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THE ANTI-CENTRO-SYMMETRIC EXTREMAL RANK SOLUTIONS OF THE MATRIX EQUATION $AX = B$

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Abstract. A matrix $A = (a_{ij}) \in R^{n\times n}$ is said to be a centro-symmetric matrix if $a_{ij} = -a_{n+1-i,n+1-j}, i, j = 1, 2, \ldots, n$. In this paper, we mainly investigate the anti-centro-symmetric maximal and minimal rank solutions to the system of matrix equation $AX = B$. We present necessary and sufficient conditions for the existence of the maximal and minimal rank solutions with anti-centro-symmetric to the system. The expressions of such solutions to this system are also given when the solvability conditions are satisfied. In addition, in corresponding the minimal rank solution set to the system, the explicit expression of the nearest matrix to a given matrix in the Frobenius norm has been provided.

1. Introduction

Throughout this paper, let $R^{n\times m}$ be The set of all $n \times m$ real matrices, $OR^{n\times n}$ be The set of all $n\times n$ orthogonal matrices. Denote by I_n the identity matrix with order n. For a matrix A, A^T, A^+, $\|A\|$ and $r(A)$ represent its transpose, Moore-Penrose inverse, Frobenius norm and rank, respectively.

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Definition 1.1. A matrix $A = (a_{ij}) \in R^{n\times n}$ is said to be an anti-centro-symmetric matrix if $a_{ij} = -a_{n+1-i,n+1-j}, i, j = 1, 2, \ldots, n$. The set of all $n \times n$ anti-centro-symmetric matrices is denoted by $ACSR^{n\times n}$.

Anti-centro-symmetric matrices have practical applications in information theory, linear system theory, linear estimate theory, and numerical analysis(see, e.g. [1-4]).

In matrix theory and applications, many problems are closely related to the ranks of some matrix expressions with variable entries, and so it is necessary to explicitly characterize the possible ranks of the matrix expressions concerned. The study on the possible ranks of matrix equations can be traced back to the late 1970s (see, e.g. [5-9]). Recently, the extremal ranks, i.e. maximal and minimal ranks, of some matrix expressions have found many applications in control theory [10,11], statistics, and economics (see, e.g. [12-14]).

In this paper, we consider the anti-centro-symmetric extremal rank solutions of the matrix equation

$$AX = B, \tag{1.1}$$

where A and B are given matrices in $R^{m\times n}$. In 1987, Uhlig [8] gave the maximal and minimal ranks of solutions to system (1.1). By applying the matrix rank method, recently, Tian [15] obtained the minimal rank of solutions to the matrix equation $A = BX + YC$. Xiao et al. [16] in 2009 considered the symmetric minimal rank solution to system (1.1). The centro-symmetric and anti-centro-symmetric matrices are two classes of important matrices and have engineering and scientific applications. The anti-centro-symmetric maximal and minimal rank solutions of the matrix equation (1.1), however, has not been considered yet. In this paper, we will discuss this problem.

We also consider the matrix nearness problem

$$\min_{X\in S_m} \left\| X - \tilde{X} \right\|_F , \tag{1.2}$$

where $\tilde{X}$ is a given matrix in $R^{n\times n}$ and S_m is the minimal rank solution set of Eq. (1.1).

The matrix nearness problem (1.2) is so-called the optimal approximation problem, which has important application in practice, and has been discussed far and wide (see, e.g., [17-22] and the references therein).

We organize this paper as follows. In Section 2, we first establish a representation for the anti-centro-symmetric matrix. Then we give necessary and sufficient conditions for the existence of anti-centro-symmetric solution to (1.1). We also give the expressions of such solutions when the solvability conditions are satisfied. In Section 3, we establish formulas of maximal and minimal ranks of anti-centro-symmetric solutions to (1.1), and present the anti-centro-symmetric extremal rank solutions to (1.1). In Section 4, we present the expression of the optimal approximation solution to the set of the minimal rank solution.

2. Anti-centro-symmetric solution to (1.1)

In this section we first establish the representations of anti-centro-symmetric matrix. Then we give the necessary and sufficient conditions for the existence of and the expressions for anti-centro-symmetric solution of Eq. (1.1).

Let e_i be the ith column of I_n and set $S_n = (e_n, e_{n-1}, \cdots, e_1)$. It is easy to see that

$$S_n^T = S_n, \quad S_n^T S_n = I.$$

Let $k=[\frac{n}{2}]$, where $[\frac{n}{2}]$ is the maximum integer which is not greater than $\frac{n}{2}$. Define D_n as

$$D_n=\frac{1}{\sqrt{2}}\begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix}(n=2k), D_n=\frac{1}{\sqrt{2}}\begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix}(n=2k+1), \tag{2.1}$$

then it is easy verified that the above matrices S_n and D_n are orthogonal matrices.

Lemma 2.1. [18] *$X\in ACSR^{2k\times 2k}$ if and only if there exist $M,H\in R^{k\times k}$ such that*

$$X=\begin{pmatrix} M & HS_k \\ -S_kH & -S_kMS_k \end{pmatrix}=D_{2k}\begin{pmatrix} 0 & M-H \\ M+H & 0 \end{pmatrix}D_{2k}^T. \tag{2.2}$$

$X\in ACSR^{(2k+1)\times(2k+1)}$ if and only if there exist $M,H\in R^{k\times k}$, $u,v\in R^{k\times 1}$ and $\alpha\in R$ such that

$$X=\begin{pmatrix} M & u & HS_k \\ -v^T & 0 & v^TS_k \\ -S_kH & -S_ku & -S_kMS_k \end{pmatrix}=D_{2k+1}\begin{pmatrix} 0 & 0 & M-H \\ 0 & 0 & -\sqrt{2}v^T \\ M+H & \sqrt{2}u & 0 \end{pmatrix}D_{2k+1}^T. \tag{2.3}$$

Lemma 2.2. [18] *Let $X\in R^{n\times n}$ and D_n with the forms of* (2.1)*, then X is the anti-centro-symmetric matrix if and only if there exist $X_1\in R^{(n-k)\times k}$ and $X_2\in R^{k\times(n-k)}$, whether n is odd or even, such that*

$$X=D_n\begin{pmatrix} 0 & X_1 \\ X_2 & 0 \end{pmatrix}D_n^T. \tag{2.4}$$

Here, we always assume $k=[\frac{n}{2}]$.

Given matrices $A_1\in R^{m\times n}$, $B_1\in R^{m\times p}$, by making generalized singular value decomposition to $[A_1,B_1]$, we have

$$A_1=M_1\Sigma_{A_1}U_1,\quad B_1=M_1\Sigma_{B_1}V_1, \tag{2.5}$$

where M_1 is an $m \times m$ nonsingular matrix, $U_1 \in OR^{n\times n}$, $V_1 \in OR^{p\times p}$,

$$\Sigma_{A_1} = \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1 - s_1 \\ s_1 \\ k_1 - r_1 \\ m - k_1 \end{matrix}, \quad \Sigma_{B_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_1} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1 - s_1 \\ s_1 \\ k_1 - r_1 \\ m - k_1 \end{matrix},$$

$k_1 = r[A_1, B_1]$, $r_1 = r(A_1)$, $s_1 = r(A_1) + r(B_1) - r[A_1, B_1]$, $S_{A_1} = diag(\alpha_1, \ldots, \alpha_{s_1})$, $S_{B_1} = diag(\beta_1, \ldots, \beta_{s_1})$, $0 < \alpha_{s_1} \leq \cdots \leq \alpha_1 < 1$, $0 < \beta_1 \leq \cdots \leq \beta_{s_1} < 1$, $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, \ldots, s_1$.

Lemma 2.3. *Given matrices $A_1 \in R^{m\times n}$, $B_1 \in R^{m\times p}$, the generalized singular value decomposition of the matrix pair $[A_1, B_1]$ is given by (2.5), then matrix equation $A_1X = B_1$ is consistent, if and only if*

$$r[A_1, B_1] = r(A_1), \tag{2.6}$$

and the expression of its general solution is

$$X = U_1^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_1}^{-1} S_{B_1} \\ Y_{31} & Y_{32} \end{bmatrix} V_1, \tag{2.7}$$

where $Y_{31} \in R^{(n-r_1)\times(p-s_1)}$, $Y_{32} \in R^{(n-r_1)\times s_1}$ are arbitrary.

Proof. With (2.5) we have

$$r(B_1 - A_1X) = r(M_1\Sigma_{B_1}V_1 - M_1\Sigma_{A_1}U_1X) = r(\Sigma_{B_1} - \Sigma_{A_1}U_1XV_1^T).$$

Let $Y = U_1XV_1^T$ and Partition Y with $Y = (Y_{ij})_{3\times 3}$, then

$$\Sigma_{B_1} - \Sigma_{A_1}Y = \begin{bmatrix} -Y_{11} & -Y_{12} & -Y_{13} \\ -S_{A_1}Y_{21} & S_{B_1} - S_{A_1}Y_{22} & -S_{A_1}Y_{23} \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1 - s_1 \\ s_1 \\ k_1 - r_1 \\ m - k_1 \end{matrix}. \tag{2.8}$$

Noting that $Y_{ij}(i = 1, 2, j = 1, 2, 3)$ are arbitrary, then

$$\min r(B_1 - A_1X) = \min r(\Sigma_{B_1} - \Sigma_{A_1}Y) = k_1 - r_1 = r(A_1, B_1) - r(A_1).$$

$A_1X = B_1$ is solvable in $R^{n\times p}$ if and only if $\min r(B_1 - A_1X) = 0$. Then matrix equation $A_1X = B_1$ is consistent, if and only if (2.6) holds. In this case, from (2.8) and $Y = U_1XV_1^T$, its general solution can be expressed as (2.7). The proof is completed.

Assume D_n with the form of (2.1), and AD_n and BD_n have the following partition form

$$AD_n = [A_2, A_3], \quad BD_n = [B_2, B_3], \tag{2.9}$$

where $A_2 \in R^{m\times(n-k)}$, $A_3 \in R^{m\times k}$, $B_2 \in R^{m\times(n-k)}$, $B_3 \in R^{m\times k}$, and the generalized singular value decomposition of the matrix pair $[A_2, B_2]$, $[A_3, B_3]$ are, respectively,

$$A_2 = M_2\Sigma_{A_2}U_2, \quad B_3 = M_2\Sigma_{B_3}V_2, \tag{2.10}$$

$$A_3 = M_3\Sigma_{A_3}U_3, \quad B_2 = M_3\Sigma_{B_2}V_3, \tag{2.11}$$

where $U_2 \in OR^{(n-k)\times(n-k)}$, $V_2 \in OR^{k\times k}$, $U_3 \in OR^{k\times k}$, $V_3 \in OR^{(n-k)\times(n-k)}$, nonsingular matrices $M_2, M_3 \in R^{m\times m}$, $k_2 = r[A_2, B_3]$, $r_2 = r(A_2)$, $s_2 = r(A_2) + r(B_3) - r[A_2, B_3]$, and $k_3 = r[A_3, B_2]$, $r_3 = r(A_3)$, $s_3 = r(A_3) + r(B_2) - r[A_3, B_2]$,

$$\Sigma_{A_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_2 - s_2 \\ s_2 \\ k_2 - r_2 \\ m - k_2 \end{matrix}, \qquad \Sigma_{B_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_3} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_2 - s_2 \\ s_2 \\ k_2 - r_2 \\ m - k_2 \end{matrix},$$

$$\Sigma_{A_3} = \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_3 - s_3 \\ s_3 \\ k_3 - r_3 \\ m - k_3 \end{matrix}, \qquad \Sigma_{B_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_2} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_3 - s_3 \\ s_3 \\ k_3 - r_3 \\ m - k_3 \end{matrix},$$

Then we can establish the existence theorems as follows.

Theorem 2.1. *Let* $A, B \in R^{m\times n}$ *and* D_n *with the form of* (2.1)*,* AD_n, BD_n *have the partition forms of* (2.9)*, and the generalized singular value decompositions of the matrix pair* $[A_2, B_3]$ *and* $[A_3, B_2]$ *are given by* (2.10) *and* (2.11)*. Then the equation* (1.1) *has an anti-centro-symmetric solution* X *if and only if*

$$r[A_2, B_3] = r(A_2), \quad r[A_3, B_2] = r(A_3), \tag{2.12}$$

and its general solution can be expressed as

$$X = D_n \begin{bmatrix} 0 & U_2^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_2} \\ Z_{31} & Z_{32} \end{bmatrix} V_2 \\ U_3^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_3} \\ W_{31} & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} D_n^T, \tag{2.13}$$

where $Z_{31} \in R^{(n-k-r_2)\times(k-s_2)}$*,* $Z_{32} \in R^{(n-k-r_2)\times s_2}$*,* $W_{31} \in R^{(k-r_3)\times(n-k-s_3)}$*,* $W_{32} \in R^{(k-r_3)\times s_3}$ *are arbitrary.*

Proof. Suppose the matrix equation (1.1) has an anti-centro-symmetric solution X, then it follows from Lemma 2.2 that there exist $X_1 \in R^{(n-k)\times k}$, $X_2 \in R^{k\times(n-k)}$ satisfying

$$X = D_n \begin{bmatrix} 0 & X_1 \\ X_2 & 0 \end{bmatrix} D_n^T \quad \text{and} \quad AX = B \tag{2.14}$$

By (2.9), that is

$$[A_2 \;\; A_3] \begin{bmatrix} 0 & X_1 \\ X_2 & 0 \end{bmatrix} = [B_2 \;\; B_3], \tag{2.15}$$

i.e.

$$A_2X_1 = B_3, \quad A_3X_2 = B_2. \tag{2.16}$$

Therefore by Lemma 2.3, (2.12) holds, and

$$X_1 = U_2^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} V_2, \quad X_2 = U_3^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} V_3, \tag{2.17}$$

where $Z_{31} \in R^{(n-k-r_2)\times(k-s_2)}$, $Z_{32} \in R^{(n-k-r_2)\times s_2}$, $W_{31} \in R^{(k-r_3)\times(n-k-s_3)}$, $W_{32} \in R^{(k-r_3)\times s_3}$ are arbitrary. Substituting (2.17) into (2.14) yields that the anti-centro-symmetric solution X of the matrix equation (1.1) can be represented by (2.13). The proof is completed.

3. Anti-centro-symmetric extremal rank solutions to (1.1)

In this section, we first derive the formulas of the maximal and minimal ranks of anti-centro-symmetric solutions of (1.1), then present the expressions of anti-centro-symmetric maximal and minimal rank solutions to (1.1).

Theorem 3.1. *Suppose that the matrix equation* (1.1) *has an anti-centro-symmetric solution X and let Ω be the set of all anti-centro-symmetric solutions of* (1.1)*. Then the extreme ranks of X are as follows:*

(1) *The maximal rank of X is*

$$\min\{k, n-k-r(A_2)+r(B_3)\} + \min\{n-k, k-r(A_3)+r(B_2)\}. \tag{3.1}$$

The general expression of X satisfying (3.1) *is*

$$X = D_n \begin{bmatrix} 0 & U_2^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} V_2 \\ U_3^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} D_n^T, \tag{3.2}$$

where $Z_{31} \in R^{(n-k-r_2)\times(k-s_2)}$, $W_{31} \in R^{(k-r_3)\times(n-k-s_3)}$ *are chosen such that* $r(Z_{31}) = \min(n-k-r_2, k-s_2)$, $r(W_{31}) = \min(k-r_3, n-k-s_3)$, $Z_{32} \in R^{(n-k-r_2)\times s_2}$, $W_{32} \in R^{(k-r_3)\times s_3}$ *are arbitrary.*

(2) *The minimal rank of X is*

$$\min_{X\in\Omega} r(X) = r(B_2) + r(B_3). \tag{3.3}$$

The general expression of X satisfying (3.3) *is*

$$X = D_n \begin{bmatrix} 0 & U_2^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & Z_{32} \end{bmatrix} V_2 \\ U_3^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} D_n^T, \tag{3.4}$$

where $Z_{32} \in R^{(n-k-r_2)\times s_2}$, $W_{32} \in R^{(k-r_3)\times s_3}$ *are arbitrary.*

Proof (1) By (2.13),

$$\max_{X\in\Omega} r(X) = \max_{Z_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} + \max_{W_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & W_{32} \end{bmatrix}, \tag{3.5}$$

$$\max_{Z_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} = s_2 + \min\{n-k-r_2, k-s_2\} \tag{3.6}$$

$$= \min\{k, n-k-r_2+s_2\} = \min\{k, n-k-r(A_2)+r(B_3)\},$$

and

$$\max_{W_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} = s_3 + \min\{k-r_3, n-k-s_3\} \tag{3.7}$$

$$= \min\{n-k, k-r_3+s_3\} = \min\{n-k, k-r(A_3)+r(B_2)\}.$$

Taking (3.6) and (3.7) into (3.5) yields (3.1).

According to the general expression of the solution in theorem 2.1, it is easy to verify the rest of part (1).

(2) By (2.13),

$$\min_{X\in\Omega} r(X) = \min_{Z_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} + \min_{W_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix}, \tag{3.8}$$

$$\min_{Z_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} = s_2 = r(B_3) \tag{3.9}$$

and

$$\min_{W_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} = s_3 = r(B_2). \tag{3.10}$$

Taking (3.9) and (3.10) into (3.8) yields (3.3).

According to the general expression of the solution in theorem 2.1, it is easy to verify the rest of part (2). The proof is completed.

4. The expression of the optimal approximation solution to the set of the minimal rank solution

From (3.4), When the solution set $S_m = \{X \mid AX = B, X \in ACSR^{n\times n}, r(X) = \min_{X\in\Omega} r(X)\}$ is nonempty, it is easy to verify that S_m is a closed convex set, therefore there exists a unique solution $\hat{X}$ to the matrix nearness Problem (1.2).

Theorem 4.1. *Given a matrix $\tilde{X}$, and the other given notations and conditions are the same as in Theorem 2.1. Let*

$$D_n^T \tilde{X} D_n = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{bmatrix}, \quad \tilde{X}_{12} \in C^{(n-k)\times k}, \quad \tilde{X}_{21} \in C^{k\times(n-k)}, \tag{4.1}$$

and we denote

$$U_2 \tilde{X}_{12} V_2^T = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \\ \tilde{Z}_{31} & \tilde{Z}_{32} \end{bmatrix}, \quad U_3 \tilde{X}_{21} V_3^T = \begin{bmatrix} \tilde{W}_{11} & \tilde{W}_{12} \\ \tilde{W}_{21} & \tilde{W}_{22} \\ \tilde{W}_{31} & \tilde{W}_{32} \end{bmatrix}. \tag{4.2}$$

If S_m is nonempty, then Problem (1.2) has a unique $\hat{X}$ which can be represented as

$$\hat{X} = D_n \begin{bmatrix} 0 & U_2^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & \tilde{Z}_{32} \end{bmatrix} V_2 \\ U_3^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & \tilde{W}_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} D_n^T, \tag{4.3}$$

where $\tilde{Z}_{32}, \tilde{W}_{32}$ are the same as in (4.2).

Proof When S_m is nonempty, it is easy to verify from (3.4) that S_m is a closed convex set. Since $R^{n\times n}$ is a uniformly convex banach space under the Frobenius norm, there exists a unique solution for Problem (1.2). By Theorem 3.1, for any $X \in S_m$, X can be expressed as

$$X = D_n \begin{bmatrix} 0 & U_2^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & Z_{32} \end{bmatrix} V_2 \\ U_3^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} D_n^T, \tag{4.4}$$

where $Z_{32} \in R^{(n-k-r_2)\times s_2}$, $W_{32} \in R^{(k-r_3)\times s_3}$ are arbitrary.

Using the invariance of the Frobenius norm under unitary transformations, we have

$$\begin{aligned} \|X - \tilde{X}\|^2 &= \left\| \begin{bmatrix} 0 & U_2^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & Z_{32} \end{bmatrix} V_2 \\ U_3^T \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} - D_n^T \tilde{X} D_n \right\|^2 \\ &= \left\| Z_{32} - \tilde{Z}_{32} \right\|^2 + \left\| W_{32} - \tilde{W}_{32} \right\|^2 + \left\| S_{A_2}^{-1} S_{B_3} - \tilde{Z}_{22} \right\|^2 + \left\| S_{A_3}^{-1} S_{B_2} - \tilde{W}_{22} \right\|^2 \\ &+ \left\| \tilde{X}_{11} \right\|^2 + \left\| \tilde{X}_{22} \right\|^2 + \left\| \tilde{Z}_{11} \right\|^2 + \left\| \tilde{Z}_{12} \right\|^2 + \left\| \tilde{Z}_{21} \right\|^2 + \left\| \tilde{Z}_{31} \right\|^2 \\ &+ \left\| \tilde{W}_{11} \right\|^2 + \left\| \tilde{W}_{12} \right\|^2 + \left\| \tilde{W}_{21} \right\|^2 + \left\| \tilde{W}_{31} \right\|^2 . \end{aligned}$$

Therefore, $\|X - \tilde{X}\|$ reaches its minimum if and only if

$$Z_{32} = \tilde{Z}_{32}, \quad W_{32} = \tilde{W}_{32}. \tag{4.5}$$

Substituting (4.5) into (4.4) yields (4.3). The proof is completed.

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