Jordan Journal of Mathematics and Statistics (JJMS) 6(4), 2013, pp.313- 339

CHEN INEQUALITIES FOR SUBMANIFOLDS OF SOME SPACE FORMS ENDOWED WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we prove Chen inequalities for submanifolds of complex space forms and respectively Sasakian space form, endowed with a semi-symmetric non-metric connection.

1. Introduction

In [9], H.A. Hayden introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. K. Yano studied in [17] some properties of a Riemannian manifold endowed with a semi-symmetric metric connection.

In [18], Agashe and Chafle introduced the idea of semi-symmetric non-metric connection on a Riemannian manifold. This was further developed by Agashe and Chafle [19], De and Kamilya [21], De, Sengupta and Binh [11], De and Sengupta [12].

On the other hand, one of the basic problems in submanifold theory is to find simple relationships between the extrinsic and intrinsic invariants of a submanifold. B. Y. Chen [5], [6], [10] established some fundamental inequalities in this respect, well-known as *Chen inequalities*.

Afterwards, many geometers studied similar problems for different submanifolds in various ambient spaces, for example see [13]-[15], [16], [1] and [4].

1991 *Mathematics Subject Classification.* 53C40, 53B05, 53B15.

Key words and phrases. Semi-symmetric non-metric connection, Chen inequality, complex space form , Sasakian space form, Ricci curvature.

Received: Oct. 6, 2012 Accepted : July 10 , 2013 .

Recently, in [2] , [3] and [8] , the authors studied Chen inequalities for submanifolds of real space forms ,complex space forms and Sasakian space forms with a semi-symmetric metric connection and Chen inequalities for submanifolds of real space forms with a semi-symmetric non-metric connections, respectively.

In this paper we will study Chen inequalities for submanifolds in complex and Sasakian space forms endowed with semi-symmetric non-metric connections, respectively. The paper is organized as follows. In Section 2, we give a brief introduction about a semi-symmetric non-metric connection, Chen Lemma and Ricci curvature. In Section 3, for submanifolds of complex space forms endowed with a semi-symmetric non-metric connection we establish Chen first inequality. In Section 4, we state a relationships between Ricci curvature of a submanifold M^n of a complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature, endowed with a semi-symmetric non-metric connection, and the squared mean curvature $\|H\|^2$ .In Section 5, for submanifolds of Sasakian space forms endowed with a semi-symmetric non-metric connection we establish Chen first inequality. In Section 6, we state a relationship between the sectional curvature of a submanifold M^n of a Sasakian space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric non-metric connection $\widetilde{\widetilde{\nabla}}$ and the squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the Ricci curvature of M^n and the squared mean curvature $\|H\|^2$.

2. **Preliminaries**

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold and $\widetilde{\widetilde{\nabla}}$ a linear connection on N^{n+p}. If the torsion tensor $\widetilde{T}$ of $\widetilde{\widetilde{\nabla}}$, defined by

$$\widetilde{T}\left(\widetilde{X},\widetilde{Y}\right) = \widetilde{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} \; - \; \widetilde{\widetilde{\nabla}}_{\widetilde{Y}}\widetilde{X} \; - \; [\widetilde{X},\widetilde{Y}],$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, satisfies

$$\widetilde{T}\left(\widetilde{X},\widetilde{Y}\right) = \phi(\widetilde{Y})\widetilde{X} - \phi(\widetilde{X})\widetilde{Y}$$

for a 1-form ϕ, then the connection $\widetilde{\nabla}$ is called a *semi-symmetric connection.*

Let g be a Riemannian metric on N^{n+p}. If $\widetilde{\nabla}g = 0$, then $\widetilde{\nabla}$ is called a *semi-symmetric metric connection* on N^{n+p}. If $\widetilde{\nabla}g \neq 0$, then $\widetilde{\nabla}$ is called a *semi-symmetric non-metric connection* on N^{n+p}.

Following [12], a semi-symmetric non-metric connection $\widetilde{\nabla}$ on N^{n+p} is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \phi(\widetilde{Y})\widetilde{X} - g(\widetilde{X},\widetilde{Y})\widetilde{P} - \eta(\widetilde{X})\widetilde{Y} - \eta(\widetilde{Y})\widetilde{X},$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric g and P is a vector field associated with the 1-form ϕ defined by

$$\phi(\widetilde{X}) = g(\widetilde{X}, P) \tag{2.1}$$

and E is a vector field associated with the 1-form

$$\eta(\widetilde{X}) = g(\widetilde{X}, E). \tag{2.2}$$

We will consider a Riemannian manifold N^{n+p} endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$ and the Levi-Civita connection denoted by $\overset{\circ}{\widetilde{\nabla}}$.

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. On the submanifold M^n we consider the induced semi-symmetric non-metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\overset{\circ}{\nabla}$.

Let $\widetilde{R}$ be the curvature tensor of N^{n+p} with respect to $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{R}}$ the curvature tensor of N^{n+p} with respect to $\overset{\circ}{\widetilde{\nabla}}$. We also denote by R and $\overset{\circ}{R}$ the curvature tensors of ∇ and $\overset{\circ}{\nabla}$, respectively, on M^n.

The Gauss formulas with respect to ∇ and $\overset{\circ}{\nabla}$ can be written as:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X, Y \in \chi(M),$$

$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \overset{\circ}{h}(X,Y), \quad X, Y \in \chi(M),$$

where $\overset{\circ}{h}$ is the second fundamental form of M^n in N^{n+p} and h is a $(0,2)$-tensor on M^n. According to the formula (17) from [22] h is also symmetric. The Gauss equation for the submanifold M^n into an $(n+p)$ dimensional Riemannian manifold N^{n+p} is

$$\begin{aligned} \overset{\circ}{\widetilde{R}}(X,Y,Z,W) &= \overset{\circ}{R}(X,Y,Z,W) + g(\overset{\circ}{h}(X,Z), \overset{\circ}{h}(Y,W)) \\ &\quad - g(\overset{\circ}{h}(X,W), \overset{\circ}{h}(Y,Z)). \end{aligned} \tag{2.3}$$

One denotes by $\overset{\circ}{H}$ the mean curvature vector of M^n in N^{n+p}.

Then the curvature tensor $\widetilde{R}$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ on N^{n+p} can be written as (see [12])

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) &= \overset{\circ}{\widetilde{R}}(X,Y,Z,W) - \alpha(Y,Z)g(X,W) \\ &\quad + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) \\ &\quad + \alpha(Y,W)g(X,Z) + \beta(Y,X)\,g(Z,W) \\ &\quad - \beta(X,Y)\,g(Z,W) + \beta(Y,Z)\,g(X,W) \\ &\quad - \beta(X,Z)\,g(Y,W), \end{aligned} \tag{2.4}$$

for any vector fields $X, Y, Z, W \in \chi(M^n)$, where α and β are $(0,2)$-tensor field defined by

$$\alpha(X,Y) = \left(\overset{\circ}{\widetilde{\nabla}}_X \phi\right) Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(P)g(X,Y), \tag{2.5}$$

$$QX = \overset{\circ}{\widetilde{\nabla}}_X P - \phi(X)P + \frac{1}{2}\phi(P)X \tag{2.6}$$

and

$$\beta(X,Y) = (\overset{\circ}{\tilde{\nabla}}_X \eta)(Y) - \eta(X)\phi(Y) + \eta(X)\eta(Y) - \phi(X)\eta(Y) + \eta(P)g(X,Y). \tag{2.7}$$

Denote by λ the trace of α and γ the trace of β.

Let $\pi \subset T_xM^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric non-metric connection ∇. For any orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_xM^n, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

Recall that the *Chen first invariant* is given by

$$\delta_{M^n}(x) = \tau(x) - \inf\{K(\pi) \mid \pi \subset T_xM^n, x \in M^n, \dim\pi = 2\},$$

(see for example [10]), where M^n is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M^n associated with a 2-plane section, $\pi \subset T_xM^n, x \in M^n$ and τ is the scalar curvature at x.

The following algebraic Lemma is well-known.

Lemma 2.1. [5] *Let $a_1, a_2, ..., a_n, b$ be $(n+1)$ $(n \ge 2)$ real numbers such that*

$$\left(\sum_{i=1}^n a_i\right)^2 = (n-1)\left(\sum_{i=1}^n a_i^2 + b\right).$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3 = ... = a_n$.

Let M^n be an n-dimensional Riemannian manifold, L a k-plane section of T_xM^n, $x \in M^n$, and X a unit vector in L.

We choose an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$.

Ones define [7] the *Ricci curvature* of L at X by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j. For each integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k on M^n is defined by:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad x \in M^n,$$

where L runs over all k-plane sections in T_xM^n and X runs over all unit vectors in L.

3. Chen First Inequality for submanifolds of complex space forms

Let N^{2m} be a Keahler manifold and J the canonical almost complex structure. The sectional curvature of N^{2m} in the direction of an invariant 2-plane section by J is called the *holomorphic sectional curvature*. If the holomorphic sectional curvature is constant $4c$ for all plane sections π of T_xN^{2m} invariant by J for any $x \in N^{2m}$, then N^{2m} is called a *complex space form* and is denoted by $N^{2m}(4c)$. The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ on $N^{2m}(4c)$ is given by

$$\begin{aligned} \overset{\circ}{\widetilde{R}}(X,Y,Z,W) &= c[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) - \\ &\quad -g(JX,Z)g(JY,W) + g(JX,W)g(JY,Z) - \\ &\quad -2g(X,JY)g(Z,JW)]. \end{aligned} \tag{3.1}$$

If $N^{2m}(4c)$ is a complex space form of constant holomorphic sectional curvature $4c$ with a semi-symmetric non-metric connection $\widetilde{\nabla}$, then from (2.4) and (3.1), the curvature tensor $\widetilde{R}$ of $N^{2m}(4c)$ can be expressed as

$$\widetilde{R}(X,Y,Z,W) = c[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) - -g(JX,Z)g(JY,W) + g(JX,W)g(JY,Z) - -2g(X,JY)g(Z,JW)] - \alpha(Y,Z)g(X,W) + +\alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + +\alpha(Y,W)g(X,Z) + \beta(Y,X)\, g(Z,W) - -\beta(X,Y)\, g(Z,W) + \beta(Y,Z)\, g(X,W) - -\beta(X,Z)\, g(Y,W). \tag{3.2}$$

Let M^n,$n \geq 3$, be an n-dimensional submanifold of an $2m$-dimensional complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature $4c$. For any tangent vector field X to M^n, we put

$$JX = TX + FX,$$

where TX and FX are the tangential and normal components of JX, respectively. We define

$$\|T\|^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j).$$

Following [20], we denote by $\Theta^2(\pi) = g^2(Te_1, e_2) = g^2(Je_1, e_2)$ and where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π. $\Theta^2(\pi)$ is a real number in [0, 1], independent of the choice of e_1, e_2.

Denote by

$$\Omega(e_2) = \beta_{22} + \eta(h(e_2, e_2)), \tag{3.3}$$

for any orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_xM^n. Detailed explanations will be given in the proof of Theorem 3.1 and Theorem 5.1.

For submanifolds of complex space form endowed with a semi-symmetric non-metric connection we establish the following optimal inequality, which will call Chen first inequality:

Theorem 3.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $2m$-dimensional complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature $4c$, endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$. We have:*

$$\tau(x) - K(\pi) \leq \frac{n-2}{2}\left[\frac{n^2}{(n-1)}\|H\|^2 + (n+1)c - 2\lambda\right] - -\Omega(e_2) + \frac{n-1}{2}(\gamma + n\eta(H)) - -[6\Theta^2(\pi) - 3\|T\|^2]\frac{c}{2} - trace\left(\alpha_{|_{\pi^\perp}}\right), \tag{3.4}$$

where π is a 2-plane section of $T_xM^n, x \in M^n$.

Proof. From [22], the Gauss equation with respect to the semi-symmetric non-metric connection is

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - -g(h(Y,Z),h(X,W)) - \eta(h(Y,Z))g(X,W) +\eta(h(X,Z))g(Y,W). \tag{3.5}$$

Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and $\{e_{n+1}, ..., e_{n+p}\}$ be orthonormal basis of T_xM^n and $T_x^\perp M^n$, respectively. For $X = W = e_i, Y = Z = e_j$, $i \neq j$, from the equation (3.2) it follows that:

$$\tilde{R}(e_i, e_j, e_j, e_i) = c[1 + 3g^2(Je_i, e_j)] - \alpha(e_i, e_i) - \alpha(e_j, e_j) + \beta(e_j, e_j). \tag{3.6}$$

From (3.5) and (3.6) we get

$$\begin{aligned} c[1+3g^2(Je_i,e_j)]-\alpha(e_i,e_i)-\alpha(e_j,e_j)+\beta(e_j,e_j) &= R(e_i,e_j,e_j,e_i)+ \\ &+g(h(e_i,e_j),h(e_i,e_j))- \\ &-g(h(e_i,e_i),h(e_j,e_j))- \\ &-\eta(h(e_j,e_j)). \end{aligned}$$

By summation after $1 \leq i,j \leq n$, it follows from the previous relation that

$$c[n^2-n+3\sum_{i,j=1}^{n} g^2(Je_i,e_j)]+(n-1)[-2\lambda+\gamma]= \tag{3.7}$$

$$=2\tau+\|h\|^2-n^2\|H\|^2-(n^2-n)\eta(H),$$

where we recall that λ is the trace of α and γ is the trace of β and denote by

$$\|h\|^2=\sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)),$$

$$H=\frac{1}{n}\mathrm{trace}h.$$

One takes

$$\varepsilon=2\tau+\frac{n^2(2-n)}{n-1}\|H\|^2-n(n-1)\eta(H)-[n^2-n+3\|T\|^2]c-(n-1)[-2\lambda+\gamma]. \tag{3.8}$$

Then, from (3.7) and (3.8) we get

$$n^2\|H\|^2=(n-1)\left(\|h\|^2+\varepsilon\right). \tag{3.9}$$

Let $x \in M^n$, $\pi \subset T_xM^n$, $\dim \pi = 2$, $\pi = sp\{e_1,e_2\}$. If $H=0$ at x, one may choose e_{n+1} to be any unit normal vector at x. From the relation (3.9) we obtain:

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2=(n-1)(\sum_{i,j=1}^{n}\sum_{r=n+1}^{n+p}(h_{ij}^r)^2+\varepsilon),$$

or equivalently,

$$(3.10)\qquad (\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\{\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \\ + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^{r})^2 + \varepsilon\}.$$

By using Lemma 2.1 we have from (3.10):

$$(3.11)\qquad 2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^{r})^2 + \varepsilon.$$

The Gauss equation for $X = Z = e_1, Y = W = e_2$ gives

$$K(\pi) = R(e_1, e_2, e_2, e_1) = c[1 + 3g^2(Je_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2) +$$

$$+\beta(e_2, e_2) + \eta(h(e_2, e_2)) + \sum_{r=n+1}^{p}[h_{11}^{r}h_{22}^{r} - (h_{12}^{r})^2] \geq$$

$$\geq c[1 + 3g^2(Je_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \beta(e_2, e_2) + \eta(h(e_2, e_2)) +$$

$$+\frac{1}{2}[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m}(h_{ij}^{r})^2 + \varepsilon] + \sum_{r=n+2}^{2m}h_{11}^{r}h_{22}^{r} - \sum_{r=n+1}^{2m}(h_{12}^{r})^2 =$$

$$= c[1 + 3g^2(Je_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \beta(e_2, e_2) + \eta(h(e_2, e_2)) +$$

$$+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{2m}(h_{ij}^{r})^2 + \frac{1}{2}\varepsilon + \sum_{r=n+2}^{2m}h_{11}^{r}h_{22}^{r} - \sum_{r=n+1}^{2m}(h_{12}^{r})^2 =$$

$$= c[1 + 3g^2(Je_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \beta(e_2, e_2) + \eta(h(e_2, e_2)) +$$

$$+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=2m}^{2m}\sum_{i,j>2}(h_{ij}^{r})^2 + \frac{1}{2}\sum_{r=2m}^{2m}(h_{11}^{r} + h_{22}^{r})^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{1}{2}\varepsilon \geq$$

$$\geq c[1 + 3g^2(Je_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \beta(e_2, e_2) + \eta(h(e_2, e_2)) + \frac{\varepsilon}{2},$$

which implies

$$K(\pi) \geq c[1 + 3g^2(Je_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \beta(e_2, e_2) + \eta(h(e_2, e_2)) + \frac{\varepsilon}{2}.$$

We remark that

$$\alpha(e_1,e_1)+\alpha(e_2,e_2)=\lambda-trace\left(\alpha_{|_{\pi^\perp}}\right).$$

By using (3.8) and (3.3) we get

$$\begin{aligned}K(\pi) \geq\ & \tau-\frac{(n-2)}{2}\left[(n+1)c-2\lambda+\frac{n^2}{(n-1)}\|H\|^2\right]+\Omega(e_2)-\\ & -\frac{(n-1)}{2}(\gamma+n\eta(H))+[6\Theta^2(\pi)-3\|T\|^2]\frac{c}{2}+trace\left(\alpha_{|_{\pi^\perp}}\right),\end{aligned}$$

which represents the inequality to prove. □

Recall the following important result (Theorem 3.2) from [22].

Proposition 3.1. *The mean curvature H of M^n with respect to the semi-symmetric non-metric connection coincides with the mean curvature $\overset{\circ}{H}$ of M^n with respect to the Levi-Civita connection if and only if the vector field P is tangent to M^n.*

Remark 1. According to the formula (17) from [22] (see also Proposition 3.2) it follows that $h=\overset{\circ}{h}$ if P is tangent to M^n.In this case inequality proved in (Theorem 3.1) becomes

$$\begin{aligned}\tau(x)-K(\pi) \leq\ & \frac{n-2}{2}\left[\frac{n^2}{(n-1)}\left\|\overset{\circ}{H}\right\|^2+(n+1)c-2\lambda\right]-\Omega(e_2)+\\ & +\frac{n-1}{2}\left(\gamma+n\eta(\overset{\circ}{H})\right)-[6\Theta^2(\pi)-3\|T\|^2]\frac{c}{2}-trace\left(\alpha_{|_{\pi^\perp}}\right).\end{aligned}$$

Theorem 3.2. *Under the same assumptions as in Theorem 3.1, if the vector field P is tangent to M^n, then the equality case of inequality from Theorem 3.1 holds at a point $x\in M^n$ if and only if there exists an orthonormal basis $\{e_1,e_2,...,e_n\}$ of T_xM^n*

and an orthonormal basis $\{e_{n+1}, ..., e_{2m}\}$ *of* $T_x^{\perp} M^n$ *such that the shape operators of* M^n *in* $N^{2m}(4c)$ *at* x *have the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a+b=\mu,$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n+2 \leq i \leq 2m,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$*,* $1 \leq i, j \leq n$ *and* $n+2 \leq r \leq 2m$*.*

Proof. The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have the equality in the Lemma.

$$h_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$h_{ij}^r = 0, \quad \forall i \neq j, i, j > 2, r = n+1, ..., 2m,$$

$$h_{11}^r + h_{22}^r = 0, \quad \forall r = n+2, ..., 2m,$$

$$h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \quad \forall j > 2,$$

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = ... = h_{nn}^{n+1}.$$

We may chose $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$ and we denote by $a = h_{11}^r, b = h_{22}^r, \mu = h_{33}^{n+1} = ... = h_{nn}^{n+1}$.

It follows that the shape operators take the desired forms. □

4. **Ricci curvature for submanifolds of complex space forms**

In this section we prove relationships between Ricci curvature of a submanifold M^n of a complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature, endowed with a semi-symmetric non-metric connection $\widetilde{\widetilde{\nabla}}$ and the squared mean curvature $\|H\|^2$. We suppose that the vector field P is tangent to M^n.

Theorem 4.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $2m$-dimensional complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature $4c$ endowed with a semi-symmetric non-metric connection $\widetilde{\widetilde{\nabla}}$ such that the vector field P is tangent to M^n. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{1}{n}[nc - 2\lambda + \gamma] - \eta(H) - \frac{3c}{n(n-1)} \|T\|^2 . \tag{4.1}$$

Proof. Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and orthonormal basis of T_xM^n. The relation (3.7) is equivalent with

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - c[n^2 - n + 3\|T\|^2] - (n-1)[-2\lambda + \gamma] - (n^2 - n)\eta(H). \tag{4.2}$$

We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and $e_1, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \tag{4.3}$$

$$A_{e_r} = (h_{ij}^r),\ i, j = 1, ..., n; r = n+2, ..., 2m, \text{trace}\, A_{e_r} = 0. \tag{4.4}$$

From (4.2), we get

$$n^2 \|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - (n-1)[-2\lambda + \gamma] - \\ -(n^2-n)\eta(H) - c[n^2 - n + 3\|T\|^2]. \tag{4.5}$$

On the other hand, since

$$0 \le \sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_{i=1}^{n} a_i^2 - 2\sum_{i<j} a_i a_j,$$

we obtain

$$n^2 \|H\|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \le n \sum_{i=1}^{n} a_i^2, \tag{4.6}$$

which implies

$$\sum_{i=1}^{n} a_i^2 \ge n \|H\|^2. \tag{4.7}$$

We have from (4.5)

$$n^2 \|H\|^2 = 2\tau + n\|H\|^2 - (n-1)[-2\lambda + \gamma] - \\ -(n^2-n)\eta(H) - c[n^2 - n + 3\|T\|^2], \tag{4.8}$$

i.e. (4.1)

$$\|H\|^2 \ge \frac{2\tau}{n(n-1)} - \frac{1}{n}[nc - 2\lambda + \gamma] - \eta(H) - \frac{3c}{n(n-1)}\|T\|^2.$$

□

Using Theorem 4.1, we obtain the following

Theorem 4.2. *Let $M^n, n \ge 3$, be an n-dimensional submanifold of an $2m$-dimensional complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature $4c$ endowed* *with a semi-symmetric non-metric connection $\widetilde{\nabla}$, such that the vector field P is tangent to M^n. Then, for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\|H\|^2 (x) \geq \Theta_k(x) - \frac{1}{n}[nc - 2\lambda + \gamma] - \frac{3c}{n(n-1)} \|T\|^2 . \tag{4.9}$$

Proof. Let $\{e_1, ...e_n\}$ be an orthonormal basis of T_xM. Denote by $L_{i_1...i_k}$ the k-plane section spanned by $e_{i_1}, ..., e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1...i_k}) = \frac{1}{2} \sum_{i \in \{i_1,...,i_k\}} Ric_{L_{i_1...i_k}}(e_i), \tag{4.10}$$

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < ... < i_k \leq n} \tau(L_{i_1...i_k}). \tag{4.11}$$

From (4.1), (4.10) and (4.11), one derives

$$\tau(x) \geq \frac{n(n-1)}{2} \Theta_k(x), \tag{4.12}$$

which implies (4.9). □

5. Chen First Inequality for submanifolds of Sasakian space forms

A $(2m+1)$-dimensional Riemannian manifold (N^{2m+1}, g) has an *almost contact metric structure* if it admits a (1, 1)-tensor field φ, a vector field ξ and a 1-form ω satisfying:

$$\varphi^2 X = -X + \omega(X)\xi \ , \ \omega(\xi) = 1$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \omega(X)\omega(Y),$$

$$g(X, \xi) = \omega(X),$$

for any vector fields X, Y on TN. Let Φ denote the fundamental 2-form in N^{2m+1},given by $\Phi(X, Y) = g(X, \varphi Y)$, for all X, Y on TN. If $\Phi = d\omega$, then N^{2m+1} is called a *contact metric manifold.* The structure of N^{2m+1} is called *normal* if

$$[\varphi, \varphi] + 2d\omega \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A *Sasakian manifold* is a normal contact metric manifold.

A plane section π in T_pN^{2m+1} is called a *φ-section* if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature of a *φ-section* is called *φ-sectional curvature.* A Sasakian manifold with constant φ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $N^{2m+1}(c)$. The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ on $N^{2m+1}(c)$ is expressed by

$$\begin{aligned}
(5.1) \quad \overset{\circ}{\widetilde{R}}(X, Y, Z, W) \quad = \quad & \frac{c+3}{4}[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)] + \\
& + \frac{c-1}{4}[\omega(X)\omega(Z)g(Y, W) - \omega(Y)\omega(Z)g(X, W) + \\
& + \omega(Y)\omega(W)g(X, Z) - \omega(X)\omega(W)g(Y, Z) + \\
& + g(X, \varphi Z)g(\varphi Y, W) - g(Y, \varphi Z)g(\varphi X, W) + \\
& + 2g(X, \varphi Y)g(\varphi Z, W)].
\end{aligned}$$

for vector fields X, Y, Z, W on $N^{2m+1}(c)$.

If $N^{2m+1}(c)$ is a $(2m+1)$-dimensional Sasakian space form of constant φ-sectional curvature c endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$, then from (2.4) and (5.1) it follows that the curvature tensor $\widetilde{R}$ of $N^{2m+1}(c)$ can be expressed as

$$
\begin{aligned}
(5.2)\quad \widetilde{R}(X,Y,Z,W) \;=\; & \frac{c+3}{4}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)]+\\
&+\frac{c-1}{4}[\omega(X)\omega(Z)g(Y,W)-\omega(Y)\omega(Z)g(X,W)+\\
&+\omega(Y)\omega(W)g(X,Z)-\omega(X)\omega(W)g(Y,Z)+\\
&+g(X,\varphi Z)g(\varphi Y,W)-g(Y,\varphi Z)g(\varphi X,W)+\\
&+2g(X,\varphi Y)g(\varphi Z,W)]-\alpha(Y,Z)g(X,W)+\\
&+\alpha(X,Z)g(Y,W)-\alpha(X,W)g(Y,Z)+\\
&+\alpha(Y,W)g(X,Z)+\beta(Y,X)\,g(Z,W)-\\
&-\beta(X,Y)\,g(Z,W)+\beta(Y,Z)\,g(X,W)-\\
&-\beta(X,Z)\,g(Y,W).
\end{aligned}
$$

Let M^n,$n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional Sasakian space form of constant φ-sectional curvature $N^{n+p}(c)$ of constant sectional curvature c. For any tangent vector field X to M^n, we put

$$\varphi X = TX + FX,$$

where TX and FX are the tangential and normal components of φX, respectively and we decompose

$$\xi = \xi^T + \xi^\perp,$$

where ξ^T and $\xi^\perp$ denotes the tangential and normal parts of ξ.

Recall that $\Theta^2(\pi) = g^2(Te_1, e_2) = g^2(\varphi e_1, e_2)$ and where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π. $\Theta^2(\pi)$ is a real number in [0, 1], independent of the choice of e_1, e_2.

For submanifolds Sasakian space forms endowed with a semi-symmetric non-metric connection we establish the following optimal inequality.

Theorem 5.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional Sasakian space form $N^{2m+1}(c)$ of constant φ-sectional curvature endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$. We have:*

$$
\begin{aligned}
(5.3)\ \tau(x) - K(\pi) \leq\ & (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c+3}{8} - \lambda\right] + \\
& + \frac{c-1}{8}\left[3\|T\|^2 - 6\Theta^2(\pi) - 2(n-1)\left\|\xi^T\right\|^2 + 2\|\xi_\pi\|^2\right] - \\
& - \Omega(e_2) + \frac{(n-1)}{2}(\gamma + n\eta(H)) - trace\left(\alpha_{|_{\pi^\perp}}\right),
\end{aligned}
$$

where π is a 2-plane section of $T_xM^n, x \in M^n$.

Proof. From [22], the Gauss equation with respect to the semi-symmetric non-metric connection is

$$
\begin{aligned}
(5.4)\qquad \widetilde{R}(X,Y,Z,W) =\ & R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - \\
& - g(h(Y,Z),h(X,W)) - \eta(h(Y,Z))g(X,W) \\
& + \eta(h(X,Z))g(Y,W).
\end{aligned}
$$

Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and $\{e_{n+1}, ..., e_{2m+1}\}$ be orthonormal basis of T_xM^n and $T_x^\perp M^n$, respectively. For $X = W = e_i, Y = Z = e_j$, $i \neq j$, from the equation (5.2) it follows that:

$$
\begin{aligned}
(5.5)\qquad \tilde{R}(e_i,e_j,e_j,e_i) =\ & \frac{c+3}{4} + \frac{c-1}{4}[-\omega(e_i)^2 - \omega(e_j)^2 + 3g^2(Te_j,e_i)] - \\
& - \alpha(e_i,e_i) - \alpha(e_j,e_j) + \beta(e_j,e_j).
\end{aligned}
$$

From (5.4) and (5.5) we get

$$
\frac{c+3}{4} + \frac{c-1}{4}[-\omega(e_i)^2 - \omega(e_j)^2 + 3g^2(Te_j,e_i)] - \alpha(e_i,e_i) - \alpha(e_j,e_j) + \beta(e_j,e_j) =
$$

$$
= R(e_i,e_j,e_j,e_i) + g(h(e_i,e_j),h(e_i,e_j))g(h(e_i,e_i),h(e_j,e_j)) - \eta(h(e_j,e_j)).
$$

By summation after $1 \leq i, j \leq n$, it follows from the previous relation that

$$(5.6)\qquad (n^2-n)\frac{c+3}{4}+\frac{c-1}{4}\left[-2(n-1)\left\|\xi^T\right\|^2+3\left\|T\right\|^2\right]+(n-1)[-2\lambda+\gamma]=$$

$$=2\tau+\|h\|^2-n^2\|H\|^2-(n^2-n)\eta(H).$$

We take

where we recall that λ is the trace of α and γ is the trace of β and denote by

$$\|h\|^2=\sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)),$$

$$H=\frac{1}{n}\mathrm{trace}h.$$

One takes

$$(5.7)\qquad \varepsilon = 2\tau+\frac{n^2(2-n)}{n-1}\|H\|^2-n(n-1)\eta(H)-(n^2-n)\frac{c+3}{4}- -\frac{c-1}{4}\left[-2(n-1)\left\|\xi^T\right\|^2+3\left\|T\right\|^2\right]-(n-1)[-2\lambda+\gamma].$$

Then, from (5.6) and (5.7) we get

$$(5.8)\qquad n^2\|H\|^2=(n-1)\left(\|h\|^2+\varepsilon\right).$$

Let $x \in M^n$, $\pi \subset T_xM^n$, $\dim \pi = 2$, $\pi = sp\{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$ and from the relation (5.8) we obtain:

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2=(n-1)(\sum_{i,j=1}^{n}\sum_{r=n+1}^{2m+1}(h_{ij}^r)^2+\varepsilon),$$

or equivalently,

$$(5.9)\qquad (\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\{\sum_{i=1}^{n}(h_{ii}^{n+1})^2+\sum_{i\neq j}(h_{ij}^{n+1})^2+ +\sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2+\varepsilon\}.$$

By using algebraic Lemma we have from the previous relation

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \varepsilon. \tag{5.10}$$

If we denote by $\xi_\pi = pr_\pi \xi$ we can write

$$-\omega(e_1)^2 - \omega(e_2)^2 = -\left\|\xi_\pi\right\|^2.$$

The Gauss equation for $X = Z = e_1, Y = W = e_2$ gives

$$K(\pi) = R(e_1,e_2,e_2,e_1) = \frac{c+3}{4} + \frac{c-1}{4}[-\left\|\xi_\pi\right\|^2 + 3g^2(Te_1,e_2)] -$$

$$-\alpha(e_1,e_1) - \alpha(e_2,e_2) + \beta(e_2,e_2) + \eta(h(e_2,e_2)) + \sum_{r=n+1}^{p}[h_{11}^r h_{22}^r - (h_{12}^r)^2] \geq$$

$$\geq \frac{c+3}{4} + \frac{c-1}{4}[-\left\|\xi_\pi\right\|^2 + 3g^2(Te_1,e_2)] - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \beta(e_2,e_2) +$$

$$+\eta(h(e_2,e_2)) + \frac{1}{2}[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \varepsilon] + \sum_{r=n+2}^{2m+1}h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m+1}(h_{12}^r)^2 =$$

$$= \frac{c+3}{4} + \frac{c-1}{4}[-\left\|\xi_\pi\right\|^2 + 3g^2(Te_1,e_2)] - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \beta(e_2,e_2) +$$

$$+\eta(h(e_2,e_2)) + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \frac{1}{2}\varepsilon + \sum_{r=n+2}^{2m+1}h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m+1}(h_{12}^r)^2 =$$

$$= \frac{c+3}{4} + \frac{c-1}{4}[-\left\|\xi_\pi\right\|^2 + 3g^2(Te_1,e_2)] - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \beta(e_2,e_2) + \eta(h(e_2,e_2)) +$$

$$+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j>2}(h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{1}{2}\varepsilon \geq$$

$$\geq \frac{c+3}{4} + \frac{c-1}{4}[-\left\|\xi_\pi\right\|^2 + 3g^2(Te_1,e_2)] - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \beta(e_2,e_2) + \eta(h(e_2,e_2)) + \frac{\varepsilon}{2},$$

which implies

$$K(\pi) \geq \frac{c+3}{4} + \frac{c-1}{4}[-\left\|\xi_\pi\right\|^2 + 3g^2(Te_1,e_2)] - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \beta(e_2,e_2) + \eta(h(e_2,e_2)) + \frac{\varepsilon}{2}.$$

We remark that

$$\alpha(e_1, e_1) + \alpha(e_2, e_2) = \lambda - trace\left(\alpha_{|_{\pi^\perp}}\right).$$

By using (5.7) and (3.3) we get

$$\begin{aligned} K(\pi) \geq\ & \tau - (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c+3}{8} - \lambda\right] - \\ & -\frac{c-1}{8}\left[3\|T\|^2 - 6\Theta^2(\pi) - 2(n-1)\left\|\xi^T\right\|^2 + 2\|\xi_\pi\|^2\right] + \Omega(e_2) - \\ & -\frac{(n-1)}{2}(\gamma + n\eta(H)) + trace\left(\alpha_{|_{\pi^\perp}}\right), \end{aligned}$$

which represents the inequality to prove. □

Corollary 5.1. *Under the same assumptions as in Theorem 5.1, if ξ is tangent to M^n, we have*

$$\begin{aligned} \tau(x) - K(\pi) \leq\ & (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c+3}{8} - \lambda\right] + \\ & +\frac{c-1}{8}\left[3\|T\|^2 - 6\Theta^2(\pi) - 2(n-1) + 2\|\xi_\pi\|^2\right] - \Omega(e_2) + \\ & +\frac{(n-1)}{2}(\gamma + n\eta(H)) - trace\left(\alpha_{|_{\pi^\perp}}\right). \end{aligned}$$

If ξ is normal to M^n, we have

$$\begin{aligned} \tau(x) - K(\pi) \leq\ & (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c+3}{8} - \lambda\right] - \Omega(e_2) + \\ & +\frac{(n-1)}{2}(\gamma + n\eta(H)) - trace\left(\alpha_{|_{\pi^\perp}}\right). \end{aligned}$$

Remark 2. According to the formula (17) from [22] (see also Proposition 3.2) it follows that $h = \overset{\circ}{h}$ if P is tangent to M^n. In this case inequality proved in (3.1) becomes

$$\begin{aligned}\tau(x) - K(\pi) \leq\ & (n-2)\left[\frac{n^2}{2(n-1)}\left\|\overset{\circ}{H}\right\|^2 + (n+1)\frac{c+3}{8} - \lambda\right] + \\ & + \frac{c-1}{8}\left[3\left\|T\right\|^2 - 6\Theta^2(\pi) - 2(n-1)\left\|\xi^T\right\|^2 + 2\left\|\xi_\pi\right\|^2\right] - \Omega(e_2) + \\ & + \frac{(n-1)}{2}\left(\gamma + n\eta(\overset{\circ}{H})\right) - trace\left(\alpha_{|_{\pi^\perp}}\right).\end{aligned}$$

Theorem 5.2. *If the vector field P is tangent to M^n, then the equality case of inequality (5.3) holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of T_xM^n and an orthonormal basis $\{e_{n+1}, ..., e_{n+p}\}$ of $T_x^\perp M^n$ such that the shape operators of M^n in $N^{2m+1}(c)$ at x have the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n + 2 \leq i \leq 2m + 1,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n + 2 \leq r \leq 2m + 1$.

Proof. The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have the equality in the Lemma.

$$h_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$h_{ij}^{r} = 0, \;\; \forall i \neq j, i, j > 2, r = n+1, ..., 2m+1,$$

$$h_{11}^{r} + h_{22}^{r} = 0, \;\; \forall r = n+2, ..., 2m+1,$$

$$h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \;\; \forall j > 2,$$

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = ... = h_{nn}^{n+1}.$$

We may chose $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$ and we denote by $a = h_{11}^{r}, b = h_{22}^{r}, \mu = h_{33}^{n+1} = ... = h_{nn}^{n+1}$.

It follows that the shape operators take the desired forms. □

6. Ricci curvature for submanifolds of Sasakian space forms

We first state a relationship between the sectional curvature of a submanifold M^n of a Sasakian space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric non-metric connection $\widetilde{\widetilde{\nabla}}$ and the squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the k-Ricci curvature of M^n (intrinsic invariant) and the squared mean curvature $\|H\|^2$ (extrinsic invariant), as another answer of the basic problem in submanifold theory which we have mentioned in the introduction.

In this section we suppose that the vector field P is tangent to M^n.

Theorem 6.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional Sasakian space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric non-metric connection $\widetilde{\widetilde{\nabla}}$ such that the vector field P*

is tangent to M^n. Then we have

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{1}{n}[-2\lambda+\gamma] - \eta(H) - \frac{c+3}{4} - \\ -\frac{c-1}{4n(n-1)}\left[-2(n-1)\left\|\xi^T\right\|^2 + 3\left\|T\right\|^2\right]. \tag{6.1}$$

Proof. Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and orthonormal basis of T_xM^n. The relation (5.6) is equivalent with

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - (n-1)[-2\lambda+\gamma] - (n^2-n)\eta(H) - \\ -(n^2-n)\frac{c+3}{4} - \frac{c-1}{4}\left[-2(n-1)\left\|\xi^T\right\|^2 + 3\left\|T\right\|^2\right]. \tag{6.2}$$

We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{n+p}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and $e_1, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \tag{6.3}$$

$$A_{e_r} = (h^r_{ij}),\ i,j = 1, ..., n; r = n+2, ..., 2m+1, \operatorname{trace} A_{e_r} = 0. \tag{6.4}$$

From (6.2), we get

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h^r_{ij})^2 - (n-1)[-2\lambda+\gamma] - \\ -(n^2-n)\eta(H) - (n^2-n)\frac{c+3}{4} - \\ -\frac{c-1}{4}\left[-2(n-1)\left\|\xi^T\right\|^2 + 3\left\|T\right\|^2\right], \tag{6.5}$$

which implies

$$n^2 \|H\|^2 = 2\tau + n\|H\|^2 - (n-1)[-2\lambda+\gamma] - (n^2-n)\eta(H) - -(n^2-n)\frac{c+3}{4} - \frac{c-1}{4}\left[-2(n-1)\left\|\xi^T\right\|^2 + 3\|T\|^2\right], \tag{6.6}$$

because $\sum_{i=1}^n a_i^2 \geq n\|H\|^2$(see (4.7)).

Last inequality represents (6.1) □

Using Theorem 6.1, we obtain the following

Theorem 6.2. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional Sasakian space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$, such that the vector field P is tangent to M^n. Then, for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\|H\|^2(x) \geq \Theta_k(x) - \frac{1}{n}[-2\lambda+\gamma] - \eta(H) - \frac{c+3}{4} - -\frac{c-1}{4n(n-1)}\left[-2(n-1)\left\|\xi^T\right\|^2 + 3\|T\|^2\right]. \tag{6.7}$$

Proof. It follows immediately from (6.1) and (4.12). □

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