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# WEIGHTED LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF MARCINKIEWICZ OPERATOR

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Abstract. In this paper, the weighted boundedness for the multilinear commutator of the Marcinkiewicz operator and the weighted Lipschitz functions are obtained.

## 1. Introduction

As the development of the singular integral operators, their commutators and multilinear operators have been well studied. In [5][11-12], the authors proved that the commutators and multilinear operators generated by the singular integral operators and $BMO$ functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. In [4][9-10], the boundedness for the commutators and multilinear operators generated by the singular integral operators and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ and Triebel-Lizorkin spaces are obtained. And the weighted boundedness for the commutators generated by the singular integral operators and $BMO$ or Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained(see [2][7]). Motivated by these, we will discuss the weighted boundedness for multilinear operators associated to the multilinear commutator of the Marcinkiewicz operator and the weighted Lipschitz functions.

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## 2. Preliminaries and Lemmas

Let us introduce some notations(see [5][8][11][14][15]). In this paper, $Q$ will denote a cube in $R^n$ with sides parallel to the axes. For a locally integrable function $f$ on $R^n$ and a cube Q, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and the sharp function of $f$ is defined by

$$f^{\#}(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy.$$

It is well-known that (see [8])

$$f^{\#}(x) \approx \sup_{Q\ni x} \inf_{c\in C} \frac{1}{|Q|}\int_Q |f(y) - c|dy.$$

For $1 \le p < \infty$ and $0 \le \eta < n$, let

$$M_{\eta,p}(f)(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|^{1-p\eta/n}}\int_Q |f(y)|^p dy\right)^{1/p},$$

The $A_p$ weight is defined by (see [8])

$$A_p = \left\{w : \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty\right\}, 1 < p < \infty,$$

$$A_1 = \{w > 0 : M(w)(x) \le Cw(x), a.e.\},$$

and $A_\infty = \cup_{p\ge 1} A_p$. We know, for $w \in A_1$, $w$ satisfies the doubly condition, that is, for any cube $Q$,

$$w(2Q) \le Cw(Q).$$

The $A(p,q)$ weight is defined by (see [10])

$$A(p,q) = \left\{w > 0 : \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)^q dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w(x)^{-p/(p-1)}dx\right)^{(p-1)/p} < \infty\right\},$$

$$1 < p, q < \infty.$$

Given a weight function $w$. For $1 < p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions $f$ such that

$$||f||_{L^p(w)} = \left( \int_{R^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

For a weight function $w$, $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(w)$ be the weighted homogeneous Triebel-Lizorkin space. For $0 < \beta < 1$, the weighted Lipschitz space $Lip_\beta(w)$ is the space of functions $f$ such that

$$||f||_{Lip_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |f(y) - f_Q| dy < \infty.$$

Given some function $b_j \in Lip_\beta(w), 1 \leq j \leq m$, we denote by $C_j^m$ the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of $j$ different elements and $\sigma(i) < \sigma(j)$ whenever $i < j$. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = \prod_{i=1}^{j} b_{\sigma(i)}$ and $||\vec{b_\sigma}||_{Lip_\beta(\omega)} = ||b_{\sigma(1)}||_{Lip_\beta(w)} \cdots ||b_{\sigma(i)}||_{Lip_\beta(w)}$.

In this paper, we will study a multilinear commutator as follows.

**Definition 2.1.** Let $0 < \gamma \leq 1$, Suppose that $S^{n-1}$ is the unit sphere in $R^n (n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x)$ be homogeneous of degree zero and satisfy the following two conditions:

(1) $\Omega(x)$ is contionuous on $S^{n-1}$ and satisfies the $Lip_\gamma$ condition on $S^{n-1}$, i.e.

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(2) $\int_{S^{n-1}} \Omega(x') dx' = 0$; The *Marcinkiewicz* multilinear commutator is defined by

$$\mu_S^{\vec{b}}(f)(x) = \left[ \int \int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x,y) = \int_{|y-z| \leq t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \left[ \prod_{j=1}^{m} (b_j(x) - b_j(z)) \right] f(z) dz.$$

Set

$$F_t(f)(y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz.$$

We also define that

$$\mu_S(f)(x) = \left( \int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

which is the Marcinkiewicz operator (see [16]).

In order to prove our conclusion, we need the following lemmas.

**Lemma 2.1.** *(see [7][9]) For* $0<\beta<1$, $w\in A_1$, $b\in Lip_\beta(w)$ *and* $1\le p\le\infty$, *we have*

$$||b||_{Lip_\beta(w)} \approx \sup_Q w(Q)^{-\frac{\beta}{n}} \left( w(Q)^{-1} \int_Q |b(x)-b_Q|^p w(y)^{1-p} dx \right)^{1/p}.$$

**Lemma 2.2.** *(see [8][9]) For* $0<\beta<1$, $w\in A_1$, $b\in Lip_\beta(w)$ *and any cube* $Q$, *we have*

$$\sup_{x\in Q} |b(x)-b_Q| \le C||b||_{Lip_\beta(w)} w(Q)^{1+\frac{\beta}{n}} |Q|^{-1}.$$

**Lemma 2.3.** *For* $0<\beta<1$, $w\in A_1$, $b\in Lip_\beta(w)$ *and any cube* $Q$, *we have, for* $\tilde{x}\in Q$,

$$|b_{2^kQ}-b_Q| \le Ckw(\tilde{x})w(2^kQ)^{\frac{\beta}{n}}||b||_{Lip_\beta(w)}.$$

**Lemma 2.4.** *(see [11]) For* $0<\beta<1$, $w\in A_1$, $1<p<\infty$ *and* $m>0$, *we have*

$$\begin{aligned} ||f||_{\dot{F}_p^{m\beta,\infty}(w)} &\approx \left|\left| \sup_{Q\ni\tilde{x}} |Q|^{-1-\frac{m\beta}{n}} \int_Q |f(x)-f_Q|dx \right|\right|_{L^p(w)} \\ &\approx \left|\left| \sup_{Q\ni\tilde{x}} \inf_{c\in C_Q} |Q|^{-1-\frac{m\beta}{n}} \int_Q |f(x)-c|dx \right|\right|_{L^p(w)}. \end{aligned}$$

**Lemma 2.5.** *(see [6]) Suppose that* $1\le s<p<n/\eta$, $1/q=1/p-\eta/n$ *and* $w\in A(p,q)$. *Then*

$$||M_{\eta,s}(f)||_{L^q(w^q)} \le C||f||_{L^p(w^p)}.$$

## 3. Proofs of Theorems

Now we shall state our theorems as follows.

**Theorem 3.1.** *Let* $0 < \beta < 1$, $w \in A_1$ *and* $b_j \in Lip_\beta(w)$ *for* $1 \le j \le m$. *Suppose* $1 < p < n/m\beta$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{m\beta}{n}$. *Then* $\mu_S^{\vec{b}}$ *is bounded from* $L^p(w)$ *to* $L^q(w^{1-m+\frac{(q-1)m\beta}{n}})$.

**Theorem 3.2.** *Let* $0 < \beta < 1$, $w \in A_1$ *and* $b_j \in Lip_\beta(w)$ *for* $1 \le j \le m$. *Suppose* $1 < p < \infty$. *Then* $\mu_S^{\vec{b}}$ *is bounded from* $L^p(w)$ *to* $\dot{F}_p^{m\beta,\infty}(w^{1-m-\frac{m\beta}{n}})$.

*Proof.* of Theorem 3.1: We first prove that for any $1 < r < \infty$ and fix a cube $Q = Q(x_0, d)$, there exists some constant $C_0 > 0$ such that for $f \in L^p(w)$,

$\frac{1}{|Q|}\int_Q |\mu_S^{\vec{b}}(f)(x) - C_0|dx \le C||\vec{b}||_{Lip_\beta(w)} w(\tilde{x})^{m+\frac{m\beta}{n}}(M_{m\beta,r}(f)(\tilde{x}) + M_{m\beta,r}(\mu_S(f))(\tilde{x}))$.

We write $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$. We will consider the cases $m = 1$ and $m > 1$, and choose $C_0 = \mu_S(((b_1)_{2Q} - b_1)f_2)(x_0)$.

We first consider the **Case** $m = 1$**.** For $C_0 = T(((b_1)_{2Q} - b_1)f_2)(x_0)$, we write $F_t^{b_1}(f)(x,y) = (b_1(x) - (b_1)_{2Q})F_t(f)(y) - F_t((b_1 - (b_1)_{2Q})f_1)(y) - F_t((b_1 - (b_1)_{2Q})f_2)(y)$.

Then

$$\begin{aligned}
& |\mu_S^{b_1}(f)(x) - C_0| \\
= \ & \big| ||\chi_{\Gamma(x)}F_t^{b_1}(f)(x,y)|| - ||\chi_{\Gamma(x_0)}F_t(((b_1)_{2Q} - b_1)f_2)(y)|| \big| \\
\le \ & ||\chi_{\Gamma(x)}F_t^{b_1}(f)(x,y) - \chi_{\Gamma(x_0)}F_t(((b_1)_{2Q} - b_1)f_2)(y)|| \\
\le \ & ||\chi_{\Gamma(x)}(b_1(x) - (b_1)_{2Q})F_t(f)(y)|| + ||\chi_{\Gamma(x)}F_t((b_1 - (b_1)_{2Q})f_1)(y)|| \\
& + ||\chi_{\Gamma(x)}F_t((b_1 - (b_1)_{2Q})f_2)(y) - \chi_{\Gamma(x_0)}F_t((b_1 - (b_1)_{2Q})f_2)(y)|| \\
= \ & A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, by Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |A(x)|dx \leq \frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}||\mu_S(f)(x)|dx \\
\leq\;& \frac{1}{|Q|}\left(\int_Q |b_1(x)-(b_1)_{2Q}|^{r'}dx\right)^{\frac{1}{r'}}\left(\int_Q |\mu_S(f)(x)|^r dx\right)^{\frac{1}{r}} \\
\leq\;& \frac{1}{|2Q|}\sup_{x\in 2Q}|b_1(x)-(b_1)_{2Q}||Q|^{\frac{1}{r'}}\left(\int_Q |\mu_S(f)(x)|^r dx\right)^{\frac{1}{r}} \\
\leq\;& \frac{C}{|Q|}||b_1||_{Lip_\beta(w)}w(Q)^{1+\frac{\beta}{n}}|Q|^{-1}|Q|^{\frac{1}{r'}}|Q|^{\frac{1}{r}-\frac{\beta}{n}}\left(\frac{1}{|Q|^{1-\frac{r\beta}{n}}}\int_Q |\mu_S(f)(x)|^r dx\right)^{\frac{1}{r}} \\
\leq\;& C||b_1||_{Lip_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{1+\frac{\beta}{n}}M_{\beta,r}(\mu_S(f))(\tilde{x}) \\
\leq\;& C||b_1||_{Lip_\beta(w)}w(\tilde{x})^{1+\frac{\beta}{n}}M_{\beta,r}(\mu_S(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, using the boundness of $\mu_S$ and Lemma 2.2, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |B(x)|dx \leq \frac{1}{|Q|}\left(\int_{R^n} |\mu_S((b_1-(b_1)_{2Q})f_1)(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}} \\
\leq\;& \frac{C}{|Q|}\left(\int_{R^n} |(b_1)(x)-(b_1)_{2Q}|^r|f_1(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}} \\
\leq\;& \frac{C}{|Q|}\sup_{x\in 2Q}|(b_1)(x)-(b_1)_{2Q}|\left(\int_{2Q} |f(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}} \\
\leq\;& \frac{C}{|Q|}||b_1||_{Lip_\beta(w)}w(2Q)^{1+\frac{\beta}{n}}|2Q|^{-1}|2Q|^{\frac{1}{r}-\frac{\beta}{n}}\left(\frac{1}{|2Q|^{1-\frac{r\beta}{n}}}\int_{2Q} |f(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}} \\
\leq\;& C||b_1||_{Lip_\beta(w)}\left(\frac{w(2Q)}{|2Q|}\right)^{1+\frac{\beta}{n}}M_{\beta,r}(f)(x) \\
\leq\;& C||b_1||_{Lip_\beta(w)}w(\tilde{x})^{1+\frac{\beta}{n}}M_{\beta,r}(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, we know that

$$\mu_S^{\vec{b}}(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x,y)|^2\frac{dydt}{t^{n+3}}\right]^{1/2},$$

By the Minkowski's inequality,we obtain

$$
\begin{aligned}
C(x) &\leq \left(\int\int_{R_+^{n+1}} ||(\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)})F_t((b_1 - (b_1)_{2Q})f_2(y)||)^2 \frac{dydt}{t^{n+3}}\right)^{1/2} \\
&\leq C\int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}||f(z)| \times \\
&\quad \left|\int\int_{|x-y|\leq t} \frac{\chi_{\Gamma(z)}(y,t)dydt}{|y-z|^{2n-2}t^{n+3}} - \int\int_{|x_0-y|\leq t} \frac{\chi_{\Gamma(z)}(y,t)dydt}{|y-z|^{2n-2}t^{n+3}}\right|^{1/2} dz \\
&\leq \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}||f(z)| \times \\
&\quad \left(\int\int_{|y|\leq t, |x+y-z|\leq t} \left|\frac{1}{|x+y-z|^{2n-2}} - \frac{1}{|x_0+y-z|^{2n-2}}\right| \frac{dydt}{t^{n+3}}\right)^{1/2} dz \\
&\leq \\
&\int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}||f(z)| \left(\int\int_{|y|\leq t, |x+y-z|\leq t} \frac{|x-x_0|}{|x+y-z|^{2n-1}} t^{-n-3}dydt\right)^{1/2} dz,
\end{aligned}
$$

note that $|x-z| \leq 2t$,$|x+y-z| \geq |x-z| - t \geq |x-z| - 3t$, when $|y| \leq t$, $|x+y-z| \leq t$, then, for $x \in Q$,

$$
\begin{aligned}
C(x) &\leq C\int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}||f(z)||x-x_0|^{1/2} \\
&\quad \times \left(\int\int_{|y|\leq t, |x+y-z|\leq t} \frac{t^{-n}dydt}{|x+y-z|^{2n+2}}\right)^{1/2} dz \\
&\leq C\int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}||f(z)||x-x_0|^{1/2} \\
&\quad \times \left(\int\int_{|y|\leq t, |x+y-z|\leq t} \frac{t^{-n}dydt}{(|x-z| - 3t)^{2n+2}}\right)^{1/2} dz \\
&\leq C\int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}||f(z)||x-x_0|^{1/2} \\
&\quad \times \left(\int_{|x-z|/2}^{\infty} \frac{dt}{(|x-z| - 3t)^{2n+2}}\right)^{1/2} dz
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2}}dz\\
&\le C\sum_{l=1}^{\infty}\int_{2^{l+1}Q\backslash 2^lQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2)}|b_1(z)-(b_1)_{2Q}||f(z)|dz\\
&\le C\sum_{l=1}^{\infty}2^{-l/2}|2^{l+1}Q|^{-1}\int_{2^{l+1}Q}|b_1(z)-(b_1)_{2Q}||f(z)|dz\\
&\le C\sum_{l=1}^{\infty}2^{-l/2}\frac{1}{|2^{l+1}Q|}\int_{2^{l+1}Q}|(b_1)_{2^{l+1}Q}-b_1(z)||f(z)|dz\\
&\quad +C\sum_{l=1}^{\infty}2^{-l/2}\frac{1}{|2^{l+1}Q|}\int_{2^{l+1}Q}|(b_1)_{2^{l+1}Q}-(b_1)_{2Q}||f(z)|dz\\
&\le C\sum_{l=1}^{\infty}2^{-l/2}\frac{1}{|2^{l+1}Q|}\left(\int_{2^{l+1}Q}|(b_1)_{2^{l+1}Q}-b_1(z)|^{r'}dz\right)^{\frac{1}{r'}}\left(\int_{2^{l+1}Q}|f(z)|^rdz\right)^{\frac{1}{r}}\\
&\quad +C\sum_{l=1}^{\infty}2^{-l/2}\frac{1}{|2^{l+1}Q|}|(b_1)_{2^{l+1}Q}-(b_1)_{2Q}|\left(\int_{2^{l+1}Q}|f(z)|^rdz\right)^{\frac{1}{r}}|2^{l+1}Q|^{\frac{1}{r'}}\\
&\le C\sum_{l=1}^{\infty}2^{-l/2}\sup_{x\in 2^{l+1}Q}|b_1(z)-(b_1)_{2^{l+1}Q}||2^{l+1}Q|^{\frac{-\beta}{n}}\left(\frac{1}{|2^{l+1}Q|^{1-\frac{r\beta}{n}}}\int_{2^{l+1}Q}|f(z)|^rdz\right)^{\frac{1}{r}}\\
&\quad +C\sum_{l=1}^{\infty}2^{-l/2}|(b_1)_{2^{l+1}Q}-(b_1)_{2Q}||2^{l+1}Q|^{\frac{-\beta}{n}}\left(\frac{1}{|2^{l+1}|^{1-\frac{r\beta}{n}}}\int_{2^{l+1}Q}|f(z)|^rdz\right)^{\frac{1}{r}}\\
&\le C\sum_{l=1}^{\infty}2^{-l/2}||b_1||_{Lip_\beta(w)}w(2^{l+1}Q)^{1+\beta/n}|2^{l+1}Q|^{-1}|2^{l+1}Q|^{-\beta/n}M_{\beta,r}(f)(\tilde{x})\\
&\quad +C\sum_{l=1}^{\infty}2^{-l/2}lw(\tilde{x})w(2^{l+1}Q)^{\frac{-\beta}{n}}||b_1||_{Lip_\beta(w)}|2^{l+1}Q|^{-\beta/n}M_{\beta,r}(f)(\tilde{x})\\
&\le C||b_1||_{Lip_\beta(w)}\sum_{l=1}^{\infty}2^{-l/2}\left(\frac{w(2^{l+1}Q)}{|2^{l+1}Q|}\right)^{1+\beta/n}M_{\beta,r}(f)(\tilde{x})\\
&\quad +C||b_1||_{Lip_\beta(w)}\sum_{l=1}^{\infty}l2^{-l/2}w(\tilde{x})\left(\frac{w(2^{l+1}Q)}{|2^{l+1}Q|}\right)^{\beta/n}M_{\beta,r}(f)(\tilde{x})\\
&\le C||b_1||_{Lip_\beta(w)}w(\tilde{x})^{1+\beta/n}M_{\beta,r}(f)(\tilde{x}),
\end{aligned}
$$

so, we obtain

$\frac{1}{|Q|}\int_Q |C(x)|dx \leq C||b_1||_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{r,\beta}(f)(\tilde{x})$.

Now, we consider the **Case** $m \geq 2$. For $b = (b_1, \cdots, b_m)$,we have,

$$F_t^{\tilde{b}}(f)(x,y) = \int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \left[\prod_{j=1}^m (b_j(x) - b_j(z))\right] f(z)dz$$

$$= \int_{|y-z|\leq t} [((b_1(x) - (b_1)_{2Q}) - (b_1(z) - (b_1)_{2Q}))\cdots$$

$$((b_m(x) - (b_m)_{2Q}) - (b_m(z) - (b_m)_{2Q}))] \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz$$

$$= \sum_{j=0}^m \sum_{\sigma\in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{|y-z|\leq t} (b(z) - (b)_{2Q})_{\sigma^c} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz$$

$$= (b_1(x) - (b_1)_{2Q})\cdots(b_m(x) - (b_m)_{2Q}) F_t(f)(y) + (-1)^m F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f)(y)$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{|y-z|\leq t} (b(z) - b(x))_{\sigma^c} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz$$

$$= (b_1(x) - (b_1)_{2Q})\cdots(b_m(x) - (b_m)_{2Q}) F_t(f)(y) + (-1)^m F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f)(y)$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} c_{m,j} (b(x) - (b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x,y),$$

so,

$$|\mu_S^{\tilde{b}}(f)(x) - \mu_S(((b_1)_{2Q} - b_1)\cdots((b_m)_{2Q} - b_m))f_2)(x_0)|$$

$$\leq ||\chi_{\Gamma(x)} F_t^{\tilde{b}}(f)(x,y) - \chi_{\Gamma(x_0)} F_t(((b_1)_{2Q} - b_1)\cdots((b_m)_{2Q} - b_m)f_2)(y)||$$

$$\leq ||\chi_{\Gamma(x)}(b_1(x) - (b_1)_{2Q})\cdots(b_m(x) - (b_m)_{2Q}) F_t(f)(y)||$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} ||\chi_{\Gamma(x)}(\tilde{b}(x) - (b_m)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x,y)||$$

$$+ ||\chi_{\Gamma(x)} F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_1)(y)||$$

$$+ ||\chi_{\Gamma(x)} F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_2)(y)$$

$$
\begin{aligned}
& -\chi_{\Gamma(x_0)} F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(y)|| \\
= \ & I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}
$$

For $I_1(x)$, by Hölder's inequality with exponent $\frac{1}{r_1}+\cdots+\frac{1}{r_m}+\frac{1}{r}=1$ and Lemma 2.2, we get,

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q |I_1(x)|dx \ \leq \ & \frac{1}{|Q|}\int_Q |\prod_{j=1}^m (b_j(x)-(b_j)_{2Q})||\mu_S(f)(x)|dx \\
\leq \ & \frac{C}{|2Q|}\prod_{j=1}^m \left(\int_{2Q} b_j(x)-(b_j)_{2Q}|^{r_j}dx\right)^{\frac{1}{r_j}} \left(\int_Q |\mu_S(f)(x)|^r dx\right)^{\frac{1}{r}} \\
\leq \ & \frac{1}{|2Q|}\prod_{j=1}^m \left(\sup_{x\in 2Q}|b_j(x)-(b_j)_{2Q}||2Q|^{\frac{1}{r_j}}\right)\left(\int_Q |\mu_S(f)(x)|^r dx\right)^{\frac{1}{r}} \\
\leq \ & \\
\frac{C}{|Q|}\prod_{j=1}^m & (||b_j||_{Lip_\beta(w)} w(2Q)^{1+\beta/n}|2Q|^{-1})|Q|^{(1-\frac{1}{r})+(\frac{1}{r}-m\beta/n)}\left(\frac{1}{|Q|^{r-rm\beta/n}}\int_Q |\mu_S(f)(x)|^r dx\right)^{\frac{1}{r}} \\
\leq \ & C||\vec{b}||_{Lip_\beta(w)} w(2Q)^{m+m\beta/n}|2Q|^{-m-m\beta/n} M_{m\beta,r}(\mu_S(f))(\tilde{x}) \\
\leq \ & C||\vec{b}||_{Lip_\beta(w)}\left(\frac{w(2Q)}{|2Q|}\right)^{m+m\beta/n} M_{m\beta,r}(\mu_S(f))(\tilde{x}) \\
\leq \ & C||\vec{b}||_{Lip_\beta(w)} w(\tilde{x})^{m+m\beta/n} M_{m\beta,r}(\mu_S(f))(\tilde{x}).
\end{aligned}
$$

For $I_2(x)$, similar to $B(x)$, using the boundness of $\mu_S$ and Lemma 2.2, we get

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q I_2(x)dx & \leq \frac{1}{|Q|}\left(\int_{R^n}|\mu_S(\prod_{j=1}^m (b_j-(b_j)_{2Q})f_1)(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}} \\
& \leq \frac{C}{|Q|}\left(\int_{R^n}|\prod_{j=1}^m (b_j(x)-(b_j)_{2Q})f_1(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}} \\
& \leq \frac{C}{|Q|}\left(\int_{2Q}|\prod_{j=1}^m (b_j(x)-(b_j)_{2Q})|^r|f(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}
\end{aligned}
$$

$$\leq \frac{C}{|Q|}\prod_{j=1}^{m}\sup_{x\in 2Q}|b_j(x)-(b_j)_{2Q}|\left(\int_{2Q}f(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}$$

$$\leq C\frac{1}{|Q|}\prod_{j=1}^{m}||b_j||_{Lip_\beta(w)}w(2Q)^{1+\beta/n}|2Q|^{-1})|2Q|^{\frac{1}{r'}+\frac{1}{r}-m\beta/n}\left(\frac{1}{|2Q|^{1-rm\beta/n}}\int_{2Q}|f(x)|^r dx\right)^{\frac{1}{r}}$$

$$\leq C||\vec{b}||_{Lip_\beta(w)}\left(\frac{w(2Q)}{|2Q|}\right)^{m+m\beta/n}M_{m\beta,r}(f)(\tilde{x})$$

$$\leq ||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+m\beta/n}M_{m\beta,r}(f)(\tilde{x}).$$

For $I_3(x)$, by Hölder's inequality and Lemma 2.2, we get

$$\frac{1}{|Q|}\int_Q|I_3(x)|dx\leq\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|}\int_{2Q}|b_j(x)-(b_j)_{2Q})_\sigma||\mu_S((b_j-(b_j)_{2Q})_{\sigma^c}f)(x)|dx$$

$$\leq \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|}\left(\int_{2Q}|b_j(x)-(b_j)_{2Q})_\sigma|^{r'}dx\right)^{\frac{1}{r'}}\left(\int_{2Q}|\mu_S((b_j-(b_j)_{2Q})_{\sigma^c}f)(x)|^r dx\right)^{\frac{1}{r}}$$

$$\leq$$

$$C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|}\sup_{x\in 2Q}|(b_j(x)-(b_j)_{2Q})_\sigma||2Q|^{\frac{1}{r'}}\sup_{x\in 2Q}|(b_j(x)-(b_j)_{2Q})_{\sigma^c}|\left(\int_{2Q}|\mu_S(f)(x)|^r dx\right)^{\frac{1}{r}}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|}||\vec{b}_\sigma||_{Lip_\beta(w)}w(2Q)^{j+j\beta/n}|2Q|^{-j}||\vec{b}_{\sigma^c}||_{Lip_\beta(w)}$$

$$\times w(2Q)^{(m-j)+(m-j)\beta/n}|2Q|^{\frac{1}{r'}+\frac{1}{r}-m\beta/n-(m-j)}\left(\frac{1}{|2Q|^{1-rm\beta/n}}\int_{2Q}|\mu_S(f)(x)|^r dx\right)^{\frac{1}{r}}$$

$$\leq C||\vec{b}||_{Lip_\beta(w)}\left(\frac{w(2Q)}{|2Q|}\right)^{m+m\beta/n}M_{m\beta,r}(\mu_S(f))(\tilde{x})$$

$$\leq C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+m\beta/n}M_{m\beta,r}(\mu_S(f))(\tilde{x}).$$

For $I_4(x)$, similar to the proof of $C(x)$ in the **Case** $m=1$, we obtain

$$I_4(x)\leq C\int_{(2Q)^c}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2)}|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})||f(z)|dz$$

$$
\begin{aligned}
&\le C\sum_{l=1}^{\infty}\int_{2^{l+1}Q\backslash 2^lQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2)}|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})||f(z)|dz\\
&\le C\sum_{l=1}^{\infty}2^{-l/2}|2^{l+1}Q|^{-1}\int_{2^{l+1}Q}|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})||f(z)|dz\\
&\le C\sum_{l=1}^{\infty}2^{-l/2}|2^{l+1}Q|^{-1}\\
&\quad\prod_{j=1}^{m}\left(\int_{2^{l+1}Q}|b_j(z)-(b_j)_{2^{l+1}Q})||f(z)|dz+|(b_j)_{2^{l+1}Q}-(b_j)_{2Q}|\right)\int_{2^{l+1}Q}|f(z)|dz\\
&\le C\sum_{l=1}^{\infty}2^{-l/2}|2^{l+1}Q|^{-1}\prod_{j=1}^{m}\left(\int_{2^{l+1}Q}|b_j(z)-(b_j)_{2^{l+1}Q})_{\sigma^c}|^{r'}dz\right)^{\frac{1}{r'}}\left(\int_{2^{l+1}Q}|f(z)|^rdz\right)^{\frac{1}{r}}\\
&\quad+|((b_j)_{2^{l+1}Q}-(b_j)_{2Q})||2^{l+1}Q|^{\frac{1}{r'}+(\frac{1}{r}-m\beta/n)}\times\left(\frac{1}{|2^{l+1}Q|^{1-rm\beta/n}}\int_{2^{l+1}Q}|f(z)|^rdz\right)^{\frac{1}{r}}\\
&\le C||\vec{b}||_{Lip_\beta(w)}w(x)^j\left(\frac{w(2^{l+1}Q)}{|2^{l+1}Q|}\right)^{(m-j)+m\beta/n}M_{m\beta,r}(f)(\tilde{x})\\
&\le C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+m\beta/n}M_{m\beta,r}(f)(\tilde{x}).
\end{aligned}
$$

Thus, we get

$$
\frac{1}{|Q|}\int_Q I_4(x)dx\le C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+m\beta/n}M_{m\beta,r}(f)(\tilde{x}).
$$

Combining all the estimates, we finish the case $m\ge 2$ .

So, for $m\ge 1$ and any cube $Q$, there exists $C_0$ such that for $f\in L^p(w)$,

$$
\frac{1}{|Q|}\int_Q|\mu_S^{\vec{b}}(f)(x)-C_0|dx\le C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}(M_{m\beta,r}(f)(\tilde{x})+M_{m\beta,r}(\mu_S(f))(\tilde{x}))
$$

and

$$
\mu_S^{\vec{b}}(f)^{\#}(\tilde{x})\le C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}(M_{m\beta,r}(f)(\tilde{x})+M_{m\beta,r}(\mu_S(f))(\tilde{x})).
$$

Now, choose $1 < r < p$ and by Lemma 2.5, we have

$$\begin{aligned}
&||\mu_S^{\vec{b}}(f)||_{L^q(w^{1-m+(q-1)m\beta/n})}\\
\leq\ & C||M(\mu_S^{\vec{b}}(f))||_{L^q(w^{1-m+(q-1)m\beta/n})}\\
\leq\ & C||(\mu_S^{\vec{b}}(f))^{\#}||_{L^q(w^{1-m+(q-1)m\beta/n})}\\
\leq\ & C||\vec{b}||_{Lip_\beta(w)}(||w^{m+m\beta/n}M_{m\beta,r}(f)||_{L^q(w^{1-m+(q-1)m\beta/n})}\\
& +||w^{m+m\beta/n}M_{m\beta,r}(\mu_S(f))||_{L^q(w^{1-m+(q-1)m\beta/n})})\\
\leq\ & C||\vec{b}||_{Lip_\beta(w)}(||M_{m\beta,r}(f)||_{L^q(w^{\frac{q}{p}})}+||M_{m\beta,r}(\mu_S(f))||_{L^q(w^{\frac{q}{p}})})\\
\leq\ & C||\vec{b}||_{Lip_\beta(w)}(||f||_{L^p(w)}+||\mu_S(f)||_{L^p(w)})\\
\leq\ & C||\vec{b}||_{Lip_\beta(w)}||f||_{L^p(w)}.
\end{aligned}$$

This completes the proof of Theorem 3.1. □

*Proof.* of Theorem 3.2: Similar to Theorem 3.1, for any $1 < r < \infty$ and cube $Q = Q(x_0, d)$, there exists some constant $C_0$ such that for $f \in L^p(w)$ and $\tilde{x} \in Q$,

$$|Q|^{-1-\frac{m\beta}{n}}\int_Q |\mu_S^{\vec{b}}(f)(x)-C_0|dx \leq C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}(M_r(f)(\tilde{x})+M_r(\mu_S(f))(\tilde{x})).$$

Further, we have

$$\sup_{Q\ni\tilde{x}}\inf_{c\in C}|Q|^{-1-\frac{m\beta}{n}}\int_Q |\mu_S^{\vec{b}}(f)(x)-c|dx \leq C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}(M_r(f)(\tilde{x})+M_r(\mu_S(f))(\tilde{x})).$$

Choose $1 < r < p$ and by lemma 2.4, we obtain

$$\begin{aligned}
&||\mu_S^{\vec{b}}(f)||_{\dot{F}_p^{m\beta,\infty}(w^{1-m-m\beta/n})} \approx \left|\left|\sup_{\tilde{x}\in Q}\inf_{c\in C}|Q|^{-1-m\beta/n}\int_Q|\mu_S^{\vec{b}}(f)(x)-c|dx\right|\right|_{L^p(w^{1-m-m\beta/n})}\\
\leq\ & C||\vec{b}||_{Lip_\beta(w)}(||w^{m+m\beta/n}M_r(f)||_{L^p(w^{1-m-m\beta/n})}+||w^{m+m\beta/n}M_r(\mu_S(f))||_{L^p(w^{1-m-m\beta/n})})\\
\leq\ & C||\vec{b}||_{Lip_\beta(w)}(||M_r(f)||_{L^p(w)}+||M_r(\mu_S(f))||_{L^p(w)})\\
\leq\ & C||\vec{b}||_{Lip_\beta(w)}(||f||_{L^p(w)}+||\mu_S(f)||_{L^p(w)})\\
\leq\ & C||\vec{b}||_{Lip_\beta(w)}||f||_{L^p(w)}.
\end{aligned}$$

This completes the proof of Theorem 3.2. □

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