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A NEW SHARP OSTROWSKI-GRÜSS TYPE INEQUALITY

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ABSTRACT. Using a Grüss type inequality for Lipschitzian type functions to obtain a sharp Ostrowski-Grüss type inequality in which a unified treatment of sharp integral inequalities for Lipschitzian type functions of mid-point, trapezoid and Simpson type is provided. Applications for cumulative distribution functions are given.

1. INTRODUCTION

In 1935, G. Grüss (see[5,p.296]) proved the following integral inequality which gives an approximation for the integral of a product of two functions in terms of the product of integrals of the two functions.

Theorem 1.1. *Let $h, g : [a, b] \to \mathbf{R}$ be two integrable functions such that $\phi \leq h(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$, where $\phi, \Phi, \gamma, \Gamma$ are real numbers. Then we have*

$$\begin{aligned} |T(h,g)| &:= \left|\frac{1}{b-a}\int_a^b h(x)g(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \\ &\leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \end{aligned} \tag{1.1}$$

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and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ *cannot be replaced by a smaller one.*

It is clear that the constant $\frac{1}{4}$ is achieved for

$$h(x) = g(x) = sgn(x - \frac{a+b}{2}).$$

.

From then on, (1.1) is well known in the literature as Grüss inequality.

In 1998, S. S. Dragomir and I. Fedotov [2, Theorem 2.1] established the following Grüss type inequality for Riemann-Stieltjes integrals:

Theorem 1.2. *Let* $h, u : [a, b] \to \mathbf{R}$ *be so that* u *is* L*-Lipschitzian on* $[a, b]$*, i.e.,*

$$|u(x) - u(y)| \leq L|x - y|$$

for all $x, y \in [a, b]$*,* h *is Riemann integrable on* $[a, b]$ *and there exist the real numbers* m, M *so that* $m \leq h(x) \leq M$ *for all* $x \in [a, b]$*. Then we have the inequality*

$$|\int_a^b h(x)\,du(x) - \frac{u(b) - u(a)}{b - a}\int_a^b h(x)\,dx| \leq \frac{1}{2}L(M - m)(b - a), \tag{1.2}$$

and the constant $\frac{1}{2}$ *is sharp.*

In [4,Theorem 2], the inequality (1.2) has been generalized and refined as

Theorem 1.3. *Let* $h, u : [a, b] \to \mathbf{R}$ *be so that* u *is* (l, L)*-Lipschitzian on* $[a, b]$*, i.e., satisfies the condition*

$$l(x_2 - x_1) \leq u(x_2) - u(x_1) \leq L(x_2 - x_1)$$

for all $a \leq x_1 \leq x_2 \leq b$ *with* $l < L$*,* h *is Riemann integrable on* $[a, b]$*. Then we have the inequality*

(1.3)
$$|\int_a^b h(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b h(x)\,dx| \le \frac{L-l}{2}\int_a^b |h(x) - \frac{1}{b-a}\int_a^b h(t)\,dt|\,dx.$$

In this paper, we will use the Grüss type inequality (1.3) to obtain a sharp Ostrowski-Grüss type inequality for Lipschitzian type functions. From which a unified treatment of sharp integral inequalities of midpoint, trapezoid and Simpson type for Lipschitzian type functions is provided.

Applications for cumulative distribution functions are given.

2. MAIN RESULTS

Theorem 2.1. *Let $u : [a,b] \to \mathbf{R}$ be (l,L)–Lipschitzian on $[a,b]$. Then for all $x \in [a,b]$ we have*

$$\begin{aligned} &|\textstyle\int_a^b u(t)\,dt - (b-a)[(1-\theta)u(x) + \theta\frac{u(a)+u(b)}{2}] \\ &+(1-\theta)(x - \frac{a+b}{2})[u(b)-u(a)]| \le \frac{L-l}{2} I(\theta,x), \end{aligned} \tag{2.1}$$

where

$$I(\theta,x) = \begin{cases} [\frac{a+b}{2} - (1-\theta)a - \theta x]^2, & a \le x \le \frac{a+(1-2\theta)b}{2(1-\theta)}, \\ [\frac{1}{4} + (\theta - \frac{1}{2})^2][(x-a)^2 + (b-x)^2], & \frac{a+(1-2\theta)b}{2(1-\theta)} < x < \frac{(1-2\theta)a+b}{2(1-\theta)}, \\ [\theta x + (1-\theta)b - \frac{a+b}{2}]^2, & \frac{(1-2\theta)a+b}{2(1-\theta)} \le x \le b \end{cases} \tag{2.2}$$

for $0 \le \theta \le \frac{1}{2}$, and

$$I(\theta,x) = \begin{cases} [\frac{a+b}{2} - \theta a - (1-\theta)x]^2, & a \le x \le \frac{a+(2\theta-1)b}{2\theta}, \\ [\frac{1}{4} + (\theta - \frac{1}{2})^2][(x-a)^2 + (b-x)^2], & \frac{a+(2\theta-1)b}{2\theta} < x < \frac{(2\theta-1)a+b}{2\theta}, \\ [(1-\theta)x + \theta b - \frac{a+b}{2}]^2, & \frac{(2\theta-1)a+b}{2\theta} \le x \le b \end{cases} \tag{2.3}$$

for $\frac{1}{2} < \theta \leq 1$.

Proof. Integrating by parts produces the identity

$$\int_a^b K(x,t)\,du(t) = (1-\theta)(b-a)u(x) + \theta(b-a)\frac{u(a)+u(b)}{2} - \int_a^b u(t)\,dt \tag{2.4}$$

where

$$K(x,t) = \begin{cases} t - [a + \theta\frac{b-a}{2}], & t \in [a,x], \\ t - [b - \theta\frac{b-a}{2}], & t \in (x,b], \end{cases} \tag{2.5}$$

Moreover,

$$\frac{1}{b-a}\int_a^b K(x,t)\,dt = (1-\theta)(x - \frac{a+b}{2}). \tag{2.6}$$

Applying the Grüss type inequality (1.3) by associating $h(t)$ with $K(x,t)$ gives

$$\begin{aligned} &|\textstyle\int_a^b K(x,t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b K(x,t)\,dt| \\ &\leq \textstyle\frac{L-l}{2}\int_a^b |K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds|\,dt. \end{aligned}$$

Then for any fixed $x \in [a,b]$ we can derive from (2.4), (2.5) and (2.6) that

$$\begin{aligned} &|\textstyle\int_a^b u(t)\,dt - (b-a)[(1-\theta)u(x) + \theta\frac{u(a)+u(b)}{2}] \\ &+(1-\theta)(x - \frac{a+b}{2})[u(b)-u(a)]| \leq \frac{L-l}{2}I(\theta,x), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} I(\theta,x) &= \textstyle\int_a^x |t - [a + \theta\frac{b-a}{2}] - (1-\theta)(x - \frac{a+b}{2})|\,dt \\ &+ \textstyle\int_x^b |t - [b - \theta\frac{b-a}{2}] - (1-\theta)(x - \frac{a+b}{2})|\,dt \\ &= \textstyle\int_a^x |t - [(1-\theta)x + \theta b - \frac{b-a}{2}]|\,dt \\ &+ \textstyle\int_x^b |t - [\theta a + (1-\theta)x + \frac{b-a}{2}]|\,dt. \end{aligned}$$

The last two integrals can be calculated as follows:

For brevity, we put

$$p_1(t) := t - [(1-\theta)x + \theta b - \frac{b-a}{2}],\ \ t \in [a,x],$$

$$p_2(t) := t - [\theta a + (1-\theta)x + \frac{b-a}{2}],\ \ t \in [x,b]$$

and denote

$$t_1 = (1-\theta)x + \theta b - \frac{b-a}{2},\ \ t_2 = \theta a + (1-\theta)x + \frac{b-a}{2}.$$

It is clear that both $p_1(t)$ and $p_2(t)$ are strictly increasing on $[a,x]$ and $[x,b]$ respectively. Moreover, we have

$$p_1(a) = (1-\theta)(b-x) - \frac{b-a}{2},\ \ p_1(x) = \frac{b-a}{2} - \theta(b-x);$$

$$p_2(x) = \theta(x-a) - \frac{b-a}{2},\ \ p_2(b) = \frac{b-a}{2} - (1-\theta)(x-a).$$

For $0 \le \theta \le \frac{1}{2}$, it is immediate that $p_1(x) > 0$ and $p_2(x) < 0$. Meanwhile, $p_1(a) \ge 0$ if $x \in [a, \frac{a+(1-2\theta)b}{2(1-\theta)}]$ and $p_1(a) < 0$ if $x \in (\frac{a+(1-2\theta)b}{2(1-\theta)}, b]$, $p_2(b) \le 0$ if $x \in [\frac{(1-2\theta)a+b}{2(1-\theta)}, b]$ and $p_2(b) > 0$ if $x \in [a, \frac{(1-2\theta)a+b}{2(1-\theta)})$.

Noticed that $\frac{a+(1-2\theta)b}{2(1-\theta)} \le \frac{(1-2\theta)a+b}{2(1-\theta)}$, there are three possible cases to be determined.

In case $x \in [a, \frac{a+(1-2\theta)b}{2(1-\theta)}]$, $p_1(t) \ge 0$ for $t \in [a,x]$ and $p_2(b) > 0$ with $t_2 \in (x,b)$ such that $p_2(t_2) = 0$. We have

$$\begin{aligned} I(\theta,x) &= \textstyle\int_a^x (t-t_1)\,dt + \int_x^{t_2}(t_2-t)\,dt + \int_{t_2}^b (t-t_2)\,dt \\ &= \tfrac{(1-2\theta)(x-a)(b-x)}{2} + \theta(\theta-1)(x-a)^2 + \tfrac{(x-a)^2+(b-x)^2}{4} \\ &= [\tfrac{1}{2}(b-x) + (\tfrac{1}{2}-\theta)(x-a)]^2 \\ &= [\tfrac{a+b}{2} - (1-\theta)a - \theta x]^2. \end{aligned} \tag{2.8}$$

In case $x \in (\frac{a+(1-2\theta)b}{2(1-\theta)}, \frac{(1-2\theta)a+b}{2(1-\theta)})$, $p_1(a) < 0$ with $t_1 \in (a,x)$ such that $p_1(t_1) = 0$ and $p_2(b) > 0$ with $t_2 \in (x,b)$ such that $p_2(t_2) = 0$. We have

$$\begin{aligned} I(\theta,x) &= \int_a^{t_1}(t_1-t)\,dt + \int_{t_1}^{x}(t-t_1)\,dt + \int_x^{t_2}(t_2-t)\,dt + \int_{t_2}^{b}(t-t_2)\,dt \\ &= [\tfrac{1}{4} + (\tfrac{1}{2}-\theta)^2][(x-a)^2 + (b-x)^2]. \end{aligned} \tag{2.9}$$

In case $x \in [\frac{(1-2\theta)a+b}{2(1-\theta)}, b]$, $p_1(a) < 0$ with $t_1 \in (a,x)$ such that $p_1(t_1) = 0$ and $p_2(t) \le 0$ for $t \in [x,b]$. We have

$$\begin{aligned} I(\theta,x) &= \int_a^{t_1}(t_1-t)\,dt + \int_{t_1}^{x}(t-t_1)\,dt + \int_x^{b}(t_2-t)\,dt \\ &= \tfrac{(1-2\theta)(x-a)(b-x)}{2} + \theta(\theta-1)(b-x)^2 + \tfrac{(x-a)^2+(b-x)^2}{4} \\ &= [\tfrac{1}{2}(x-a) + (\tfrac{1}{2}-\theta)(b-x)]^2 \\ &= [\theta x + (1-\theta)b - \tfrac{a+b}{2}]^2. \end{aligned} \tag{2.10}$$

For $\frac{1}{2} < \theta \le 1$, it is immediate that $p_1(a) < 0$ and $p_2(b) > 0$. Meanwhile, $p_1(x) \le 0$ if $x \in [a, \frac{a+(2\theta-1)b}{2\theta}]$ and $p_1(x) > 0$ if $x \in [\frac{a+(2\theta-1)b}{2\theta}, b]$, $p_2(x) \ge 0$ if $x \in [\frac{(2\theta-1)a+b}{2\theta}, b]$ and $p_2(x) < 0$ if $x \in [a, \frac{(2\theta-1)a+b}{2\theta})$.

Noticed that $\frac{a+(2\theta-1)b}{2\theta} \le \frac{(2\theta-1)a+b}{2\theta}$, there are three possible cases to be determined.

In case $x \in [a, \frac{a+(2\theta-1)b}{2\theta}]$, $p_1(t) \le 0$ for $t \in [a,x]$ and $p_2(x) < 0$ with $t_2 \in (x,b)$ such that $p_2(t_2) = 0$. We have

$$\begin{aligned} I(\theta,x) &= \int_a^{x}(t_1-t)\,dt + \int_x^{t_2}(t_2-t)\,dt + \int_{t_2}^{b}(t-t_2)\,dt \\ &= \tfrac{(2\theta-1)(x-a)(b-x)}{2} + \theta(\theta-1)(x-a)^2 + \tfrac{(x-a)^2+(b-x)^2}{4} \\ &= [\tfrac{1}{2}(b-x) + (\theta-\tfrac{1}{2})(x-a)]^2 \\ &= [\tfrac{a+b}{2} - \theta a - (1-\theta)x]^2. \end{aligned} \tag{2.11}$$

In case $x \in (\frac{a+(2\theta-1)b}{2\theta}, \frac{(2\theta-1)a+b}{2\theta})$, $p_1(x) > 0$ with $t_1 \in (a,x)$ such that $p_1(t_1) = 0$ and $p_2(x) < 0$ with $t_2 \in (x,b)$ such that $p_2(t_2) = 0$. We have

$$
\begin{aligned}
I(\theta, x) &= \int_a^{t_1}(t_1-t)\,dt + \int_{t_1}^{x}(t-t_1)\,dt + \int_x^{t_2}(t_2-t)\,dt + \int_{t_2}^{b}(t-t_2)\,dt \\
&= [\tfrac{1}{4}+(\theta-\tfrac{1}{2})^2][(x-a)^2+(b-x)^2].
\end{aligned} \tag{2.12}
$$

In case $x \in [\frac{(2\theta-1)a+b}{2\theta}, b]$, $p_1(x) > 0$ with $t_1 \in (a, x)$ such that $p_1(t_1) = 0$ and $p_2(t) \geq 0$ for $t \in [x, b]$. We have

$$
\begin{aligned}
I(\theta, x) &= \int_a^{t_1}(t_1-t)\,dt + \int_{t_1}^{x}(t-t_1)\,dt + \int_x^{b}(t-t_2)\,dt \\
&= \tfrac{(2\theta-1)(x-a)(b-x)}{2} + \theta(\theta-1)(b-x)^2 + \tfrac{(x-a)^2+(b-x)^2}{4} \\
&= [\tfrac{1}{2}(x-a)+(\theta-\tfrac{1}{2})(b-x)]^2 \\
&= [(1-\theta)x+\theta b-\tfrac{a+b}{2}]^2.
\end{aligned} \tag{2.13}
$$

Consequently, the inequality (2.1) with (2.2) and (2.3) follows from (2.7), (2.8), (2.9), (2.10), (2.11), (2.12) and (2.13).

The Proof is completed.

Remark 1. It is not difficult to prove that the inequality (2.1) with (2.2) and (2.3) is sharp in the sense that we can construct the function u to attain the equality in (2.1) with (2.2) and (2.3).

Indeed, if $0 \leq \theta \leq \frac{1}{2}$ then we may choose u such that

$$
u(t) = \begin{cases} L(t-a), & a \leq t < x, \\ l(t-x)+(x-a)L, & x \leq t < t_2, \\ L(t-t_2+x-a)+(t_2-x)l, & t_2 \leq t \leq b \end{cases}
$$

for any $x \in [a, \frac{a+(1-2\theta)b}{2(1-\theta)}]$, and

$$u(t)=\begin{cases} l(t-a), & a\le t<t_1,\\ L(t-t_1)+(t_1-a)l, & t_1\le t<x,\\ l(t-x+t_1-a)+(x-t_1)L, & x\le t<t_2\\ L(t-t_2+x-t_1)+(t_2-x+t_1-a)l, & t_2\le t\le b\end{cases}$$

for any $x\in(\frac{a+(1-2\theta)b}{2(1-\theta)},\frac{(1-2\theta)a+b}{2(1-\theta)})$, and

$$u(t)=\begin{cases} l(t-a), & a\le t<t_1,\\ L(t-t_1)+(t_1-a)l, & t_1\le t<x,\\ l(t-x+t_1-a)+(x-t_1)L, & x\le t\le b\end{cases}$$

for any $x\in[\frac{(1-2\theta)a+b}{2(1-\theta)},b]$, and if $\frac{1}{2}<\theta\le 1$ then we may choose u such that

$$u(t)=\begin{cases} l(t-a), & a\le t<t_2,\\ L(t-t_2)+(t_2-a)l, & t_2\le t\le b\end{cases}$$

for any $x\in[a,\frac{a+(2\theta-1)b}{2\theta}]$, and

$$u(t)=\begin{cases} l(t-a), & a\le t<t_1,\\ L(t-t_1)+(t_1-a)l, & t_1\le t<x,\\ l(t-x+t_1-a)+(x-t_1)L, & x\le t<t_2\\ L(t-t_2+x-t_1)+(t_2-x+t_1-a)l, & t_2\le t\le b\end{cases}$$

for any $x\in(\frac{a+(2\theta-1)b}{2\theta},\frac{(2\theta-1)a+b}{2\theta})$, and

$$u(t)=\begin{cases} l(t-a), & a\le t<t_1,\\ L(t-t_1)+(t_1-a)l, & t_1\le t\le b\end{cases}$$

for any $x\in[\frac{(2\theta-1)a+b}{2\theta},b]$.

It is clear that the above all $u(t)$ satisfy the condition of the Theorem.

Remark 2. Let $\theta=0$. Then for all $x\in[a,,b]$, we have

$$|\int_a^b u(t)\,dt-(b-a)u(x)+(x-\frac{a+b}{2})[u(b)-u(a)]|\le\frac{(L-l)(b-a)^2}{8}. \tag{2.14}$$

It should be noted that (2.14) is a sharp Ostrowski-Grüss type inequality with a uniform bound independent of x, and in particular, if we choose in (2.14), $x=\frac{a+b}{2}$, we get a sharp midpoint inequality

$$|\int_a^b u(t)\,dt-(b-a)u(\frac{a+b}{2})|\le\frac{(L-l)(b-a)^2}{8}.$$

Remark 3. Let $\theta=1$. Then we get a sharp trapezoid inequality

$$|\int_a^b u(t)\,dt-\frac{b-a}{2}[u(a)+u(b)]|\le\frac{(L-l)(b-a)^2}{8}.$$

Remark 4. Let $\theta=\frac{1}{2}$. Then for all $x\in[a,b]$, we have

(2.15)
$$|\int_a^b u(t)\,dt-\frac{1}{2}[(b-a)u(x)+(x-a)u(a)+(b-x)u(b)]|\le\frac{L-l}{8}[(x-a)^2+(b-x)^2].$$

Remark 5. Let $x=\frac{a+b}{2}$. Then for all $\theta\in[0,1]$, we have

(2.16)
$$|\int_a^b u(t)\,dt-(b-a)[(1-\theta)u(\frac{a+b}{2})+\theta\frac{u(a)+u(b)}{2}]|\le\frac{(L-l)(b-a)^2}{4}[\frac{1}{4}+(\theta-\frac{1}{2})^2].$$

It should be noted that taking $x = \frac{a+b}{2}$ in (2.15) or $\theta = \frac{1}{2}$ in (2.16) is equivalent to taking both these values in (2.1) which produces a sharp averaged midpoint-trapezoid inequality as

$$|\int_a^b u(t)\,dt - \frac{b-a}{4}[u(a) + 2u(\frac{a+b}{2}) + +u(b)]| \leq \frac{(L-l)(b-a)^2}{16}. \tag{2.17}$$

Remark 6. Let $\theta = \frac{1}{3}$. Then for all $x \in [a, b]$, we have

$$|\int_a^b u(t)\,dt - \frac{b-a}{4}[u(a)+4u(x)++u(b)] + \frac{2}{3}(x - \frac{a+b}{2})[u(b)-u(a)]| \leq \frac{L-l}{2} I(\frac{1}{3}, x), \tag{2.18}$$

where

$$I(\frac{1}{3}, x) = \begin{cases} \frac{1}{36}[(x-a)+3(b-x)]^2, & a \leq x \leq \frac{3a+b}{4}, \\ \frac{5}{18}[(x-a)^2+(b-x)^2], & \frac{3a+b}{4} < x < \frac{a+3b}{4}, \\ \frac{1}{36}[3(x-a)+(b-x)]^2, & \frac{3a+b}{4} \leq x \leq b. \end{cases} \tag{2.19}$$

It should be noted that (2.18) with (2.19) is a sharp generalized Simpson type inequality for unprescribed x, and in particular, if we choose in (2.18) and (2.19), $x = \frac{a+b}{2}$, we get a sharp Simpson inequality

$$|\int_a^b u(t)\,dt - \frac{b-a}{6}[u(a) + 4u(\frac{a+b}{2}) + +u(b)]| \leq \frac{5(L-l)(b-a)^2}{72}. \tag{2.20}$$

It is interesting to note that from (2.17) and (2.20) we can conclude that the averaged midpoint-trapezoid quadrature rule has a better estimation of error than the well-known Simpson quadrature rule.

Corollary 2.1. *Let $u : [a, b] \to \mathbf{R}$ be L-lipschitzian on $[a, b]$. Then for all $x \in [a, b]$, we have*

$$|\int_a^b u(t)\,dt-(b-a)[(1-\theta)u(x)+\theta\frac{u(a)+u(b)}{2}]+(1-\theta)(x-\frac{a+b}{2})[u(b)-u(a)]| \leq LI(\theta, x),$$

where $I(\theta, x)$ is given in (2.2) and (2.3).

Proof. It is immediate by taking $l = -L$ in Theorem 2.1.

3. APPLICATIONS FOR CUMULATIVE DISTRIBUTION FUNCTIONS

Now we consider some applications for cumulative distribution functions.

Let X be a random variable having the probability density function $f : [a, b] \to \mathbf{R}$ and the cumulative distribution function $F(x) = Pr(X \leq x)$, i.e.,

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b].$$

$E(X)$ is the expectation of X. Then we have

Theorem 3.1. *With the above assumptions and if there exist constants M, m such that*

$$0 \leq m \leq f(t) \leq M$$

for all $t \in [a, b]$, then for all $x \in [a, b]$ we have the inequality

$$(3.1)\quad |E(X)-b+(b-a)[(1-\theta)Pr(X \leq x)+\frac{\theta}{2}]+(1-\theta)(x-\frac{a+b}{2})| \leq \frac{M-m}{2}I(\theta, x),$$

where $I(\theta, x)$ is given by (2.2) for $0 \leq \theta \leq \frac{1}{2}$ and (2.3) for $\frac{1}{2} \leq \theta \leq 1$, respectively.

Proof. It is easy to find that the function $F(x) = \int_a^x f(t)\,dt$ is (m, M)-Lipschitzian on $[a, b]$. So, by Theorem 1 we get

$$|\int_a^b F(t)\,dt - (b-a)[(1-\theta)F(x) + \theta\frac{F(a)+F(b)}{2}]$$
$$+(1-\theta)(x-\frac{a+b}{2})[F(b)-F(a)]| \leq \frac{M-m}{2} I(\theta,x)$$

with (2.2) and (2.3).

As $F(a)=0$, $F(b)=1$ and

$$\int_a^b F(t)\,dt = b - E(X),$$

then we can easily deduce the inequality (3.1).

Corollary 3.1. *Under the assumption of Theorem 3.1, we have*

$$|E(X) - \frac{a+b}{2}| \leq \frac{M-m}{8}(b-a)^2. \tag{3.2}$$

Proof. We set $\theta = 1$ in (3.1) to get (3.2).

Remark 7. It should be noted that the inequality (3.2) improves the inequality (5.4) in [1].

Corollary 3.2. *Under the assumption of Theorem 3.1, we have*

$$|E(X) + (b-a)Pr(X \leq x) - x - \frac{b-a}{2}| \leq \frac{M-m}{8}(b-a)^2. \tag{3.3}$$

for all $x \in [a,b]$.

Proof. We set $\theta = 0$ in (3.1) to get (3.3).

Remark 8. It should be noted that the inequality (3.3) improves the inequality (5.18) in [1].

If in (3.3), we choose $x = a$ or $x = b$, then we recapture the inequality (3.2). If in (3.3), we choose $x = \frac{a+b}{2}$, then we get the inequality

$$|E(X) + (b-a)Pr(X \leq \frac{a+b}{2}) - b| \leq \frac{M-m}{8}(b-a)^2. \tag{3.4}$$

The inequality (3.4) is an improvement of inequality (5.21) in [1].

Corollary 3.3. *Under the assumption of Theorem 3.1, we have*

$$|E(X) + \frac{b-a}{2}Pr(X \leq x) - \frac{b+x}{2}| \leq \frac{M-m}{4}[(x - \frac{a+b}{2})^2 + \frac{(b-a)^2}{4}] \tag{3.5}$$

for all $x \in [a, b]$.

Proof. We set $\theta = \frac{1}{2}$ in (3.1) to get (3.5).

Remark 9. It should be noted that the inequality (3.5) improves the inequality (5.22) in [1].

If in (3.5), we choose $x = a$ or $x = b$, then we recapture the inequality (3.2). If in (3.5), we choose $x = \frac{a+b}{2}$, then we get the inequality

$$|E(X) + \frac{b-a}{2}Pr(X \leq \frac{a+b}{2}) - \frac{a+3b}{4}| \leq \frac{M-m}{16}(b-a)^2. \tag{3.6}$$

The inequality (3.6) is an improvement of inequality (5.30) in [1].

Corollary 3.4. *Under the assumption of Theorem 3.1, we have*

$$|Pr(X \leq \frac{a+b}{2}) - \frac{1}{2}| \leq \frac{M-m}{4}(b-a). \tag{3.7}$$

Proof. Using the triangle inequality, we get

$$
\begin{aligned}
& |Pr(X \leq \tfrac{a+b}{2}) - \tfrac{1}{2}| \\
= \ & |Pr(X \leq \tfrac{a+b}{2}) - \tfrac{1}{2} + \tfrac{1}{b-a}(E(X) - \tfrac{a+b}{2}) - \tfrac{1}{b-a}(E(X) - \tfrac{a+b}{2})| \\
\leq \ & |Pr(X \leq \tfrac{a+b}{2}) - \tfrac{b-E(X)}{b-a}| + \tfrac{1}{b-a}|E(X) - \tfrac{a+b}{2}|,
\end{aligned}
$$

and then the inequality (3.7) follows from (3.2) and (3.5).

Remark 10. Finally, we would like to point out that Theorem 3.1 provides a unified treatment and different proofs for some previous results due to D. Y. Hwang (see [3], Theorem 1, Theorem 11, Theorem 13).

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