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ON GENERALIZED HERMITE HADAMARD'S INEQUALITY

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ABSTRACT. The object is to construct the log-convex and the exponential convex functions via functional generalization of Hermite Hadamard's inequality for some special classes of continuous functions defined on compact interval in $\mathbb{R}$. Constructed n-exponentially convex functions are used to obtain the generalization of already discovered mean with positive weights p and q, and prove their monotonicity and also introduce several classes of Stolarsky means called Stolarsky type means. Prove Minkowsky type inequalities for new discovered means as applications of Lyponuve type inequalities of constructed log-convex functions.

1. INTRODUCTION

If $f : I \to \mathbb{R}$ is a convex function on I and $a, b \in I$ such that $a < b$, then the following double inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequality in (1.1) is due to C.Hemite, he obtained it in 1881.
On November 22, 1881, he sent to the journal of elementary mathematics "*Mathesis*", where published in 1883.

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A leading expert in history and complex analysis was not apparently aware of C. Hermite work and wrote that first inequality in (1.1) was proved by Hadamard in 1893.

In 1906, Fejer while studying trigonometric polynomials, obtain inequality which generalize (1.1).

If $\mathrm{w} : [a, b] \to (0, \infty)$ is an integrable function such that $\mathrm{w}(a + b - x) = \mathrm{w}(x)$ and $f : [a, b] \to \mathbb{R}$ is convex, then (see [11, p. 138])

$$f\left(\frac{a+b}{2}\right)\int_a^b \mathrm{w}(x)dx \le \int_a^b f(x)\mathrm{w}(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b \mathrm{w}(x)dx. \tag{1.2}$$

Obviously, for $\mathrm{w} = 1$ in (1.2), Hermite-Hadamard inequality (1.1) is obtained.

In 1974, D.S Mitronović found (1.1) in "*Mathesis*" as a short note. Due to these historical facts, (1.1) referred to as Hermite-Hadamard inequality (see [14, p.137].

It is very important to observe that inequalities (1.1) give upper and lower bounds for the average of conve x function f.

In 1976, Vasić and Lacković [17] and Lupas [11] (see also [14, 12]) obtained a generalization of Hermite Hadamard inequality

Theorem 1.1. ([14, Theorem 5.11]) *Assume that p, q be positive real numbers and a, $b, \in \mathbb{R}$ such that $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous convex function, then the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \le \frac{1}{2y}\int_{\tau-y}^{\tau+y} f(x)dx \le \frac{pf(a)+qf(b)}{p+q}, \tag{1.3}$$

hold for if and only if $\tau = (pa + qb)/(p + q)$ and

$$0 < y \le \frac{b-a}{p+q}\min\{p, q\}. \tag{1.4}$$

Proof. See [14, p. 143]. □

In 1991, Brenner and Alzer [5] obtained the following generalization of Theorem 1.1, which is in fact a Fejer type variant:

Theorem 1.2. [14] *If* $\mathrm{w} : [a,b] \to [0,\infty)$ *is integrable and symmetric with respect to* $\tau = (pa+qb)/(p+q)$ *with positive numbers* p *and* q, y *(*y *is given in* (1.4)*) and* $f : [a,b] \to \mathbb{R}$ *is convex function,then*

(1.5)
$$f\left(\frac{pa+qb}{p+q}\right)\int_{\tau-y}^{\tau+y} \mathrm{w}(x)dx \le \int_{\tau-y}^{\tau+y} f(x)\mathrm{w}(x)dx \le \frac{pf(a)+qf(b)}{p+q}\int_{\tau-y}^{\tau+y} \mathrm{w}(x)dx.$$

In 1986, Pečarić and Beesack [3] (see also [14]) generalized Theorem 1.1 for isotonic normalized linear functionals .

In order to state the result, we need to define some notations.

Let E be a nonempty set and let L be a class of functions $f : E \to \mathbb{R}$ having the properties:

(L_1) $\alpha f + \beta g \in L, \forall \alpha, \beta \in \mathbb{R} \forall f, g \in L$.

(L_2) $1 \in L$, that is if $f(t) = 1$ for $t \in E$, then $f \in L$.

A function $A : L \to \mathbb{R}$ is a positive (isotonic) linear functional if it satisfies the properties:

(A_1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g), \forall f, g \in L,\ \forall \alpha, \beta \in \mathbb{R}$

(A_2) $A(f) \ge 0\ \forall f \in L,\ f(t) \ge 0$ on E (A is positive or isotonic).

If additionally the condition $A(1) = 1$ is satisfied, we say that A is positive normalized lineal functional on L.

Theorem 1.3. [14, Theorem 5.13] *Let* L *satisfying* (L_1) *and* (L_2) *and let* A *be positive normalized linear functional. If* $f : T \supseteq [a,b] \to \mathbb{R}$, $a < b$ *is continuous convex function and* $g \in L$ *such that* $f(g) \in L$ *then*

$$f\left(\frac{pa+qb}{p+q}\right) \le A(f(g)) \le \frac{pf(a)+qf(b)}{p+q}. \tag{1.6}$$

holds for $p = p_g,\ q = q_g$ *are nonnegative real numbers (with* $p+q>0$ *such that*

$$A(g) = \frac{pa+qb}{p+q}.$$

Remark 1. (i) Note that Theorem 1.2, (thus Theorem 1.1 for $\mathrm{w} = 1$) can be obtained as a special case of Theorem 1.3. Namely, for given p and q, τ and w as in Theorem 1.2 and y satisfying (1.4) such that $\int_{\tau-y}^{\tau+y} \mathrm{w}(x)dx \neq 0$, define $E = [a, b]$, $L = B(E)$ (space of all bounded, Remain integrable functions on $[a, b]$), $g = e_1$ and

$$A(f(e_1)) = \frac{1}{\overline{\mathrm{W}}} \int_{\tau-y}^{\tau+y} \mathrm{w}(x) f(x) dx. \tag{1.7}$$

Observe that A is a normalized isotonic linear functional and

$$A(g) = A(e_1) = \frac{1}{\overline{\mathrm{W}}} \int_{\tau-y}^{\tau+y} x\mathrm{w}(x) dx = \tau = \frac{pa + qb}{p + q}.$$

(ii) Let $E = [a, b] \times [a, b]$, $L = B(E)$ (space of all bounded, integrable functions on $[a, b] \times [a, b]$) , $g = e_1$ and normalized isotonic linear functional

$$A(f) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y) dx dy. \tag{1.8}$$

satisfying

$$A(g) = A(e_1) = \frac{1}{(b-a)^2} \int_a^b \int_a^b (tx + (1-t)y) dx dy = \frac{a+b}{2}.$$

(iii) As a special $E = [a, b]$, $L = B(E)$ (space of all bounded, integrable functions on $[a, b]$) , $g = e_1$ and

$$A(f) = \frac{1}{2(b-a)} \int_a^b \left[f(ta + (1-t)x) + f(tb + (1-t)y) \right] dx. \tag{1.9}$$

It is easy to verify that A is a normalized isotonic linear functional and

$$A(g) = A(e_1) = \frac{1}{2(b-a)} \int_a^b (t(a+b)(1-t)(x+y)) dx = \frac{a+b}{2}.$$

Let L satisfying (L_1) and (L_2) and . If $f : T \supseteq [a, b] \to \mathbb{R}, : a < b$ is continuous function, $g \in L$ such that $a \leq g(t) \leq b$ on $t \in E$ and $f(g) \in L$ and $A : L \to \mathbb{R}$ is positive normalized linear functional such that

$$A(g) = \frac{pa + qb}{p + q}.$$

Let us define $H_i : C([a,b]) \to \mathbb{R}$ two linear functionals as follows:

$$H_1(f) = \frac{pf(a)+qf(b)}{p+q} - A(f(g)), \tag{1.10}$$

$$H_2(f) = A(f(g)) - f\left(\frac{pa+qb}{p+q}\right), \tag{1.11}$$

satisfy the property

$$f \in C[a,b] \text{ is convex } \Rightarrow H_i(f) \geq 0. \tag{1.12}$$

In 1975, Stolarsky [16] (see also [2, 14, 12]) consider Cauchy mean value theorem and applied functions $x \mapsto x^u$ and $x \mapsto x^v$ (u, v are nonzero real numbers) on interval $[a,b]$ (a, b are positive reals) to produces

$$\xi = \left(\frac{u}{v}\frac{b^v - a^v}{b^u - a^u}\right)^{1/(v-u)}$$

and $a < \xi < b$, thus $\xi = E_{u,v}(a,b)$, $uv(u-v) \neq 0$ is mean of a and b. He showed that this mean can be extended continuously and get the form :

$$E_{u,v}(a,b) = \begin{cases} \left(\frac{u}{v}\frac{b^v-a^v}{b^u-a^u}\right)^{1/(v-u)}, & vu(v-u) \neq 0, \\ \left(\frac{1}{u}\frac{b^u-a^u}{\log b - \log a}\right)^{1/u}, & u \neq 0,\ v = 0, \\ \exp\left(-\frac{1}{u} + \frac{b^u \log b - a^u \log a}{b^u - a^u}\right), & v = u \neq 0, \\ \sqrt{ab}, & v = u = 0. \end{cases} \tag{1.13}$$

This mean is called the Storalsky mean.

Stolarsky proved that the function $E_{r,s}(a,b)$ is increasing in both parameters r and s, that is for $r \geq p$ and $s \geq q$:

$$E_{r,s}(a,b) \geq E_{p,q}(a,b). \tag{1.14}$$

These means, since their invention, are generalized in various directions. However, in 2010 [2] Stolarsky means are recognized as application of the linear functional

$$f \mapsto \frac{f(x)-f(y)}{x-y},\ x \neq y \tag{1.15}$$

on the family of functions $\{\varphi_r : r \in \mathbb{R}\}$ (defined on $(0,\infty)$)

$$\varphi_r(x) = \begin{cases} x^r/r, & r \neq 0; \\ \log x, & r = 0. \end{cases} \tag{1.16}$$

Since functional defined above is nonnegative on monotonically increasing functions, and $\frac{d\varphi_r}{dx}(x) = x^{r-1} \geq 0,\ r \in \mathbb{R}$, then using Cauchy mean-value theorem and log-convexity we get construction and monotonicity property of Stolarsky means, as is showed in [2]. In that paper, this idea is further extended via application of functionals defined by differences of (1.1) on a family of convex functions $\{f_r : r \in \mathbb{R}\}$ (defined on $(0,\infty)$

$$f_r(x) = \begin{cases} \frac{x^r}{r(r-1)} & r \neq 1, 0, \\ -\ln x & r = 0, \\ x \ln x & r = 1, \end{cases} \tag{1.17}$$

another two classes of Stolarsky type means are constructed and monotonicity property is proved again using log-convexity.

In [12], weighted version of Stolarsky type means form [2] is obtained and three classes of monotonous generalized Stolarsky type meas.

In this paper we generalize means from [12] (thus from [2])in several directions. First, Hermite-Hadamard functionals are generalized through generalized Hermite-Hadamard inequality (1.6) (thus (1.5)). Second, these functionals are applied on new families (aside of (1.17)) of convex functions which give us quite different means. Third, it is showed that log-convexity can be shifted on finer classes such as n-exponentially convex and exponentially convex functions. Also, our approach give us non-trivial examples of exponentially convex functions.

2. Mean Value Theorems

Theorem 2.1. *Let $f \in C^2([a,b])$ and $H_i\,(i=1,2)$ be non-negative linear functionals defined as in* (1.10) *and* (1.11)*. Then there exists $\xi \in [a,a]$ such that*

$$H_i(f) = \frac{f''(\xi)}{2} H_i(e_2),\ i=1,2, \tag{2.1}$$

where $e_2 = x^2$.

Proof. Since $f \in C^2([a,b])$, Wiestrass Theorem yields that $m = \min_{x\in[a,b]} f''(x),\ M = \max_{x\in[a,b]} f''(x)$. Let us observe that functions $\varphi_1(x) = f(x) - m\frac{x^2}{2} = f(x) - mg_2(x)$ and $\varphi_2(x) = M\frac{x^2}{2} - f(x) = Mg_2(x) - f(x)$ are convex.The condition (1.12) implies that $H_i(\varphi_j) \geq 0,\ i=1,2,\ j=1,2.$ We conclude that

$$\frac{m}{2} H_i(e_2) \leq H_i(f) \leq \frac{M}{2} H_i(e_2),\ i=1,2.$$

From this, we have (2.1). □

Theorem 2.2. *Let $f,\, h \in C^2([a,b])$ such that second derivative of h is nonzero on $[a,b]$ and $H_i,(i=1,2)$ linear functionals defined as in* (1.10)-(1.11)*. Then there exits $\xi \in [a,a]$ such that*

$$\frac{f''(\xi)}{h''(\xi)} = \frac{H_i(f)}{H_i(h)}. \tag{2.2}$$

Proof. We prove the claim for the functional H_1. Let us define $\varphi \in C^2([a,a])$ by

$$\varphi(x) = ch(x) - df(x),$$

where

$$c = H_1(f)$$

and

$$d = H_1(h).$$

Apply Theorem 2.1 on $f = \varphi$ to obtaine

$$0 = \left(c\frac{h''(\xi)}{2} - d\frac{f''(\xi)}{2} \right) H_1(e_2).$$

This applies that

$$\frac{f''(\xi)}{h''(\xi)} = \frac{c}{d}$$

which is the claim. □

Remark 2. If the function $\frac{f''}{h''}$ inverse on $[a, b]$, then Theorem 2.2 enables us to define various types of means that is, from (2.2) we have

$$\xi = \left(\frac{f''}{h''} \right)^{-1} \left(\frac{H_i(f)}{H_i(h)} \right).$$

The number $a \leq \xi \leq b$ we call mean on $[a, b]$.

3. n-Exponential convexity related to Hermite-Hadamard's differences

In 1929 Bernstein [4] introduced exponentially convex functions and identify as a subclass of convex functions (also subclass of log-convex functions) on a given open interval. General results about exponential convexity can be found in [1] and [7].
In 2011 J.Pečarić and J.Perić [10] introduced n-exponentially convex functions.

Definition 3.1. For arbitrary natural number n, a function $f : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\tfrac{u_i+u_j}{2}\right) \geq 0$$

holds for all choices of $\xi_i \in \mathbb{R},\ u_i \in I,\ i = 1, \ldots, n$.

A function $f : I \to \mathbb{R}$ is $n-$exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 3. Noted that 1-exponentially convex functions in the Jensen sense are non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N},\ n \geq k$.

Proposition 3.1. *If f is n-exponentially convex in the Jensen sense on I then the matrix*

$$\left[f\left(\frac{u_i + u_j}{2} \right) \right]_{i,j=1}^{n}$$

is positive semi-definite. By Gramm's inequality

$$\det \left[f\left(\frac{u_i + u_j}{2} \right) \right]_{i,j=1}^{k} \geq 0,$$

for $1 \leq k \leq n$, and $u_i \in I,\ i = 1, \ldots, k$.

Corollary 3.1. *(i) If $f : I \to (0, \infty)$ is 2-exponentially convex in the Jensen sense then f is a* log-*convex function in the Jensen sense on I.*
(ii) If $f : I \to (0, \infty)$ is 2-exponentially convex then f is a log-convex function on I.

Definition 3.2. A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense if it is n-exponentially convex in the Jensen sense for every $n \in \mathbb{N}$.

A function $f : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Proposition 3.2. [7] *Let $\mathcal{E}$ denote the set of all exponentially convex functions on an open interval J.*

(i) *$\mathcal{E}$ is a convex cone i.e. $\alpha f + \beta h \in \mathcal{E}$ for $f, h \in \mathcal{E}$ and $\alpha,\ \beta \geq 0$.*
(ii) *$\mathcal{E}$ is closed under multiplication i.e. $f \cdot h \in \mathcal{E}$ for $f, h \in \mathcal{E}$.*

Proof. (i)-part follows directly from the definition. (ii)-part is a consequence of the next theorem (see [7]). □

One of the main aspects of exponentially convex functions is its integral representation given in [1, p.211].

Theorem 3.1. *(*see [7]*) The function $f : I \to \mathbb{R}$ is exponentially convex if and only if*

$$f(x) = \int_{-\infty}^{\infty} e^{tu} d\sigma(t), \quad u \in I \tag{3.1}$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

A nontrivial example of exponentially convex as a consequence of above Theorem is given below.

Example 3.1. *For every $\alpha > 0$ the function $f : (0, \infty) \to \mathbb{R}$ defined as*

$$f(u) = u^{-\alpha}$$

is exponentially convex on $(0, \infty)$, since

$$u^{-\alpha} = \int_{-\infty}^{\infty} e^{-ut} d\left(\frac{-(-t)^{\alpha}}{\alpha\Gamma(\alpha)} 1_{(-\infty,0)}(t)\right)$$

(see [7] *and* [15, p.210] *).*

Remark 4. Every exponentially convex function is n-exponentially convex by definition. Converse is not generally true, since for example $f(x) = e^{x^3 - x}$ is 2-exponentially convex on $(0, 1)$ and not exponentially convex function on $(0, 1)$ (see [7] for details).

Definition 3.3. The second order divided difference of a function $f : [a, b] \to \mathbb{R}$ at mutually different points $u_0, u_1, u_2 \in [a, b]$ is defined recursively by

$$\begin{aligned} f[u_i] &= f(u_i),\ i = 0, 1, 2 \\ f[u_i, u_{i+1}] &= \frac{f(u_{i+1}) - f(u_i)}{u_{i+1} - u_i} :, i = 0, 1 \\ f[u_0, u_1, u_2] &= \frac{f[u_1, u_2] - f[u_0, u_1]}{u_2 - u_0}. \end{aligned} \tag{3.2}$$

The value $f[u_0, u_1, u_2]$ is independent of the order of the points u_0, u_1 and u_2 . This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $u_1 \to u_0$ in (3.2), we get

$$\lim_{u_1 \to u_0} f[u_0, u_1, u_2] = f[u_0, u_0, u_2] = \frac{f(u_2) - f(u_0) - f'(u_0)(u_2 - u_0)}{(u_2 - u_0)^2}$$

provided f'exists, and furthermore, taking the limits $u_i \to u_0, : i = 1, 2$ in (3.2), we get

$$\lim_{u_2 \to u_0} \lim_{u_1 \to u_0} f[u_0, u_1, u_2] = f[u_0, u_0, u_0] = \frac{f''(u_0)}{2}$$

provided that f'' exists.

Theorem 3.2. *Let* $\mathbf{F} = \{f_u : [a, b] \to \mathbb{R},\ u \in I\}$, *be a family of functions from* $C([a, b])$, *such that* $u \mapsto f_u[x_0, x_1, x_2]$ *is* n*-exponentially convex in the Jensen sense on* I *for every triplet of distinct points* $x_0,\ x_1,\ x_2 \in I$. *If* $H_i,\ i = 1, 2$ *are linear functionals defined as in* (1.10) *and* (1.11), *then the functions* $u \mapsto H_i(f_u)$ *are* n*-exponentially convex in the Jensen sense on* I. *If the function* $u \mapsto H_i(f_u)$ *are continuous on* I, *then it is* n*-exponentially convex on* I.

Proof. We proof the claim for the case $i = 1$. In the cases $i = 2, 3$ the proof is similar. For $\xi_k \in R,\ k = 1, \ldots, n$ and $u_k \in I,\ k = 1, \ldots, n$, we define the function

$$h(u) = \sum_{k,l=1}^{n} \xi_i \xi_j f_{\frac{u_k + u_l}{2}}(x).$$

Using the assumption that the function $u \to f_u[x_0, x_1, x_2]$ is n-exponentially convex in the Jensen sense, we have

$$h[x_0, x_1, x_2] = \sum_{k,l=1}^{n} \xi_i \xi_j f_{\frac{u_k + u_l}{2}}[x_0, x_1, x_2] \geq 0,$$

which in turn implies that h is a convex function on $[a, b]$ and therefore from the condition (1.12) we have

$$H_1(h) = \sum_{k,l=1}^{n} \xi_k \xi_l H_1(f_{\frac{u_k+u_l}{2}}) \geq 0.$$

Hence we conclude that the function $u \mapsto H_1(f_u)$ is n-exponentially convex on I in the Jensen sense. If the function $u \mapsto H_1(f_u)$ is continuous also on I, then $u \mapsto H_1(f_u)$ is n-exponentially convex by definition. □

We derive an immediate consequence of the above Theorem.

Corollary 3.2. *Let* $\mathbf{F} = \{f_u : [a, b] \to \mathbb{R},\ u \in I\}$, *be a family of functions from* $C([a, b])$, *such that* $u \mapsto f_u [x_0, x_1, x_2]$ *is exponentially convex in the Jensen sense on* I *for every triplet of distinct points* $x_0,\ x_1,\ x_2 \in I$. *If* H_i, $i = 1, 2$ *are linear functionals defined as in* (1.10) *and* (1.11), *then the functions* $u \mapsto H_i(f_u)$ *are exponentially convex in the Jensen sense on* I. *If the functions* $u \mapsto H_i(f_u)$ *are continuous on* I, *then it is exponentially convex on* I.

Corollary 3.3. *Let* $\mathbf{F} = \{f_u : [a, b] \to \mathbb{R},\ u \in I\}$, *be a family of functions from* $C([a, b])$, *such that* $u \mapsto f_u [x_0, x_1, x_2]$ *is* log-*convex in the Jensen sense on* I *for every triplet of distinct points* $x_0,\ x_1,\ x_2 \in I$. *If* H_i, $i = 1, 2$ *are linear functionals defined as in* (1.10) *and* (1.11), *then the functions* $u \mapsto H_i(f_u)$ *are* log-*convex in the Jensen sense on* I. *Then the following statements hold:*

(*i*) *If the functions* $u \mapsto H_i(f_u)$ *are continuous on* I, *then it is* log *convex on* I *and for* $t, u, v \in I$ *such that* $t < u < v$ *we have,*

$$(H_i (f_u))^{v-t} \leq (H_i (f_t))^{v-u} (H_i (f_v))^{u-t} . \tag{3.3}$$

(*ii*) *If the functions* $u \mapsto H_i(f_u)$ *are strictly positive and differentiable on* I, *then for every* $u, v, u, t \in I$ *such that* $u \leq r,\ v \leq t$,

$$E_{u,v}(H_i; \mathbf{F}) \leq E_{r,t}(H_i; \mathbf{F}), \tag{3.4}$$

holds, where

$$E_{u,v}(H_i;\mathbf{F}) = \begin{cases} \left(\frac{H_i(f_u)}{H_i(f_v)}\right)^{1/(u-v)}, & u \neq v, \\ \exp\left(\frac{\frac{d}{du}(H_i(f_u))}{H_i(f_u)}\right), & u = v, \end{cases} \tag{3.5}$$

$f_u \in \mathbf{F}$.

Proof. We give prove for $i = 1$.

(i) This is an immediate consequence of Theorem 3.2 and (ii)-part Corollary 3.1 .

(ii) Since by (i) the function $u \mapsto H_1(f_u)$ is log-convex on I, that is, the function $u \mapsto H_1(f_u)$ is convex on I, we get

$$\frac{\log H_1(f_u) - \log H_1(f_u)}{u-v} \leq \frac{\log H_1(f_r) - \log H_1(f_t)}{r-t} \tag{3.6}$$

for $u \leq r, v \leq t, u \neq v, u \neq t$, and there from we conclude that

$$E_{u,v}(H_i;\mathbf{F}) \leq E_{r,t}(H_i;\mathbf{F}).$$

For the case $u = v$, consider $\lim\limits_{v \to u}$ in (3.6) and conclude that

$$E_{u,u}(H_1;\mathbf{F}) \leq E_{r,t}(H_i;\mathbf{F}) \tag{3.7}$$

The case $r = t$ can be treated similarly.

□

Remark 5. Note that the results from Theorem 3.2, Corollary 3.2 and Corollary 3.3 still holds when two of the points x_0, x_1, $x_2 \in [a, b]$ coincides for a family of differentiable functions f_u such that $u \mapsto f_u[x_0, x_1, x_2]$ is n-exponentially convex in the Jensen'sense (exponentially convex in the Jensen'sense), further, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained using Theorem 3.2, Corollaries 3.2, 3.3 and appropriate characterizations of convex functions.

4. Examples And Applications

We define some notations, which will be used in the sequel.

Assume that A is an isotonic linear functional defined on L and $g \in L$ such that $g^u, \log g, g^u \log g \in L, : u \neq 0$. The classical mean of order $u \in \mathbb{R}$ of g is defined as:

$$M_u(A,g) = \begin{cases} (A(g^u))^{1/u}, & u \neq 0, \\ \exp\left(A(\log g)\right), & u = 0, \end{cases} \tag{4.1}$$

We state here some useful conditions for isotonic linear functional A, which will be needed for this section:

$$\lim_{t\to t_0} A(g^t) = A(g^{t_0});$$
$$\lim_{\Delta t\to 0} \frac{A(g^{t+\Delta t}) - A(g^t)}{\Delta t} = \lim_{\Delta t\to 0} \frac{A(g^{t+\Delta t} - g^t)}{\Delta t};$$
$$A\left(\lim_{\Delta t\to 0} \frac{g^t\left(g^{\Delta t}-1\right)}{\Delta t}\right) = A(g^t \log g).$$

Example 4.1. *Let a, b positive real numbers and $I = (0,\infty)$ and a family of functions $\mathbf{F}_1 = \{f_u; u \in I\}$ from $C[a,b]$ defined with*

$$f_u(x) = \frac{e^{-x\sqrt{u}}}{u}.$$

Since $\frac{d^2 f_u}{dx^2}(x) = e^{-x\sqrt{t}} \geq 0$, f_u is convex function on $[a,b]$ for each $u \in I$ and $u \mapsto e^{-x\sqrt{u}}$ is exponentially convex being Laplace transform of nonnegative function (see [7]). Corollary 3.2 yields that $u \mapsto H_i(f_u)$ are exponentially convex functions. Expression corresponding to (3.5):

$$E_{u,t}(H_1;\mathbf{F}_1) = \begin{cases} \left(\frac{H_1(f_u)}{H_1(f_v)}\right)^{1/(u-v)}, & u \neq v, \\ \exp\left(-\frac{1}{u} - \frac{1}{2\sqrt{u}} \frac{\frac{pae^{-a\sqrt{u}}+qbe^{-b\sqrt{u}}}{p+q} - A(ge^{-\sqrt{u}g})}{\frac{pe^{-a\sqrt{u}}+qe^{-b\sqrt{u}}}{p+q} - A(e^{-\sqrt{u}g})}\right), & q = u, \end{cases}$$

$$E_{u,v}(H_2;\mathbf{F}) = \begin{cases} \left(\frac{H_2(f_u)}{H_2(f_v)}\right)^{1/(u-v)}, & u \neq v, \\ \exp\left(-\frac{1}{u} - \frac{1}{2\sqrt{u}} \frac{A(ge^{-\sqrt{u}g}) - \frac{pa+qb}{p+q}\exp\left(-\sqrt{u}\frac{pa+qb}{p+q}\right)}{A(e^{-\sqrt{u}g}) - \exp\left(-\sqrt{u}\frac{pa+qb}{p+q}\right)}\right), & q = u. \end{cases}$$

It is interesting to observe that using Theorem 2.2, these Stolarsky type quotient can be used to defined new class of Stolarsky type means:

$$R_{u,v}(H_i,\mathbf{F}_1) := (\sqrt{u}+\sqrt{v})\log E_{u,v}(H_i,\mathbf{F}_1),\ i=1,2,$$

Example 4.2. *Let a,b positive real numbers and $I=(0,\infty)$ and a family of functions $\mathbf{F}_2=\{f_u; u\in I\}$ from $C[a,b]$ defined with*

$$f_u(x)=\begin{cases}\frac{u^{-x}}{(\log u)^2}, & u\neq 1,\\ \frac{x^2}{2}, & u=1.\end{cases}$$

As $\frac{d^2}{dx^2}f_u(x)=u^{-x}=e^{-x\log u}\geq 0$ shows that f_u is convex function on $[a,b]$ for every $u\in I$ and $u\mapsto \frac{d^2}{dx^2}h_u(x)=u^{-x}$ is exponentially convex by Example 3.1. Corollary 3.2 yields that $u\mapsto H_i(f_u)$ are exponentially convex functions. Stolarsky quotient corresponding to (3.5)*:*

$$E_{u,v}(H_1;\mathbf{F}_2)=\begin{cases}\left(\frac{H_1(f_u)}{H_1(f_v)}\right)^{1/(u-v)}, & u\neq v,\\ \exp\left(-\frac{2}{u\log u}-\frac{1}{u}\frac{\frac{pau^{-a}+qbu^{-b}}{p+q}-A(gu^{-g})}{\frac{pu^{-a}+qu^{-b}}{p+q}-A(u^{-g})}\right), & v=u\neq 1,\\ \exp\left(-\frac{1}{3}\frac{\frac{pa^3+qb^3}{p+q}-M_3^3(A,g)}{\frac{pa^2+qb^2}{p+q}-M_2^2(A,g)}\right), & v=u=1.\end{cases}$$

$$E_{u,v}(H_2;\mathbf{F}_2)=\begin{cases}\left(\frac{H_2(f_u)}{H_2(f_v)}\right)^{1/(u-v)}, & u\neq v,\\ \exp\left(-\frac{2}{u\log u}-\frac{1}{u}\frac{A(gu^{-g})-\frac{pa+qb}{p+q}\exp\left(-\frac{pa+qb}{p+q}\ln u\right)}{A(u^{-g})-\exp\left(-\frac{pa+qb}{p+q}\ln u\right)}\right), & v=u\neq 1,\\ \exp\left(-\frac{1}{3}\frac{M_3^3(A,g)-\left(\frac{pa+qb}{p+q}\right)^3}{M_2^2(A,g)-\left(\frac{pa+qb}{p+q}-\right)^2}\right), & v=u=1,\end{cases}$$

We can convert the above Stolarsky quotient into Stolarsky type means:

$$S_{u,v}(H_i,\mathbf{F}_2) := L(u,v)\log E_{u,v}(H_i;\mathbf{F}_2),\ i=1,2,$$

where $L(u,v)=\frac{u-v}{\log u-\log v}$ $u\neq v$ and $L(u,u)=u$ is the logarithmic mean.

Example 4.3. *Let a, b real numbers and $I = \mathbb{R}$ and a family of functions $\mathbf{F}_3 = \{f_u; u \in I\}$ from $C[a,b]$ defined with*

$$f_u(x) = \begin{cases} \frac{1}{u^2} e^{ux}, & u \neq 0, \\ \frac{1}{2} x^2, & u = 0, \end{cases}$$

We have $\frac{d^2}{dx^2} f_u(x) = e^{ux} > 0$ which shows that f_u is convex on $[a,b]$ for every $u \in I$ and $u \mapsto \frac{d^2}{dx^2} f_u(x) = e^{ux}$ by definition is exponentially convex function on I. Corollary 3.2 yields that $u \mapsto H_i(f_u)$ are exponentially convex functions. Expression corresponding to (3.5)*:*

$$E_{u,v}(H_1; \mathbf{F}_3) = \begin{cases} \left(\frac{H_1(f_u)}{H_1(f_v)}\right)^{1/(u-v)}, & u \neq v, \\ \exp\left(-\frac{2}{u} + \frac{\frac{pae^{ua}+qbe^{ub}}{p+q} - A(g\exp(ug))}{\frac{pe^{ua}+qe^{ub}}{p+q} - M_u^u(A,\exp(g))}\right), & v = u \neq 0, \\ \exp\left(\frac{1}{3} \frac{\frac{pa^3+qb^3}{p+q} - M_3^3(A,g)}{\frac{pa^2+qb^2}{p+q} - M_2^2(A,g)}\right), & v = u = 0. \end{cases}$$

$$E_{u,v}(H_2; \mathbf{F}_3) = \begin{cases} \left(\frac{H_1(f_u)}{H_1(f_v)}\right)^{1/(u-v)}, & u \neq v, \\ \exp\left(-\frac{2}{u} + \frac{A(g\exp(ug)) - \frac{pa+qb}{p+q}\exp\left(u\frac{pa+qb}{p+q}\right)}{M_u^u(A,\exp(g)) - \exp\left(u\frac{pa+qb}{p+q}\right)}\right), & v = u \neq 0, \\ \exp\left(\frac{1}{3} \frac{\frac{pa^3+qb^3}{p+q} - M_3^3(A,g)}{\frac{pa^2+qb^2}{p+q} - M_2^2(A,g)}\right), & v = u = 0, \end{cases}$$

Which shows that $E_{u,v}(H_i; \Omega_1)$, $i = 1, 2$ is mean on $[e^a, e^b]$. According to Theorem 2.2, we have mean values:

$$e^{\xi} = \left[\frac{H_i(f_u)}{H_i(f_v)}\right]^{1/(u-v)}, \tag{4.2}$$

for some $\xi \in [a,b]$.

Example 4.4. *Let a, b positive real numbers and $I = \mathbb{R}$ and a family of functions $\mathbf{F}_4 = \{f_u; u \in I\}$ from $C[a,b]$ defined with* (1.17)*.Since $\frac{d^2}{dx^2}(f_u(x)) = x^{u-2} = e^{(u-2)\ln x}$, therefore $u \mapsto \frac{d^2}{dx^2}(f_u(x)) = x^{u-2} = e^{(u-2)\ln x}$ by definition is exponentially convex* *function on I. Corollary 3.2 yields that $u \mapsto H_i(f_u)$ are exponentially convex functions. Expression corresponding to* (3.5)*:*

$$(4.3)\qquad E_{u,v}(H_1;\mathbf{F}_4)=\begin{cases}\left(\frac{H_1(f_u)}{H_1(f_v)}\right)^{1/(u-v)}, & u\neq v,\\ \exp\left(\frac{2-u}{u(u-1)}+\frac{\frac{pa^u\ln a+qb^u\ln b}{p+q}-A(g^u\ln g)}{\frac{pa^u+qb^u}{p+q}-M_u^u(A,g)}\right), & v=u\neq 0,1,\\ \exp\left(1+\frac{1}{2}\frac{\frac{p\ln^2 a+q\ln^2 b}{p+q}-A(\ln^2 g)}{\frac{p\ln a+q\ln b}{p+q}-\ln M_0(g)}\right), & v=u=0,\\ \exp\left(-1+\frac{1}{2}\frac{\frac{pa\ln^2 a+qb\ln^2 b}{p+q}-A(g\ln^2 g)}{\frac{pa\ln a+qb\ln b}{p+q}-A(g\ln g)}\right), & v=u=1.\end{cases}$$

$$(4.4)\quad E_{u,v}(H_2;\mathbf{F}_4)=\begin{cases}\left(\frac{H_1(f_u)}{H_1(f_v)}\right)^{1/(u-v)}, & u\neq v,\\ \exp\left(\frac{2-u}{u(u-1)}+\frac{A(g^u\ln g)-\left(\frac{pa+qb}{p+q}\right)^u\ln\left(\frac{pa+qb}{p+q}\right)}{M_u^u(A,g)-\left(\frac{pa+qb}{p+q}\right)^u}\right), & v=u\neq 0,1,\\ \exp\left(1+\frac{1}{2}\frac{A(\ln^2 g)-\ln^2\left(\frac{pa+qb}{p+q}\right)}{\ln M_0(g)-\ln\left(\frac{pa+qb}{p+q}\right)}\right), & v=u=0,\\ \exp\left(-1+\frac{1}{2}\frac{A(g\ln^2 g)-\frac{pa+qb}{p+q}\ln^2\left(\frac{pa+qb}{p+q}\right)}{A(g\ln g)-\frac{pa+qb}{p+q}\ln\left(\frac{pa+qb}{p+q}\right)}\right), & v=u=1,\end{cases}$$

As

$$\left(\frac{f_u''}{f_v''}\right)(x)=x^{u-v}$$

is invertible, that is

$$(4.5)\qquad a\leq\left(\frac{H_i(f_u)}{H_i(f_v)}\right)^{1/u-v}\leq b,$$

which shows that $E_{u,v}(H_i,\mathbf{F}_4)$, $(i=1,2)$ is mean on $[a,b]$.

The replacement $g\rightarrow g^s$, $u\rightarrow\frac{u}{s}$ and $p\rightarrow\frac{p}{s}$, $(s\neq 0)$ and $a\rightarrow a^s$, $b\rightarrow b^s$, for $s>0$and $a\rightarrow b^s$, $b\rightarrow a^s$, for $s<0$ in (4.5)*:*

$$(4.6)\qquad \min\{a^s,b^s\}\leq\left(\frac{H_i(f_{\frac{u}{s}})}{H_i(f_{\frac{v}{s}})}\right)^{s/u-v}\leq\max\{a^s,b^s\},\ \ s(u-v)\neq 0$$

enable to produce monotonic 3*-parametrs mean from* (4.5)*. We define this new generalized mean as*

$$(4.7)\qquad E_{u,v;s}(H_i,\mathbf{F}_4)=\begin{cases}\left(E_{\frac{u}{s},\frac{v}{s}}(H_i,\mathbf{F}_4)\right)^{\frac{1}{s}}, & s\neq 0;\\ E_{u,v}(\widetilde{H}_i,\mathbf{F}_3), & s=0,\end{cases}$$

$\widetilde{H}_i,\ i=1,2$ *are the linear functionals* (1.10) *and* (1.11) *acting on* $C([\log a, \log b])$.

Let us define functions $\phi_{i,j} : I \to \mathbb{R}$ by

$$\phi_{i,j}(u) = H_i(f_u), \quad f_u \in \mathbf{F}_j,\ i =,1,2, j = 1,2,3,4 \tag{4.8}$$

$I = (0,\infty)$ or $I = \mathbb{R}$ according as families $\mathbf{F}_1,\ \mathbf{F}_2$, or $\mathbf{F}_1,\ \mathbf{F}_1$.

Theorem 4.1. *If* $\phi_{i,j},\ i=1,2,\ j=1,2,3,3$ *are functions defined as in* (4.8) *on* I.

(i) *For all* $u_k \in I,\ k=1,2,...,n$, *matrix* $\left[\phi_{i,j}\left(\frac{u_k+u_l}{2}\right)\right]_{k,l=1}^{n}$ *is positive semi-definite matrix. Particularly*

$$\det\left[\phi_{i,j}\left(\frac{u_k+u_l}{2}\right)\right]_{k,l=1}^{m} \geq 0, \quad m = 1,\dots,n. \tag{4.9}$$

(i) *For* $u,\ v, r,\ t \in I$ *such that* $u \leq r,\ v \leq t$, *we have*

$$E_{u,v}(H_i, \mathbf{F}_j) \leq E_{r,t}(H_i, \mathbf{F}_j) \tag{4.10}$$

where

$$E_{u,v}(H_i; \mathbf{F}_i) = \begin{cases} \left(\frac{\phi_{i,j}(u)}{\phi_{i,j}(v)}\right)^{1/(u-v)}, & u \neq v, \\ \exp\left(\frac{\frac{d}{du}\phi_{i,j}(u)}{\phi_{i,j}(u)}\right), & u = v. \end{cases} \tag{4.11}$$

Proof. (i) This directly follows from Proposition 3.1 □

(ii) This conclusion is obtained by (ii)-part Corollary 3.3.

Theorem 4.2. *Assume that* p, q *are nonnegative real numbers and* $r,\ s$ *are positive. If* A *is normalized positive linear functional on* L *and* $g^r \in L$, *then the following inequalities holds*

$$\frac{1}{2(r+s)(2r-1)(r+s-1)}\left[\frac{pa^{2r}+qb^{2r}}{p+q} - M_{2r}^{2r}(g,A)\right]\left[\frac{pa^{r+s}+qb^{r+s}}{p+q} - M_{r+s}^{r+s}(g,A)\right]$$
$$\leq \frac{1}{(2r+s)(r-1)(2r+s-1)}\left[\frac{pa^{r}+qb^{r}}{p+q} - M_{r}^{r}(g,A)\right]\left[\frac{pa^{2r+s}+qb^{2r+s}}{p+q} - M_{2r+s}^{2r+s}(g,A)\right]$$

and

$$\frac{1}{2(r+s)(2r-1)(r+s-1)}\left[M_{2r}^{2r}(g,A)-\left(\frac{pa+qb}{p+q}\right)^{2r}\right]\left[M_{r+s}^{r+s}(g,A)-\left(\frac{pa+qb}{p+q}\right)^{r+s}\right]$$
$$\leq\frac{1}{(2r+s)(r-1)(2r+s-1)}\left[M_{r}^{r}(g,A)-\left(\frac{pa+qb}{p+q}\right)^{r}\right]\left[M_{2r+s}^{2r+s}(g,A)-\left(\frac{pa+qb}{p+q}\right)^{2r+s}\right].$$

The reverse inequalities hold for $r>0,\ s>0$.

Proof. The inequality (3.3) can be written as

$$H_i\left(f_u\right)\leq\left(H_i\left(f_t\right)\right)^{(v-u)/(v-t)}\left(H_i\left(f_v\right)\right)^{(u-t)/(v-t)}. \tag{4.12}$$

By the arithmetic-geometric mean inequality

$$H_i\left(f_u\right)\leq\frac{v-u}{v-t}\left(H_i\left(f_t\right)\right)+\frac{u-t}{v-t}\left(H_i\left(f_v\right)\right). \tag{4.13}$$

□

Remark 6. Assume that p,q are nonnegative real numbers, r is positive. If A is normalized positive linear functional and $g^r\in L$, then

$$\frac{1}{4(2r-1)^2}\left[\frac{pa^{2r}+qb^{2r}}{p+q}-M_{2r}^{2r}(g,A)\right]^2$$
$$\leq\frac{1}{3(r-1)(3r-1)}\left[\frac{pa^{r}+qb^{r}}{p+q}-M_{r}^{r}(g,A)\right]\left[\frac{pa^{3r}+qb^{3r}}{p+q}-M_{3r}^{3r}(g,A)\right]$$

Theorem 4.3. *Assume that* p,q *are nonnegative real numbers,* $r,\ s$ *are positive such that* $\frac{1}{r}+\frac{1}{s}=1$. *If* A *is normalized positive linear functional and* $g,\ f,\ g^r,\ f^r,\ \log g,\ \log f,\ f^r\log g,\ g^r\log f\in L$ *then*

$$\frac{r}{s}\left(\frac{pa^{1/r}+qb^{1/r}}{p+q}-\frac{M(fg,A)}{M_s^s(g,A)}\right)\leq\left(\frac{rA(f^r\ln g)-s\mathring{M}_r^r(f,A)}{M_s^s(g,A)}-\frac{p\ln a+q\ln b}{p+q}\right)^{1/s}$$
$$\times\left(\frac{pa\ln a+qb\ln b}{p+q}-\frac{r\mathring{M}_r^r(g,A)-sA(g^p\ln f)}{M_s^s(g,A)}\right)^{1/r}$$

Proof. Setting $i=1,\ t=0,\ v=1$ in (3.3) for $f_u \in \mathbf{F}_4$

$$\frac{pa^u+qb^u}{p+q} - M_u^u(A,g) \le \left(\ln M_0(g) - \frac{p\ln a + q\ln b}{p+q}\right)^{1-u} \times \left(\frac{pa\ln a + qb\ln b}{p+q} - A(g\ln g)\right)^u .$$

For $1-u = 1/s,\ u = 1/r$ and $A(g^u) = \frac{A(\mathrm{w}g^u)}{A(\mathrm{w})},\ g = g^r f^{-s}$, $\mathrm{w} = f^s/A(g^s)$ the previous inequality reduces to the required inequality. □

Theorem 4.4. *Suppose that $p,q,a,b,\ (a<b),\ r,u,v$ are nonnegative real numbers, such that $r<u<v$ and $g^r \in L\ (r \neq 0)$ then following holds*

$$\frac{\frac{1}{u(u-s)}\left[\frac{pa^u+qb^u}{p+q} - M_u^s(g,A)\right] - \frac{1}{v(v-s)}\left[\frac{pa^v+qb^v}{p+q} - M_v^s(g,A)\right]}{\frac{1}{r(r-s)}\left[\frac{pa^r+qb^r}{p+q} - M_r^s(g,A)\right] - \frac{1}{v(v-s)}\left[\frac{pa^v+qb^v}{p+q} - M_v^s(g,A)\right]} \le \frac{u-v}{t-v}$$

$$\frac{\frac{1}{u(u-s)}\left[M_u^s(g,A) - \left(\frac{pa^s+qb^s}{p+q}\right)^{u/s}\right] - \frac{1}{v(v-s)}\left[M_v^s(g,A) - \left(\frac{pa^s+qb^s}{p+q}\right)^{v/s}\right]}{\frac{1}{r(r-s)}\left[M_r^s(g,A) - \left(\frac{pa^s+qb^s}{p+q}\right)^{r/s}\right] - \frac{1}{v(v-s)}\left[M_v^s(g,A) - \left(\frac{pa^s+qb^s}{p+q}\right)^{v/s}\right]} \le \frac{u-v}{t-v}$$

Proof. Let $f_u \in \mathbf{F}_4$, the inequality (3.3) can be written as

$$H_i(f_u) \le (H_i(f_t))^{(v-u)/(v-t)} (H_i(f_v))^{(u-t)/(v-t)},\ i=1,2.$$

By the arithmetic-geometric mean inequality

$$H_i(f_u) \le \frac{v-u}{v-t} H_i(f_t) + \frac{u-t}{v-t} H_i(f_v),\ i=1,2.$$

$$H_i(f_u) - H_i(f_v) \le \frac{v-u}{v-t}\left(H_i(f_t) - H_i(f_v)\right),\ i=1,2.$$

□

4.1. **Alzer-Brenner mean value.** Suppose that $p,\ q,$ and y (y is given in (1.4)) are positive numbers, $f : [a, b] \to \mathbb{R}$ be convex function and $\mathrm{w} : [a, b] \to [0, \infty)$ be integrable function symmetric about the line $x = \tau = (pa + qb)/(p + q)$.
Define lineal functionals $\bar{H}_i : C([a, b]) \to \mathbb{R},\ i = 1, 2$ with

$$\bar{H}_1(f) = \frac{pf(a) + qf(b)}{p + q} \int_{\tau-y}^{\tau+y} \mathrm{w}(x)dx - \int_{\tau-y}^{\tau+y} f(x)\mathrm{w}(x)dx \tag{4.14}$$

$$\bar{H}_2(f) = \int_{\tau-y}^{\tau+y} f(x)\mathrm{w}(x)dx - f\left(\frac{pa + qb}{p + q}\right) \int_{\tau-y}^{\tau+y} \mathrm{w}(x)dx. \tag{4.15}$$

By Theroem 1.2, linear functionals $\bar{H}_i,\ i = 1, 2$ satisfy (1.12), imediatly all cosequence are obtainable. Particularly, explicit form of $\bar{E}_{u,v;s}(\bar{H}_1, \mathbf{F}_4) := E_{u,v;s}(\bar{H}_1, \mathbf{F}_4)$ is given as follows

$$\bar{E}_{u,v;s}(\bar{H}_1, \mathbf{F}_4) = \begin{cases} \left(\frac{\bar{H}_1(f_{\frac{u}{s}})}{\bar{H}_1(f_{\frac{v}{s}})}\right)^{1/(u-v)}, & s(u-v) \neq 0, \\ \exp\left(\frac{2s-u}{u(u-s)} + \frac{\frac{pa^u \ln a + qb^u \ln b}{p+q} - \frac{1}{\bar{\mathrm{w}}} \int_{\tau_s-y_s}^{\tau_s+y_s} x^u \ln x \mathrm{w}(x)dx}{\frac{pa^u+qb^u}{p+q} - \frac{1}{\bar{\mathrm{w}}} \int_{\tau_s-y_s}^{\tau_s+y_s} x^u \mathrm{w}(x)dx}\right), & u = v \neq 0, s\ s \neq 0, \\ \exp\left(\frac{1}{s} + \frac{1}{2} \frac{\frac{p \ln^2 a + q \ln^2 b}{p+q} - \frac{1}{\bar{\mathrm{w}}} \int_{\tau_s-y_s}^{\tau_s+y_s} \ln^2 x \mathrm{w}(x)dx}{\int_{\tau_s-y_s}^{\tau_s+y_s} \ln x \mathrm{w}(x)dx - \frac{p \ln a + q \ln b}{p+q}}\right), & u = v = 0,\ s \neq 0, \\ `\exp\left(-\frac{1}{s} + \frac{1}{2} \frac{\frac{pa^s \ln^2 a + qb^s \ln^2 b}{p+q} - \int_{\tau_s-y_s}^{\tau_s+y_s} x \ln^2 x \mathrm{w}(x)dx}{\frac{pa^s \ln a + qb^s \ln b}{p+q} - \int_{\tau_s-y_s}^{\tau_s+y_s} x \ln x \mathrm{w}(x)dx}\right), & u = v = s \neq 0, \\ \left(\frac{\tilde{H}_1(g_u)}{\tilde{H}_1(g_v)}\right)^{1/(u-v)}, & u \neq v,\ s = 0, \\ \exp\left(-\frac{2}{u} + \frac{\frac{pa^u \ln a + qb^u \ln b}{p+q} - \frac{1}{\bar{\mathrm{w}}} \int_{\tau_0-y_0}^{\tau_0+y_0} xe^{ux} \mathrm{w}(x)dx}{\frac{pa^u+qb^u}{p+q} - \frac{1}{\bar{\mathrm{w}}} \int_{\tau_0-y_0}^{\tau_0+y_0} e^{ux} \mathrm{w}(x)dx}\right), & u = v \neq 0,\ s = 0, \\ \exp\left(\frac{1}{3} \frac{\frac{pa^3+qb^3}{p+q} - \frac{1}{\bar{\mathrm{w}}} \int_{\tau_0-y_0}^{\tau_0+y_0} x^3 \mathrm{w}(x)dx}{\frac{pa^2+qb^2}{p+q} - \frac{1}{\bar{\mathrm{w}}} \int_{\tau_0-y_0}^{\tau_0+y_0} x^2 \mathrm{w}(x)dx}\right), & u = v = s = 0, \end{cases}$$

where $\tilde{H}_1$ is linear functional defined as in (4.14) acting on $C([\log a, \log b]),\ g_u \in \mathbf{F}_3$ and

$$\tau_s = \begin{cases} \left(\frac{pa^s+qb^s}{p+q}\right)^{\frac{1}{s}}, & s \neq 0, \\ \log \sqrt[p+q]{a^p b^q}, & s = 0. \end{cases} \qquad y_s \leq \begin{cases} \frac{|b^s-a^s|}{p+q} \min\{p, q\}, & s \neq 0, \\ \frac{\log b - \log a}{p+q} \min\{p, q\}, & s = 0. \end{cases}$$

Note that there are misprint in the condition of τ_s given in [12].

Remark 7. For the special case w $=1$, related consequences for the family of functions $\mathbf{F}_4$ are obtained in [12].

For w $=1$, $p=q$, $y=(b-a)/2$, $u\to u-1$, $v\to v-1$, all the related results for the family of functions $\mathbf{F}_4$ are given in [2].

4.2. **The complete symmetric mean.** The $u-$th complete symmetric polynomial mean (the complete symmetric mean) of the positive real n-tuple $\mathbf{x}$ is defined by see [6].

$$Q_n^{[u]}(\mathbf{x})=\left({q_n}^{[u]}(\mathbf{x})\right)^{\frac{1}{u}}=\left(\frac{{c_n}^{[u]}(x)}{\binom{n+u-1}{u}}\right)^{\frac{1}{u}}, \tag{4.16}$$

where $c_n^{[0]}(\mathbf{x})=1$ and $c_n^{[u]}(\mathbf{x})=\sum\limits_{(i_1,\ldots,i_n)}(\prod\limits_{i=1}^{n}x_i^{i_j})$ and the sum is taken over all $\binom{n+u-1}{u}$ nonnegative integer n-tuples $(i_1,\ldots,i_n)$ with $\sum\limits_{j=1}^{n}i_j=u,(u\neq 0)$. It is known that complete symmetric polynomial mean can expressed via integral (see [8]):

$$Q_n^{[u]}(\mathbf{x})=\left(\int\limits_{\Delta_{n-1}}\left(\sum_{j=1}^{n}x_iu_i\right)^{u}d\mu(\mathbf{u})\right)^{\frac{1}{u}}, \quad u\in\mathbb{N}\cup\{0\}, \tag{4.17}$$

where $\Delta_n=\left\{(u_0,\ldots,u_{n-1}):u_i\geq 0,\sum_{i=0}^{n-1}u_i\leq 1\right\}$, $u_n=1-\sum_{i=0}^{n-1}u_i$ and μ represents a probability measure such that $d\mu(\mathbf{u})=(n-1)!du_1,\ldots,du_{n-1}$. If we put $M_u(g,A)=Q_n^{[u]}(\mathbf{x})$ in (4.3), (4.4) and (4.7), we get new mean values:

$$\hat{E}_{u,v}^{[n]}(i,\mathbf{x}):=E_{u,v}(H_i,\mathbf{F}_4) \tag{4.18}$$

and

$$\hat{E}_{u,v}^{[n;s]}(i,\mathbf{x}):=E_{u,v;s}(H_i,\mathbf{F}_4) \tag{4.19}$$

respectively. It is interesting to observe that for $r,t,u,v,s\in\mathbb{R}$, $n\in\mathbb{N}$ such that, $v\leq t,\ u\leq r$, we have

$$\hat{E}_{u,v}^{[n;s]}(i,\mathbf{x})\leq\hat{E}_{r,t}^{[n;s]}(i,\mathbf{x}). \tag{4.20}$$

4.2.1. ***Whiteley means***. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\rho = (\rho_1, \ldots, \rho_n)$ be positive real n-tuples and $u \in \mathbb{N} \cup \{0\}$.The generalized complete symmetric polynomials are defined in a way:

$$\sum_{u=0}^{\infty} h_n^{[u,\rho]}(\mathbf{x}) t^u = \prod_{i=1}^{n} \frac{1}{(1 - x_i t)^{\rho_i}},$$

for $|t|$ small enough.

The *generalized Whiteley mean* is now defined in a manner

$$\mathcal{H}_n{}^{[u,\rho]}(\mathbf{x}) = \left(\frac{h_n^{[u,\rho]}(\mathbf{x})}{\binom{\sum_{i=1}^{n} \rho_i + u - 1}{u}} \right), \quad u \in \mathbb{N} \cup \{0\}, \ \rho = (\rho_1, \ldots, \rho_n). \tag{4.21}$$

The generalized Whitely means can also be written in the following fashion:

$$\mathcal{H}_n{}^{[u,\rho]}(\mathbf{x}) = \left(\int\limits_{\Delta_{n-1}} \left(\sum_{i=1}^{n} x_i u_i \right)^u d\mu(\mathbf{u}) \right)^{\frac{1}{u}}, \tag{4.22}$$

with a probability measure

$$d\mu(\mathbf{u}) = \frac{\Gamma\left(\sum\limits_{i=1}^{n} \rho_i \right)}{\prod\limits_{i=1}^{n} \Gamma(\rho_i)} \prod_{i=1}^{n} u_i^{\rho_i - 1} du_1, \ldots, du_{n-1}. \tag{4.23}$$

Similar to above examples, for $M_u(g, A) = \mathcal{H}_n{}^{[u,\rho]}(\mathbf{x})$ we define means:

$$E^{\star\,[n]}_{u,v}(i, \mathbf{x}) := E_{u,v}(H_i, \mathbf{F}_4) \tag{4.24}$$

and

$$E^{\star\,[n;s]}_{u,v}(i, \mathbf{x}) := E_{u,v;s}(H_i, \mathbf{F}_4) \tag{4.25}$$

also for $t, r, u, v, s \in \mathbb{R}$, $n \in \mathbb{N}$ such that, $u \leq r$, $v \leq t$, we have

$$E^{\star\,[n]}_{u,v}(i, \mathbf{x}) \leq E^{\star\,[n]}_{r,t}(i, \mathbf{x}) \tag{4.26}$$

and

$$E^{\star\,[n;s]}_{u,v}(i, \mathbf{x}) \leq E^{\star\,[n;s]}_{r,t}(i, \mathbf{x}) \tag{4.27}$$

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