implication algebra.*

Proof. We have $1 \in F$, since F is filter. F is finite, so $0_F = \bigwedge_{x \in F} x \in F$. Now, let $x, y \in F$. $x \leq x \vee y$ and $x \in F$, so $x \vee y \in F$. Moreover, by (I2), (P8) and (P11),

$$y \leq x \to y = 1 \wedge (x \to y) = (x \to x) \wedge (x \to y) = x \to (x \wedge y)$$

Now, since $x, y \in F$, we conclude that $x \to (x \wedge y) \in F$ and $x \wedge y \in F$. Also, by (P4) and (P8), $x' = x \to 0_F \geq 0_F$ and $x' \in F$ since $0_F \in F$. Finally, since $y \leq x \to y$ and $y \in F$, we have $x \to y \in F$. Therefore, F is closed under $\vee, \wedge, \to$ and $'$. Hence $(F, \wedge, \vee, \to, ', 0_F, 1)$ is a lattice implication algebra. □

Lemma 4.2. *Let $F_1 \neq F_2$ be two non-trivial minimal filters of L. Then,*
(i) $F_1 \cap F_2 = \{1\}$
(ii) If $a \in F_1$ and $b \in F_2$, then $a \to b = b$ and $b \to a = a$.

Proof. (i) $F_1 \cap F_2$ is a filter and $F_1 \neq F_2$, hence $F_1 \cap F_2 \subsetneq F_1$. Now, $F_1 \cap F_2 = \{1\}$ since F_1 is minimal.
(ii) Let $a \in F_1$ and $b \in F_2$. $a \vee b \in F_1$ since $a \leq a \vee b$ and F_1 is filter. Similarly, $a \vee b \in F_2$. Hence $a \vee b \in F_1 \cap F_2 = \{1\}$ and so $a \vee b = 1$. From Lemma 3.1(ii), it directly is followed that $a \to b = b$ and $b \to a = a$. □

Theorem 4.2. *Let $A = \{a_1, a_2, \cdots, a_n\}$ be the set of all distinct co-atoms of L. Then*

$$L \cong [a_1) \times [a_2) \times \cdots \times [a_n)$$

Proof. By Lemma 4.1, for all $1 \leq i \leq n$, the filter $[a_i)$ is a lattice implication algebra and so by Theorem 2.3, $[a_1) \times [a_2) \times \cdots \times [a_n)$ is a lattice implication algebra. Moreover, by Theorem 4.1, $L = [A)$. Now, it is enough to prove that $[A) \cong [a_1) \times [a_2) \times \cdots \times [a_n)$. For this, let $\varphi : [a_1) \times [a_2) \times \cdots \times [a_n) \longrightarrow [A)$ be defined by $\varphi(x_1, x_2, \ldots, x_n) = x_1 \wedge x_2 \wedge \ldots \wedge x_n$. Since for any $1 \leq i \leq n$, so $x_i \in [a_i)$, there exists $m_i \in \mathbb{N}$ such that

$$a_i{}^{m_i} \to x_i = 1 \tag{4.1}$$

and so by (P11), (I1) and (P1),

$$\begin{aligned}
& a_n{}^{m_n} \to (a_{n-1}{}^{m_{n-1}} \to (\cdots(a_3{}^{m_3} \to (a_2{}^{m_2} \to (a_1{}^{m_1} \to (x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_n)))) \\
& \cdots)) \\
= \ & a_n{}^{m_n} \to (a_{n-1}{}^{m_{n-1}} \to (\cdots(a_3{}^{m_3} \to (a_2{}^{m_2} \to ((a_1^{m_1} \to x_1) \wedge (a_1{}^{m_1} \to (x_2 \wedge x_3 \wedge \\
& \cdots \wedge x_n))))) \cdots)) \\
= \ & a_n{}^{m_n} \to (a_{n-1}{}^{m_{n-1}} \to (\cdots(a_3{}^{m_3} \to (a_2{}^{m_2} \to (1 \wedge (a_1{}^{m_1} \to (x_2 \wedge x_3 \wedge \cdots \wedge x_n)) \\
&))) \cdots)) \\
= \ & a_n{}^{m_n} \to (a_{n-1}{}^{m_{n-1}} \to (\cdots(a_3{}^{m_3} \to (a_2{}^{m_2} \to (a_1^{m_1} \to (x_2 \wedge x_3 \wedge \cdots \wedge x_n)))) \cdots)) \\
= \ & a_n{}^{m_n} \to (a_{n-1}{}^{m_{n-1}} \to (\cdots(a_3{}^{m_3} \to (a_1{}^{m_1} \to (a_2^{m_2} \to (x_2 \wedge x_3 \wedge \cdots \wedge x_n)))) \cdots)) \\
= \ & a_n{}^{m_n} \to (a_{n-1}{}^{m_{n-1}} \to (\cdots(a_3{}^{m_3} \to (a_1{}^{m_1} \to ((a_2^{m_2} \to x_2) \wedge (a_2^{m_2} \to (x_3 \wedge \cdots \wedge \\
& x_n))))) \cdots)) \\
= \ & a_n{}^{m_n} \to (a_{n-1}{}^{m_{n-1}} \to (\cdots(a_3{}^{m_3} \to (a_1{}^{m_1} \to (1 \wedge (a_2^{m_2} \to (x_3 \wedge \cdots \wedge x_n))))) \cdots)) \\
= \ & a_n{}^{m_n} \to (a_{n-1}{}^{m_{n-1}} \to (\cdots(a_3{}^{m_3} \to (a_1{}^{m_1} \to (a_2^{m_2} \to (x_3 \wedge \cdots \wedge x_n)))) \cdots)) \\
= \ & a_n{}^{m_n} \to (a_{n-1}{}^{m_{n-1}} \to (\cdots(a_1{}^{m_1} \to (a_2{}^{m_2} \to (a_3^{m_3} \to (x_3 \wedge \cdots \wedge x_n)))) \cdots)) \\
\vdots \ & \\
= \ & a_1{}^{m_1} \to (a_2{}^{m_2} \to (\cdots(a_{n-1}{}^{m_{n-1}} \to (a_n{}^{m_n} \to x_n)) \cdots)) \\
= \ & a_1{}^{m_1} \to (a_2{}^{m_2} \to (\cdots(a_{n-1}{}^{m_{n-1}} \to 1) \cdots)) = 1
\end{aligned}$$

Hence, $x_1 \wedge x_2 \wedge \ldots \wedge x_n \in [A)$ and so φ is well-defined.

Now, we prove that φ is a homomorphism. Let $(x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n) \in [a_1) \times [a_2) \times \cdots \times [a_n)$. By Theorem 2.2, it is enough to prove that

$$\varphi((x_1, x_2, \cdots, x_n) \to (y_1, y_2, \cdots, y_n)) = \varphi((x_1, x_2, \cdots, x_n)) \to \varphi((y_1, y_2, \cdots, y_n))$$

$$\varphi((0_{[a_1)}, 0_{[a_2)}, \cdots, 0_{[a_n)})) = 0$$

First we show that

$$(*) \qquad (x_1 \wedge \cdots \wedge x_n) \to (y_1 \wedge \cdots \wedge y_n) = (x_1 \to y_1) \wedge \cdots \wedge (x_n \to y_n)$$

By (P11) and (L2),

$$\begin{aligned}
& (x_1 \wedge \cdots \wedge x_n) \to (y_1 \wedge \cdots \wedge y_n) \\
= \;& ((x_1 \wedge \cdots \wedge x_n) \to y_1) \wedge \cdots \wedge ((x_1 \wedge \cdots \wedge x_n) \to y_n) \\
= \;& ((x_1 \to y_1) \vee \cdots \vee (x_n \to y_1)) \wedge \cdots \wedge ((x_1 \to y_n) \vee \cdots \vee (x_n \to y_n)) \\
= \;& ((x_1 \to y_1) \vee y_1 \vee \cdots \vee y_1) \wedge \cdots \wedge (y_n \vee \cdots \vee y_n \wedge (x_n \to y_n)) \text{ , by Lemma 4.2(ii)} \\
= \;& (x_1 \to y_1) \wedge \cdots \wedge (x_n \to y_n) \text{ , since } y_i \leq x_i \to y_i
\end{aligned}$$

Hence we have (*). So

$$\begin{aligned}
\varphi((x_1, \cdots, x_n)) \to \varphi((y_1, \cdots, y_n)) &= (x_1 \wedge \cdots \wedge x_n) \to (y_1 \wedge \cdots \wedge y_n) \\
&= (x_1 \to y_1) \wedge \cdots \wedge (x_n \to y_n) \text{ , by (*)} \\
&= \varphi((x_1 \to y_1), \cdots, (x_n \to y_n)) \\
&= \varphi((x_1, \cdots, x_n) \to (y_1, \cdots, y_n))
\end{aligned}$$

Now, let $x \in L$. By Theorem 4.1, $L = [a_1, a_2, \ldots, a_n)$, so there exist $n \in \mathbb{N}$ and minimal numbers $m_1, m_2, \ldots, m_n \in \mathbb{N} \cup \{0\}$, such that

$$(4.2) \qquad {a_1}^{m_1} \to ({a_2}^{m_2} \to (\ldots ({a_n}^{m_n} \to x) \ldots)) = 1$$

Now, let for any $1 \le i \le n$,

(4.3)
$$x_i = {a_1}^{m_1} \to ({a_2}^{m_2} \to (\cdots({a_{i-1}}^{m_{i-1}} \to ({a_{i+1}}^{m_{i+1}} \to (\cdots({a_n}^{m_n} \to x)\cdots)))\cdots))$$

Hence, by (P8) and (I1), for all $1 \le i \le n$, $x \le x_i$ and

$$\begin{aligned}
& {a_i}^{m_i} \to x_i \\
= \; & {a_i}^{m_i} \to ({a_1}^{m_1} \to ({a_2}^{m_2} \to (\cdots({a_{i-1}}^{m_{i-1}} \to ({a_{i+1}}^{m_{i+1}}(\cdots({a_n}^{m_n} \to x)\cdots)))\cdots))) \\
= \; & {a_1}^{m_1} \to ({a_2}^{m_2} \to (\ldots({a_{i-1}}^{m_{i-1}} \to ({a_i}^{m_i} \to ({a_{i+1}}^{m_{i+1}}(\ldots({a_n}^{m_n} \to x)\cdots))))\cdots)) \\
= \; & 1
\end{aligned}$$

and so $x_i \in [a_i)$. Hence $(x_1, x_2, \ldots, x_n) \in [a_1) \times [a_2) \times \ldots \times [a_n)$. Now, we claim that $x = x_1 \wedge x_2 \wedge \ldots \wedge x_n$, that is $\varphi(x_1, x_2, \cdots, x_n) = x$. For all $i \in \{1, 2, ..., n\}$, $x \le x_i$ so $x \le x_1 \wedge x_2 \wedge \ldots \wedge x_n$. Now, we should prove that $x_1 \wedge x_2 \wedge \ldots \wedge x_n \le x$ or equivalently $(x_1 \wedge x_2 \wedge \ldots \wedge x_n) \to x = 1$. First we show that for any $1 \le i \le n$,

(4.4)
$$ {a_1}^{m_1} \to (\ldots({a_{i-1}}^{m_{i-1}} \to ({a_{i+1}}^{m_{i+1}} \to (\ldots({a_n}^{m_n} \to a_i)\ldots)))\ldots)) = a_i$$

For any $i \le j \le n$ and $j \ne i$, $a_i \vee a_j = 1$, so by Lemma 3.1(ii), $a_j \to a_i = a_i$ and so

$$\begin{aligned}
& {a_1}^{m_1} \to (\ldots({a_{i-1}}^{m_{i-1}} \to ({a_{i+1}}^{m_{i+1}} \to (\ldots({a_n}^{m_n} \to a_i)\ldots)))\ldots) \\
= \; & {a_1}^{m_1} \to (\ldots({a_{i-1}}^{m_{i-1}} \to ({a_{i+1}}^{m_{i+1}} \to (\ldots({a_n}^{m_n-1} \to (a_n \to a_i))\ldots)))\ldots) \\
= \; & {a_1}^{m_1} \to (\ldots({a_{i-1}}^{m_{i-1}} \to ({a_{i+1}}^{m_{i+1}} \to (\ldots({a_n}^{m_n-1} \to a_i)\ldots)))\ldots) \\
= \; & {a_1}^{m_1} \to (\ldots({a_{i-1}}^{m_{i-1}} \to ({a_{i+1}}^{m_{i+1}} \to (\ldots({a_n}^{m_n-2} \to (a_n \to a_i))\ldots)))\ldots) \\
& \vdots \\
= \; & a_i
\end{aligned}$$

Moreover, we must show that for any $1 \leq i \leq n$,

(4.5)

$$a_i = {a_1}^{m_1} \rightarrow (\cdots ({a_{i-1}}^{m_{i-1}} \rightarrow ({a_i}^{m_i - 1} \rightarrow ({a_{i+1}}^{m_{i+1}} \rightarrow (\cdots ({a_n}^{m_n} \rightarrow x) \cdots)))) \cdots)$$

By using (I1), repeatedly, we have

$$\begin{aligned} & a_i \rightarrow ({a_1}^{m_1} \rightarrow (\cdots ({a_{i-1}}^{m_{i-1}} \rightarrow ({a_i}^{m_i - 1} \rightarrow ({a_{i+1}}^{m_{i+1}} \rightarrow (\cdots ({a_n}^{m_n} \rightarrow x) \cdots)))) \\ & \cdots)) \\ = \; & {a_1}^{m_1} \rightarrow (\cdots ({a_{i-1}}^{m_{i-1}} \rightarrow ({a_i}^{m_i} \rightarrow ({a_{i+1}}^{m_{i+1}} \rightarrow (\cdots ({a_n}^{m_n} \rightarrow x) \cdots)))) \cdots) \\ = \; & 1 \end{aligned}$$

Then,

$$a_i \leq {a_1}^{m_1} \rightarrow (\cdots ({a_{i-1}}^{m_{i-1}} \rightarrow ({a_i}^{m_i - 1} \rightarrow ({a_{i+1}}^{m_{i+1}} \rightarrow (\cdots ({a_n}^{m_n} \rightarrow x) \cdots)))) \cdots)$$

Now since a_i is a co-atom, we have

$${a_1}^{m_1} \rightarrow (\cdots ({a_{i-1}}^{m_{i-1}} \rightarrow ({a_i}^{m_i - 1} \rightarrow ({a_{i+1}}^{m_{i+1}} \rightarrow (\cdots ({a_n}^{m_n} \rightarrow x) \cdots)))) \cdots) = 1 \text{ or } a_i$$

If ${a_1}^{m_1} \rightarrow (\cdots ({a_{i-1}}^{m_{i-1}} \rightarrow ({a_i}^{m_i - 1} \rightarrow ({a_{i+1}}^{m_{i+1}} \rightarrow (\cdots ({a_n}^{m_n} \rightarrow x) \cdots)))) \cdots) = 1$, then we have a contradiction by (4.2). Hence ${a_1}^{m_1} \rightarrow (\cdots ({a_{i-1}}^{m_{i-1}} \rightarrow ({a_i}^{m_i - 1} \rightarrow ({a_{i+1}}^{m_{i+1}} \rightarrow (\cdots ({a_n}^{m_n} \rightarrow x) \cdots)))) \cdots) = a_i$ and so we have (4.5). Now, let $z =$

$(x_1 \wedge x_2 \wedge \ldots \wedge x_n) \to x$. Hence for any $1 \le i \le n$,

$$\begin{aligned}
& z \to a_i \\
= \;& ((x_1 \wedge x_2 \wedge \cdots \wedge x_n) \to x) \to a_i \\
= \;& ((x_1 \to x) \vee (x_2 \to x) \vee \cdots \vee (x_i \to x) \vee \cdots \vee (x_n \to x)) \to a_i \quad \text{, by (L2)} \\
= \;& ((x_1 \to x) \to a_i) \wedge ((x_2 \to x) \to a_i) \wedge \cdots \wedge ((x_i \to x) \to a_i) \wedge \cdots \wedge ((x_n \to x) \\
& \to a_i)) \quad \text{, by (L1)} \\
\le \;& (x_i \to x) \to a_i \\
= \;& (x_i \to x) \to ({a_1}^{m_1} \to (\cdots({a_{i-1}}^{m_{i-1}} \to ({a_i}^{m_i-1} \to ({a_{i+1}}^{m_{i+1}} \to (\cdots({a_n}^{m_n} \to x) \\
& \cdots))))\cdots)) \quad \text{, by (4.5)} \\
= \;& {a_1}^{m_1} \to (\cdots({a_{i-1}}^{m_{i-1}} \to ({a_i}^{m_i-1} \to ({a_{i+1}}^{m_{i+1}} \to (\cdots({a_n}^{m_n} \to ((x_i \to x) \to x)) \\
& \cdots))\cdots) \\
= \;& {a_1}^{m_1} \to (\cdots({a_{i-1}}^{m_{i-1}} \to ({a_i}^{m_i-1} \to ({a_{i+1}}^{m_{i+1}} \to (\cdots({a_n}^{m_n} \to (x_i \vee x))\cdots))\cdots) \\
= \;& {a_1}^{m_1} \to (\cdots({a_{i-1}}^{m_{i-1}} \to ({a_i}^{m_i-1} \to ({a_{i+1}}^{m_{i+1}} \to (\cdots({a_n}^{m_n} \to x_i)\cdots))\cdots) \quad , \\
& \text{since } x \le x_i \\
= \;& {a_1}^{m_1} \to (\cdots({a_{i-1}}^{m_{i-1}} \to ({a_i}^{m_i-1} \to ({a_{i+1}}^{m_{i+1}} \to (\cdots({a_n}^{m_n} \to ({a_1}^{m_1} \to (\cdots \\
& ({a_{i-1}}^{m_{i-1}} \to ({a_{i+1}}^{m_{i+1}} \to (\cdots({a_n}^{m_n} \to x)\cdots)))\cdots)))\cdots))))\cdots) \quad \text{, by (4.3)} \\
= \;& {a_1}^{m_1} \to (\cdots({a_{i-1}}^{m_{i-1}} \to ({a_{i+1}}^{m_{i+1}} \to (\cdots({a_n}^{m_n} \to ({a_1}^{m_1} \to (\cdots \\
& ({a_{i-1}}^{m_{i-1}} \to ({a_i}^{m_i-1} \to ({a_{i+1}}^{m_{i+1}} \to (\cdots({a_n}^{m_n} \to x)\cdots))))\cdots)))\cdots)))\cdots) \\
= \;& {a_1}^{m_1} \to (\cdots({a_{i-1}}^{m_{i-1}} \to ({a_{i+1}}^{m_{i+1}} \to (\cdots({a_n}^{m_n} \to a_i)\cdots)))\cdots) \quad \text{, by (4.5)} \\
= \;& a_i \quad \text{, by (4.4)}
\end{aligned}$$

Hence, for any $1 \le i \le n$,

$$z \to a_i \leq a_i < 1 \tag{4.6}$$

Now, if $z \neq 1$, then there exists a co-atom $a_j \in A$, such that $z \leq a_j$ and so $z \to a_j = 1$, which is a contradiction by (4.6). Hence $z = 1$ and so $(x_1 \wedge x_2 \wedge \cdots \wedge x_n) \to x = 1$. Therefore, $x_1 \wedge x_2 \wedge \cdots \wedge x_n = x$. Now, if $x = 0$ then $x_1 \wedge x_2 \wedge \cdots \wedge x_n = 0$. $0_{[a_i)} \leq x_i$, Since $x_i \in [a_i)$ and so

$$0 \leq 0_{[a_1)} \wedge \cdots \wedge 0_{[a_n)} \leq x_1 \wedge \cdots \wedge x_n = 0$$

and this implies that

$$\varphi((0_{[a_1)}, 0_{[a_2)}, \cdots, 0_{[a_n)})) = 0_{[a_1)} \wedge 0_{[a_2)} \wedge \cdots \wedge 0_{[a_n)} = 0$$

Therefore, φ is a lattice implication homomorphism.

Now, let $x \in L$. By the above argument, there exist $n \in \mathbb{N}$ and $x_i \in L$, for $i \in \{1, 2, ..., n\}$, such that $x_1 \wedge x_2 \wedge \cdots \wedge x_n = x$. Hence $\varphi(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n = x$ and so φ is onto.

In the following, we should prove that φ is one-to-one. Let $(x_1, \cdots, x_n), (y_1, \cdots, y_n) \in [a_1) \times \cdots \times [a_n)$ and $\varphi((x_1, \cdots, x_n)) = \varphi((y_1, \cdots, y_n))$. Hence, $x_1 \wedge \cdots \wedge x_n = y_1 \wedge \cdots \wedge y_n$. $y_i \leq x_i \to y_i$, for all $1 \leq i \leq n$, so by (*);

$$1 = (x_1 \wedge \cdots \wedge x_n) \to (y_1 \wedge \cdots \wedge y_n) = (x_1 \to y_1) \wedge \cdots \wedge (x_n \to y_n)$$

Thus, $x_i \to y_i = 1$ and so $x_i \leq y_i$, for any $1 \leq i \leq n$. Similarly, we can prove that $y_i \leq x_i$ for any $1 \leq i \leq n$ and this implies that $x_i = y_i$, for any $1 \leq i \leq n$. Hence, $(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$ and so φ is an isomorphism. Therefore,

$$L \cong [a_1) \times [a_2) \times \ldots \times [a_n)$$

$\square$

Corollary 4.1. *Let $A = \{a_1, a_2, \cdots, a_n\}$ be the set of all distinct co-atoms of L. Then,*

(i) For any filter F of L, there exist $a_{i_1}, a_{i_2}, \cdots, a_{i_m} \in A$ such that

$$F \cong [a_{i_1}) \times [a_{i_2}) \times \cdots \times [a_{i_m})$$

(ii) For any $a_{i_1}, a_{i_2}, \cdots, a_{i_m} \in A$, there exists a filter F of L such that

$$F \cong [a_{i_1}) \times [a_{i_2}) \times \cdots \times [a_{i_m})$$

(iii) The number of non-isomorphic filters of L is at most equal to 2^n.

Proof. (i) By Theorem 4.2, $L \cong [a_1) \times [a_2) \times \cdots \times [a_n)$. Now, let F be a filter of L. Then there exists a filter F' of $[a_1) \times [a_2) \times \cdots \times [a_n)$ such that $F \cong F'$. By Lemma 2.4, for any $1 \leq i \leq n$, there exists a filter F'_i of $[a_i)$ such that $F' = F'_1 \times F'_2 \times \cdots \times F'_n$. Now, since for any $1 \leq i \leq n$, $[a_i)$ is a minimal filter and so is a chain, we have $F' = \{1\}$ or $[a_i)$. Now, since $\{1\} \times F'_i \cong F'_i$, so there exist $a_{i_1}, a_{i_2}, \cdots, a_{i_m} \in A$ such that

$$F \cong F' \cong F'_{i_1} \times F'_{i_2} \times \cdots \times F'_{i_m} = [a_{i_1}) \times [a_{i_2}) \times \cdots \times [a_{i_m})$$

(ii) It is easy to see that $F' = [a_{i_1}) \times [a_{i_2}) \times \cdots \times [a_{i_m})$ is a filter of $[a_1) \times [a_2) \times \cdots \times [a_n)$ Now, by Theorem 4.2, we have $L \cong [a_1) \times [a_2) \times \cdots \times [a_n)$, so there exists a filter F of L such that $F \cong F' = [a_{i_1}) \times [a_{i_2}) \times \cdots \times [a_{i_m})$.
(iii) By Theorem 4.2, (i) and (ii) the proof is clear. □

Corollary 4.2. *If F is a filter of L, then $|F| \,\big|\, |L|$.*

Proof. If $A = \{a_1, a_2, \cdots, a_n\}$ be the set of all distinct co-atoms of L, then by Theorem 4.2, $L \cong [a_1) \times [a_2) \times \cdots \times [a_n)$ and so $|L| = |[a_1)|.|[a_2)|.\cdots.|[a_n)|$. Moreover, by Corollary 4.1, there exist $a_{i_1}, a_{i_2}, \cdots, a_{i_m} \in A$ such that $1 \leq m \leq n$ and

$$F \cong [a_{i_1}) \times [a_{i_2}) \times \cdots \times [a_{i_m})$$

and so $|F| = |[a_{i_1})|.|[a_{i_2})|.\cdots.|[a_{i_m})|$. Now, since for any $1 \leq j \leq m$, there exists $1 \leq k \leq m$ such that $[a_{i_j}) = [a_k)$ and so $|[a_{i_j})| = |[a_k)|$. Hence

$$|F| = |[a_{i_1})|.|[a_{i_2})|.\cdots.|[a_{i_m})| \big| |[a_1)|.|[a_2)|.\cdots.|[a_n)| = |L|$$

□

Corollary 4.3. *There is only one lattice implication algebra of order any prime number p, up to isomorphism.*

Proof. Let $|L| = p$, while p is a prime number. We know that if $|L| \geq 2$, then there exists at least one co-atom in L. We claim that, in this case there is only one co-atom in L. Let $A = \{a_1, a_2, \cdots, a_n\}$ $(n \geq 2)$ be the set of all distinct co-atoms of L, by contrary. Then by Theorem 4.2, $L \cong [a_1) \times [a_2) \times \cdots \times [a_n)$ and $1 < |[a_i)| < |L|$. But, by Corollary 4.2, $|[a_i)| \big| |L| = p$ and so $|[a_i)| = 1$ or p, which is impassible. Then L has only one co-atom a and so by Theorem 3.1(ii), $L = [a)$. Thus by Theorem 3.1(i), L is a chain. Since we have only one chain of order p up to isomorphism, there exists only one lattice implication algebra of order p, up to isomorphism. □

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