

PROPERTY (m) FOR BOUNDED LINEAR OPERATORS

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ABSTRACT. A bounded linear operator T acting on a Banach space satisfies property (m) if $\sigma(T) \setminus \sigma_{ub}(T) = E^0(T)$, where $\sigma_{ub}(T)$ is the upper semi-Browder spectrum of T , $\sigma(T)$ is the usual spectrum of T and $E^0(T)$ is the set of isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues of finite multiplicity. In this paper we introduce and study new properties (m) , and (gm) , which are related to Weyl type theorems. These properties are also studied in the framework of polaroid operators.

1. Introduction and Preliminary

In this paper we shall introduce properties which are related to Weyl type theorem for bounded linear operators $T \in \mathcal{L}(\mathcal{X})$, defined on a complex Banach space \mathcal{X} . These properties, that we call *property (m)* , means that the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues of finite multiplicity are exactly those points λ of the spectrum for which $T - \lambda$ is an upper semi-Browder (see Definition 2.1) and we call *property (gm)* , means that the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues are exactly those points λ of the spectrum for which $T - \lambda$ is a left Drazin invertible (see Definition 2.1). Properties (m) and (gm) are related to a variant of Weyl type theorems. We shall characterize properties (m) and (gm) in several ways and we shall also describe the relationships of it with the other variants of

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Weyl type theorems. Our main tool is localized version of the single valued extension property. Also, we consider the properties (m) and (gm) in the frame of polaroid type operators.

Throughout this paper, \mathcal{X} denotes an infinite-dimensional complex Banach space, $\mathcal{L}(\mathcal{X})$ the algebra of all bounded linear operators on \mathcal{X} . For an operator $T \in \mathcal{L}(\mathcal{X})$ we shall denote by $\alpha(T)$ the dimension of the *kernel* $\ker(T)$, and by $\beta(T)$ the codimension of the *range* $\mathfrak{R}(T)$. The closure of a set S will be denoted by \overline{S} and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. Here and elsewhere in this paper, for $A \subset \mathbb{C}$, $\text{iso } A$ denotes the set of all isolated points of A and $\text{acc } A$ denotes the set of all points of accumulation of A .

Let

$$SF_+(\mathcal{X}) := \{T \in \mathcal{L}(\mathcal{X}) : \alpha(T) < \infty \text{ and } \mathfrak{R}(T) \text{ is closed}\}$$

be the class of all *upper semi-Fredholm* operators, and let

$$SF_-(\mathcal{X}) := \{T \in \mathcal{L}(\mathcal{X}) : \beta(T) < \infty\}$$

be the class of all *lower semi-Fredholm* operators. The class of all *semi-Fredholm* operators is defined by $SF_{\pm}(\mathcal{X}) := SF_+(\mathcal{X}) \cup SF_-(\mathcal{X})$, while the class of all *Fredholm* operators is defined by $SF(\mathcal{X}) := SF_+(\mathcal{X}) \cap SF_-(\mathcal{X})$. If $T \in SF_{\pm}(\mathcal{X})$, the *index* of T is defined by

$$\text{ind}(T) := \alpha(T) - \beta(T).$$

Recall that a bounded operator T is said *bounded below* if it injective and has closed range. Evidently, if T is bounded below then $T \in SF_+(\mathcal{X})$ and $\text{ind}(T) \leq 0$. Define

$$SF_+^-(\mathcal{X}) := \{T \in SF_+(\mathcal{X}) : \text{ind}(T) \leq 0\},$$

and

$$SF_-^+(\mathcal{X}) := \{T \in SF_-(\mathcal{X}) : \text{ind}(T) \geq 0\}.$$

The set of *Weyl* operators is defined by

$$W(\mathcal{X}) := SF_+^-(\mathcal{X}) \cap SF_-^+(\mathcal{X}) = \{T \in SF(\mathcal{X}) : \text{ind}(T) = 0\}.$$

The classes of operators defined above generate the following spectra. Denote by

$$\sigma_a(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not bounded below}\}$$

the *approximate point spectrum*, and by

$$\sigma_s(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective}\}$$

the *surjectivity spectrum* of $T \in \mathcal{L}(\mathcal{X})$. The *Weyl spectrum* is defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin W(\mathcal{X})\},$$

the *Weyl essential approximate point spectrum* is defined by

$$\sigma_{SF_+^-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathcal{X})\},$$

while the *Weyl essential surjectivity spectrum* is defined by

$$\sigma_{SF_-^+}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SF_-^+(\mathcal{X})\},$$

Obviously, $\sigma_w(T) = \sigma_{SF_+^-}(T) \cup \sigma_{SF_-^+}(T)$ and from basic Fredholm theory we have

$$\sigma_{SF_+^-}(T) = \sigma_{SF_-^+}(T^*) \quad \sigma_{SF_-^+}(T) = \sigma_{SF_+^-}(T^*).$$

Note that $\sigma_{SF_+^-}(T)$ is the intersection of all approximate point spectra $\sigma_a(T + K)$ of compact perturbations K of T , while $\sigma_{SF_-^+}(T)$ is the intersection of all surjectivity spectra $\sigma_s(T + K)$ of compact perturbations K of T , see, for instance, [1, Theorem 3.65].

Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $\mathfrak{R}(T^q) = \mathfrak{R}(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$. It

is well known that if $a(T)$ and $d(T)$ are both finite then $a(T) = d(T)$ [22, Proposition 1.49]. Moreover, $0 < a(T - \lambda) = d(T - \lambda) < \infty$ precisely when λ is a pole of the resolvent of T , see Dowson [22, Theorem 1.54].

The class of all *upper semi-Browder* operators is defined by

$$B_+(\mathcal{X}) := \{T \in SF_+(\mathcal{X}) : a(T) < \infty\},$$

while the class of all *lower semi-Browder* operators is defined by

$$B_-(\mathcal{X}) := \{T \in SF_+(\mathcal{X}) : d(T) < \infty\}.$$

The class of all *Browder* operators is defined by

$$B(\mathcal{X}) := B_+(\mathcal{X}) \cap B_-(\mathcal{X}) = \{T \in SF(\mathcal{X}) : a(T), d(T) < \infty\}.$$

We have

$$B(\mathcal{X}) \subseteq W(\mathcal{X}), \quad B_+(\mathcal{X}) \subseteq SF_+^-(\mathcal{X}), \quad B_-(\mathcal{X}) \subseteq SF_-^+(\mathcal{X}),$$

see [1, Theorem 3.4]. The *Browder spectrum* of $T \in \mathcal{L}(\mathcal{X})$ is defined by

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B(\mathcal{X})\},$$

the *upper Browder spectrum* is defined by

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B_+(\mathcal{X})\},$$

and analogously the *lower Browder spectrum* is defined by

$$\sigma_{lb}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B_-(\mathcal{X})\}.$$

Clearly, $\sigma_b(T) = \sigma_{ub}(T) \cup \sigma_{lb}(T)$ and $\sigma_w(T) \subseteq \sigma_b(T)$.

For $T \in B(X)$ and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $\mathfrak{R}(T^n)$ viewed as a map from $\mathfrak{R}(T^n)$ into $\mathfrak{R}(T^n)$ (in particular, $T_{[0]} = T$). If for some integer n the range space $\mathfrak{R}(T^n)$ is closed and $T_{[n]}$ is an upper (a lower) semi-Fredholm operator, then T is called an upper (a lower) semi-B-Fredholm operator. In this case

the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [12]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator T is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Weyl operator}\}$.

An operator $T \in \mathcal{L}(\mathcal{X})$ is called Drazin invertible if it has a finite ascent and descent. The Drazin spectrum $\sigma_D(T)$ of an operator T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Drazin invertible}\}$. Define also the set $LD(X)$ by $LD(X) = \{T \in \mathcal{L}(\mathcal{X}) : a(T) < \infty \text{ and } \Re(T^{a(T)+1}) \text{ is closed}\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin LD(X)\}$. Following [15], an operator $T \in \mathcal{L}(\mathcal{X})$ is said to be left Drazin invertible if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda) < \infty$. Let $\pi_a(T)$ denotes the set of all left poles of T and let π_a^0 denotes the set of all left poles of T of finite rank. From Theorem 2.8 of [15] it follows that if $T \in \mathcal{L}(\mathcal{X})$ is left Drazin invertible, then T is an upper semi-B-Fredholm operator of index less than or equal to 0.

Let $\pi(T)$ be the set of all poles of the resolvent of T and let $\pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\pi^0(T) = \{\lambda \in \pi(T) : \alpha(T - \lambda) < \infty\}$. According to [24], a complex number λ is a pole of the resolvent of T if and only if $0 < \max\{a(T - \lambda), d(T - \lambda)\} < \infty$. Moreover, if this is true then $a(T - \lambda) = d(T - \lambda)$.

According also to [24], the space $\Re((T - \lambda)^{a(T - \lambda)+1})$ is closed for each $\lambda \in \pi(T)$. Hence we have always $\pi(T) \subset \pi_a(T)$ and $\pi^0(T) \subset \pi_a^0(T)$. We say that Browders theorem holds for $T \in \mathcal{L}(\mathcal{X})$ if $\Delta(T) = \pi^0(T)$, and that a-Browders theorem holds for $T \in \mathcal{L}(\mathcal{X})$ if $\Delta_a(T) = \pi_a^0(T)$. Following [14], we say that generalized Weyl's theorem holds for $T \in \mathcal{L}(\mathcal{X})$ if $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T) = \{\lambda \in \text{iso}\sigma(T) : \alpha(T - \lambda) > 0\}$ is the set of all isolated eigenvalues of T , and that

generalized Browder's theorem holds for $T \in \mathcal{L}(\mathcal{X})$ if $\Delta^g(T) = \pi(T)$. It is proved in Theorem 2.1 of [10] that generalized Browder's theorem is equivalent to Browder's theorem. In [15, Theorem 3.9], it is shown that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(T) = \pi(T)$, it is proved in Theorem 2.9 of [17] that generalized Weyl's theorem is equivalent to Weyl's theorem.

Let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators,

$$SBF_+^-(X) = \{T \in SBF_+(X) : \text{ind}(T) \leq 0\}.$$

The upper B-Weyl spectrum of T is defined by $\sigma_{SBF_+^-(T)} = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(X)\}$. We say that generalized a-Weyl's theorem holds for $T \in \mathcal{L}(\mathcal{X})$ if $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-(T)} = E_a(T)$, where $E_a(T) = \{\lambda \in \text{iso } \sigma_a(T) : \alpha(T - \lambda) > 0\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in \mathcal{L}(\mathcal{X})$ obeys generalized a-Browders theorem if $\Delta_a^g(T) = \pi_a(T)$. It is proved in [10, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [15, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(T) = \pi_a(T)$ it is proved in [17, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

Following [28], we say that $T \in \mathcal{L}(\mathcal{X})$ possesses property (w) if $\Delta_a(T) = E^0(T)$. The property (w) has been studied in [1, 5, 28]. In Theorem 2.8 of [5], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. We say that $T \in \mathcal{L}(\mathcal{X})$ possesses property (gw) if $\Delta_a^g(T) = E(T)$. Property (gw) has been introduced and studied in [11]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [11] that an operator possessing property (gw) possesses property (w) but the converse is not true in general. According to [18], an operator $T \in \mathcal{L}(\mathcal{X})$ is said to possess property (gb) if $\Delta_a^g(T) = \pi(T)$, and is said to

possess property (b) if $\Delta_a(T) = \pi^0(T)$. It is shown in Theorem 2.3 of [18] that an operator possessing property (gb) possesses property (b) but the converse is not true in general, see also [30, 29]. Following [9], we say an operator $T \in \mathcal{L}(\mathcal{X})$ is said to be satisfies property (R) if $\pi_a^0(T) = E^0(T)$. In Theorem 2.4 of [9], it is shown that T satisfies property (w) if and only if T satisfies a -Browder's theorem and T satisfies property (R).

The *single valued extension property* plays an important role in local spectral theory, see the recent monograph of Laursen and Neumann [26] and Aiena [1]. In this article we shall consider the following local version of this property, which has been studied in recent papers, [5, 25] and previously by Finch [23].

Let $H(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [23] we say that $T \in \mathcal{L}(\mathcal{X})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ , the only analytic function $f : U_\lambda \rightarrow \mathcal{X}$ which satisfies the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathcal{L}(\mathcal{X})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathcal{L}(\mathcal{X})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [25, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Theorem 1.1. [3, Theorem 1.3] *If $T \in SF_\pm(\mathbb{X})$ the following statements are equivalent:*

- (i) T has SVEP at λ_0 ;
- (ii) $a(T - \lambda_0) < \infty$;
- (iii) $\sigma_a(T)$ does not cluster at λ_0 ;
- (iv) $H_0(T - \lambda_0)$ is finite dimensional.

By duality we have

Theorem 1.2. *If $T \in SF_{\pm}(\mathbb{X})$ the following statements are equivalent:*

- (i) T^* has SVEP at λ_0 ;
- (ii) $d(T - \lambda_0) < \infty$;
- (iii) $\sigma_s(T)$ does not cluster at λ_0 .

2. Property (m)

Definition 2.1. Let $T \in \mathcal{L}(\mathcal{X})$. We say that T satisfies

- (i) property (m) if $\sigma(T) \setminus \sigma_{ub}(T) = E^0(T)$.
- (ii) property (gm) if $\sigma(T) \setminus \sigma_{LD}(T) = E(T)$.

Theorem 2.2. *Let $T \in \mathcal{L}(\mathcal{X})$. If T satisfies property (m). Then T satisfies property (R).*

Proof. Assume that T satisfies property (m), then $\sigma(T) \setminus \sigma_{ub}(T) = E^0(T)$. If $\lambda \in \pi_a^0(T) = \sigma_a(T) \setminus \sigma_{ub}(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{ub}(T) = E^0(T)$ and so $\pi_a^0(T) \subseteq E^0(T)$. To show the opposite inclusion, let $\lambda \in E^0(T)$ be arbitrary. Then λ is an eigenvalue of T isolated in $\sigma(T)$. Since T satisfies property (m), it follows that $\lambda \in \sigma(T) \setminus \sigma_{ub}(T)$ and $T - \lambda \in B_+(\mathcal{X})$. As $\lambda \in E^0(T)$, then $\lambda \in \text{iso}\sigma_a(T)$ and so $\lambda \in \sigma_a(T)$. Therefore, $\lambda \in \sigma_a(T) \setminus \sigma_{ub}(T) = \pi_a^0(T)$. Hence $E^0(T) \subseteq \pi_a^0(T)$ and so, $E^0(T) = \pi_a^0(T)$, i.e., T satisfies property (R). \square

The following example shows the converse of the previous theorem does not hold in general.

Example 2.3. Consider the operator $T = R \oplus S$ that defined on $\mathcal{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, where R is the right unilateral shift operator and $S(x_1, x_2, \dots) = (x_2/2, x_3/3, \dots)$. Then $\sigma(T) = D(0, 1)$, where $D(0, 1)$ is the unit disc of \mathbb{C} . Hence, $\text{iso}\sigma(T) = \emptyset$ and so, $E^0(T) = E(T) = \emptyset$. Moreover, $\sigma_a(T) = \sigma_{SF_+^-}(T) = \sigma_{ub}(T) = C(0, 1) \cup \{0\}$, where

$C(0, 1)$ is the unit circle of \mathbb{C} . Since $\sigma_a(T) \setminus \sigma_{ub}(T) = \emptyset = E^0(T)$, then T satisfies property (R). On the other hand, since $\sigma(T) \setminus \sigma_{ub}(T) \neq E^0(T)$, then T does not satisfy property (m).

Theorem 2.4. *Let $T \in \mathcal{L}(\mathcal{X})$. Then T satisfies property (m) if and only if T satisfies property (R) and $\sigma(T) = \sigma_a(T)$.*

Proof. If T satisfies property (m), then T satisfies property (R) by Theorem 2.2. Hence $E^0(T) = \sigma_a(T) \setminus \sigma_{ub}(T) = \sigma(T) \setminus \sigma_{ub}(T)$. This implies that $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies property (R) and $\sigma(T) = \sigma_a(T)$. Then

$$E^0(T) = \sigma_a(T) \setminus \sigma_{ub}(T) = \sigma(T) \setminus \sigma_{ub}(T).$$

That is, T satisfies property (m). □

Theorem 2.5. *Let $T \in \mathcal{L}(\mathcal{X})$. If T satisfies property (gm), then T satisfies property (m).*

Proof. Assume that T satisfies property (gm), then $\sigma(T) \setminus \sigma_{LD}(T) = E(T)$. If $\lambda \in \sigma(T) \setminus \sigma_{ub}(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{LD}(T) = E(T)$. Since $\lambda \in \text{iso}\sigma(T)$ and $T - \lambda \in B_+(\mathcal{X})$, then $\lambda \in E^0(T)$ and so $\sigma(T) \setminus \sigma_{ub}(T) \subseteq E^0(T)$. To show the opposite inclusion, let $\lambda \in E^0(T)$ be arbitrary. Then λ is an eigenvalue of T isolated in $\sigma(T)$. Since T satisfies property (gm), it follows that $\lambda \in \sigma(T) \setminus \sigma_{LD}(T)$ and $T - \lambda$ is a left Drazin invertible. As $\alpha(T - \lambda)$ is finite, we conclude that $T - \lambda \in B_+(\mathcal{X})$. Hence $\lambda \in \sigma(T) \setminus \sigma_{ub}(T)$. Therefore, $\sigma(T) \setminus \sigma_{ub}(T) = E^0(T)$, i.e., T satisfies property (m). □

The converse of the Theorem 2.5 is not true in general as shown by the following example.

Example 2.6. Let Q be defined for each $x = \{\xi_i\} \in \ell^1(\mathbb{N})$ by

$$Q(\xi_1, \xi_2, \dots) = (0, \alpha_1 \xi_2, \alpha_2 \xi_3, \dots, \alpha_{k-1} \xi_k, \dots),$$

where $\{\alpha_i\}$ is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$.

Define T on $\mathcal{X} = \ell^1(\mathbb{N}) \oplus \ell^1(\mathbb{N})$ by $T = Q \oplus 0$. Then $\sigma(T) = \sigma_a(T) = \{0\}$, $E(T) = \{0\}$, $E^0(T) = \emptyset$. It follows from Example 3.12 of [15] that $\Re(T^n)$ is not closed for all $n \in \mathbb{N}$. This implies that $\sigma_{SF_+^-}(T) = \sigma_{ub}(T) = \sigma_{LD}(T) = \sigma_{SBF_+^-}(T) = \{0\}$. We then have $\sigma(T) \setminus \sigma_{LD}(T) = \emptyset \neq E(T) = \{0\}$ and $\sigma(T) \setminus \sigma_{ub}(T) = E^0(T)$. Hence T satisfies property (m) , but T does not satisfies property (gm) .

Theorem 2.7. *Let $T \in \mathcal{L}(\mathcal{X})$. Then the following assertions hold.*

- (i) *If T satisfies property (m) , then Weyl's theorem hold for T .*
- (ii) *If T satisfies property (gm) , then generalized Weyl's theorem hold for T .*

Proof. (i) Assume that T satisfies property (m) , then $\Delta(T) = E^0(T)$. If $\lambda \in \Delta(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{ub}(T) = E^0(T)$ and so $\Delta(T) \subseteq E^0(T)$. Conversely, if $\lambda \in E^0(T)$ is arbitrary, then $\lambda \in \text{iso } \sigma(T)$, T and T^* has SVEP at λ . Since T satisfies property (m) , then $T - \lambda \in B_+(\mathcal{X})$. The SVEP of T and T^* implies that by Remark 1.2 of [5] $a(T - \lambda) = d(T - \lambda) < \infty$. As $\alpha(T - \lambda)$ is finite, it follows from Theorem 3.4 of [1] that $\alpha(T - \lambda) = \beta(T - \lambda) < \infty$, so $\lambda \in \pi^0(T) = \sigma(T) \setminus \sigma_b(T)$. Hence $\lambda \in \Delta(T)$ and this implies that $E^0(T) \subseteq \Delta(T)$. Therefore, $\Delta(T) = E^0(T)$, that is, Weyl's theorem holds for T .

(ii) Suppose that T satisfies property (gm) and let $\lambda \in \Delta^g(T)$. Since $\sigma_{LD}(T) \subseteq \sigma_{BW}(T)$, then $\lambda \notin \sigma_{LD}(T)$. If $\alpha(T - \lambda) = 0$, as $\lambda \notin \sigma_{BW}(T)$, then $T - \lambda$ will be invertible. But this impossible since $\lambda \in \sigma(T)$. Hence $\alpha(T - \lambda) > 0$ and $\lambda \in \sigma(T)$. As T satisfies property (gm) , then $\lambda \in E(T)$. This implies that $\Delta^g(T) \subseteq E(T)$. To show the opposite inclusion, let $\lambda \in E(T)$ be arbitrary. Since T satisfies property (gm) , then $T - \lambda$ is an upper semi-B-Fredholm with $\text{ind}(T - \lambda) \leq 0$. On the other hand, as $\lambda \in E(T)$, then λ is isolated in $\sigma(T)$, and hence T^* has SVEP at λ . This implies

by Theorem 2.11 of [2] that $\text{ind}(T - \lambda) = 0$, and $\lambda \notin \sigma_{BW}(T)$. Hence $\Delta^g(T) = E(T)$, that is, T satisfies generalized Weyl's theorem. \square

The converse of Theorem 2.7 does not hold in general as shown by the following example.

Example 2.8. Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift and

$$U(x_1, x_2, \dots) := (0, x_2, x_3, \dots) \text{ for all } (x_n) \in \ell^2(\mathbb{N}).$$

If $T := R \oplus U$ then $\sigma(T) = \sigma_w(T) = \sigma_D(T) = \sigma_{BW}(T) = \sigma_b(T) = D(0, 1)$, where $D(0, 1)$ is the unit disc of \mathbb{C} . Hence $\text{iso}\sigma(T) = \emptyset$, then $E^0(T) = E(T) = \emptyset$. Then T satisfies generalized Weyl's theorem and hence Weyl's theorem, since $\Delta^g(T) = E(T)$ and $\Delta(T) = E^0(T)$. Moreover, $\sigma_a(T) = C(0, 1) \cup \{0\}$, where $C(0, 1)$ is the unit circle of \mathbb{C} . Hence $\sigma_{ub}(T) = \sigma_{SF_+^-}(T) = \sigma_{LD}(T) = \sigma_{SBF_+^-}(T) = C(0, 1)$. Since $\sigma(T) \setminus \sigma_{ub}(T) \neq E^0(T)$ and $\sigma(T) \setminus \sigma_{LD}(T) \neq E(T)$. Then T does not satisfy property (m) nor property (gm).

The following example show that property (w) and property (gw) does not imply property (m).

Example 2.9. Consider the operator $T = R \oplus S$ that defined on $\mathcal{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, where R is the right unilateral shift operator and $S(x_1, x_2, \dots) = (x_2/2, x_3/3, \dots)$. Then $\sigma(T) = D(0, 1)$, where $D(0, 1)$ is the unit disc in \mathbb{C} . Hence, $\text{iso}\sigma(T) = \emptyset$ and so, $E^0(T) = E(T) = \emptyset$. Moreover, $\sigma_a(T) = \sigma_{SF_+^-}(T) = \sigma_{ub}(T) = C(0, 1) \cup \{0\}$, where $C(0, 1)$ is the unit circle of \mathbb{C} . Since $\Delta_a(T) = \emptyset = E^0(T)$ then T satisfies property (w) and since $\Delta_a^g(T) = \emptyset = E(T)$ then T satisfies property (gw). On the other hand, since $\sigma(T) \setminus \sigma_{ub}(T) \neq E^0(T)$, then T does not satisfy property (m).

Theorem 2.10. *Let $T \in \mathcal{L}(\mathcal{X})$. Then the following assertions hold,*

- (i) *T satisfies property (m) if and only if Weyl's theorem holds for T and $\sigma_{ub}(T) = \sigma_w(T)$.*
- (ii) *T satisfies property (gm) if and only if generalized Weyl's theorem holds for T and $\sigma_{BW}(T) = \sigma_{LD}(T)$.*

Proof. (i) If T satisfies property (m) , then it follows from Theorem 2.7 part (i) that T satisfies Weyl's theorem. Hence $E^0(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T) \setminus \sigma_{ub}(T)$. This implies that $\sigma_w(T) = \sigma_{ub}(T)$. Conversely, assume that T satisfies Weyl's theorem and $\sigma_w(T) = \sigma_{ub}(T)$. Then

$$E^0(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T) \setminus \sigma_{ub}(T).$$

That is, T satisfies property (m) .

(ii) If T satisfies property (gm) , then it follows from Theorem 2.7 part (ii) that T satisfies generalized Weyl's theorem. Hence $E(T) = \sigma(T) \setminus \sigma_{BW}(T) = \sigma(T) \setminus \sigma_{LD}(T)$. This implies that $\sigma_{BW}(T) = \sigma_{LD}(T)$. Conversely, assume that T satisfies generalized Weyl's theorem and $\sigma_{BW}(T) = \sigma_{LD}(T)$. Then

$$E(T) = \sigma(T) \setminus \sigma_{BW}(T) = \sigma(T) \setminus \sigma_{LD}(T).$$

That is, T satisfies property (gm) . □

Example 2.11. Let T be defined as in Example 2.8. Then T satisfies generalized a -Weyl's theorem since $\Delta_a^g(T) = \{0\} = E_a(T)$ and hence a -Weyl's theorem.

The next result shows that the equivalence of property (m) , property (R) , property (b) , property (w) , Weyl's theorem and a -Weyl's theorem is true whenever we assume that T^* has SVEP at the points $\lambda \notin \sigma_{SF_+}^-(T)$.

Theorem 2.12. *Let $T \in \mathcal{L}(\mathcal{X})$. If T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$. Then property (w), property (b), property (R), property (m), Weyl's theorem and a -Weyl's theorem are equivalent for T .*

Proof. Assume that T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$. Then it follows from [1, Corollary 2.5] that $\sigma(T) = \sigma_a(T)$ and by Corollary 3.53 of [1], we then have $\sigma_w(T) = \sigma_b(T) = \sigma_{SF_+^-}(T) = \sigma_{ub}(T)$. Then the SVEP of T^* at every $\lambda \notin \sigma_{SF_+^-}(T)$ entails that a -Browder's theorem (and hence Browder's theorem) holds for T , see [6, Theorem 2.3]. Then it follows by Theorem 2.19 of [9] that

$$\pi^0(T) = E^0(T), \pi_a^0(T) = E_a^0(T), E^0(T) = \pi_a^0(T)$$

are equivalent. Hence

$$\pi^0(T) = \Delta(T) = E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \Delta_a(T) = \pi_a^0(T) = E_a^0(T).$$

Therefore, property (w), property (b), property (R), property (m), Weyl's theorem and a -Weyl's theorem are equivalent for T . \square

Dually, we have

Theorem 2.13. *Let $T \in \mathcal{L}(\mathcal{X})$. If T has SVEP at every $\lambda \notin \sigma_{SF_+^+}(T)$. Then property (w), property (b), property (R), property (m), Weyl's theorem and a -Weyl's theorem are equivalent for T^* .*

Proof. Assume that T has SVEP at every $\lambda \notin \sigma_{SF_+^+}(T)$. Then it follows from [1, Corollary 2.5] that $\sigma(T^*) = \sigma(T) = \sigma_s(T) = \sigma_a(T^*)$ and by Corollary 3.53 of [1], we then have $\sigma_w(T^*) = \sigma_w(T) = \sigma_b(T) = \sigma_{SF_+^+}(T) = \sigma_{lb}(T) = \sigma_{ub}(T^*) = \sigma_{SF_+^+}(T^*)$. Then the SVEP of T at every $\lambda \notin \sigma_{SF_+^+}(T)$ entails that a -Browder's theorem (and hence Browder's theorem) holds for T^* , see [6, Theorem 2.3]. Then it follows by Theorem 2.20 of [9] that

$$\pi^0(T^*) = E^0(T^*), \pi_a^0(T^*) = E_a^0(T^*), E^0(T^*) = \pi_a^0(T^*)$$

are equivalent. Hence

$$\begin{aligned}\pi^0(T^*) &= \Delta(T^*) = E^0(T^*) = \Delta_a(T^*) = E_a^0(T^*) \\ &= \sigma_a(T^*) \setminus \sigma_{ub}(T^*) = \pi_a^0(T^*).\end{aligned}$$

Therefore, property (w), property (b), property (R), property (m), Weyl's theorem and a -Weyl's theorem are equivalent for T^* . \square

Theorem 2.14. *Suppose that T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$. Then the following assertions are equivalent:*

- (i) $E(T) = \pi(T)$;
- (ii) $E_a(T) = \pi_a(T)$;
- (iii) $E(T) = \pi_a(T)$.

Consequently, property (gw), property (gb), property (gm), generalized a -Weyl's theorem and generalized Weyl's theorem are equivalent for T .

Proof. Suppose that T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$. We prove first the equality $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. If $\lambda \notin \sigma_{SBF_+^-}(T)$ then $T - \lambda$ is an upper semi-B-Fredholm operator and $\text{ind}(T - \lambda) \leq 0$. As T^* has SVEP, then it follows from Corollary 2.8 of [16] that $T - \lambda$ is a B-Weyl operator and so $\lambda \notin \sigma_{BW}(T)$. Therefore, $\sigma_{SBF_+^-}(T) \subseteq \sigma_{BW}(T)$. Since the other inclusion is always verified, we have the equality. Now we prove that $\sigma_D(T) = \sigma_{BW}(T)$. Since $\sigma_{SBF_+^-}(T) \subseteq \sigma_{SF_+^-}(T)$ is always verified. Then T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$. This implies that T satisfies Browder's theorem. As we know from Theorem 2.1 of [10] that Browder's theorem is equivalent to generalized Browder's theorem, we have $\sigma_{BW}(T) = \sigma_D(T)$. On the other hand, as T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$, then $\sigma(T) = \sigma_a(T)$. From this we deduce that $E(T) = E_a(T)$ and

$$\pi_a(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_D(T) = \pi(T),$$

from which the equivalence of (i), (ii) and (iii) easily follows. To show the last statement observed that the SVEP of T^* at the points $\lambda \notin \sigma_{SBF_+^-}(T)$ entails that generalized a -Browder's theorem (and hence generalized Browder's theorem) holds for T , see [19, Corollary 2.7]. Therefor,

$$\pi(T) = \Delta^g(T) = E(T) = \sigma(T) \setminus \sigma_{LD}(T) = \Delta_a^g(T) = E_a(T).$$

That is, property (gm), property (gb), property (gw), generalized a -Weyl's theorem and generalized Weyl's theorem are equivalent for T . \square

Dually, we have

Theorem 2.15. *Suppose that T has SVEP at every $\lambda \notin \sigma_{SBF_-^+}(T)$. Then the following assertions are equivalent:*

- (i) $E(T^*) = \pi(T^*)$;
- (ii) $E_a(T^*) = \pi_a(T^*)$;
- (iii) $E(T^*) = \pi_a(T^*)$.

Consequently, property (gb), property (gw), generalized a -Weyl's theorem and generalized Weyl's theorem are equivalent for T^ .*

Proof. Suppose that T has SVEP at every $\lambda \notin \sigma_{SBF_-^+}(T)$. We prove first the equality $\sigma_{SBF_-^+}(T^*) = \sigma_{BW}(T^*)$. If $\lambda \notin \sigma_{SBF_-^+}(T)$ then $T - \lambda$ is a lower semi-B-Fredholm operator and $\text{ind}(T - \lambda) \geq 0$. As T has SVEP, then it follows from Theorem 2.5 of [16] that $T - \lambda$ is a B-Weyl operator and so $\lambda \notin \sigma_{BW}(T)$. As $\sigma_{BW}(T) = \sigma_{BW}(T^*)$. Then $\lambda \notin \sigma_{BW}(T^*)$. So $\sigma_{BW}(T^*) \subseteq \sigma_{SBF_-^+}(T)$. As $\sigma_{SBF_-^+}(T) = \sigma_{SBF_+^-}(T^*)$, then $\sigma_{BW}(T^*) \subseteq \sigma_{SBF_+^-}(T^*)$. Since the other inclusion is always verified, it then follows that $\sigma_{BW}(T^*) = \sigma_{SBF_+^-}(T^*)$. Now we show that $\sigma_{BW}(T^*) = \sigma_D(T^*)$. Since we have always $\sigma_{SBF_-^+}(T) \subseteq \sigma_{SF_-^+}(T)$, then T has SVEP at every $\lambda \in \sigma_{SF_-^+}(T)$. Hence T^* satisfies generalized Browder's theorem. So $\sigma_D(T^*) = \sigma_{BW}(T^*)$. Finally, we have $\sigma_{BW}(T^*) = \sigma_{SBF_+^-}(T^*) = \sigma_D(T^*)$ and $\sigma(T^*) = \sigma_a(T^*)$, from which we obtain

$E(T^*) = E_a(T^*)$ and $\pi(T^*) = \pi_a(T^*)$. The SVEP at every $\lambda \in \sigma_{SBF_-^+}(T)$ ensure by Corollary 2.7 of [19] that generalized a -Browder's theorem (and hence generalized Browder's theorem) holds for T^* . Hence

$$\pi(T^*) = \Delta^g(T^*) = E(T^*) = \sigma(T^*) \setminus \sigma_{LD}(T^*) = \Delta_a^g(T^*) = E_a(T^*).$$

That is, property (gb) , property (gm) , property (gw) , generalized a -Weyl's theorem and generalized Weyl's theorem are equivalent for T^* . \square

3. Property (m) for polaroid type operators

In this section we consider classes of operators for which the isolated points of the spectrum are poles of the resolvent.

An operator $T \in \mathcal{L}(\mathcal{X})$ is said to be polaroid if isolated point of $\sigma(T)$ is a pole of the resolvent of T . $T \in \mathcal{L}(\mathcal{X})$ is said to be a -polaroid if every isolated of $\sigma_a(T)$ is a pole of the resolvent of T .

It is easily seen that if T is a -polaroid, then T is polaroid, while in general the converse is not true. It is well known that λ is a pole of the resolvent of T if and only if λ is a pole of the resolvent of T^* . Since $\sigma(T) = \sigma(T^*)$ we then have

$$(3.1) \quad T \text{ is polaroid if and only if } T^* \text{ is polaroid.}$$

From the proof of Theorem 2.14 we know that if T^* has SVEP, then $\sigma(T) = \sigma_a(T)$. Therefore, if T^* has SVEP then

$$(3.2) \quad T \text{ is } a\text{-polaroid if and only if } T \text{ is polaroid.}$$

If T has SVEP, we know that $\sigma(T^*) = \sigma_a(T^*)$. Therefore, if T has SVEP, then

$$(3.3) \quad T^* \text{ } a\text{-polaroid} \Leftrightarrow T^* \text{ polaroid} \Leftrightarrow T \text{ polaroid.}$$

Theorem 3.1. *Suppose that $T \in \mathcal{L}(\mathcal{X})$ is a -polaroid and $\sigma(T) = \sigma_a(T)$. Then T satisfies property (m) .*

Proof. Note first that if T is a -polaroid then $E_a^0(T) = \pi^0(T)$. In fact, if $\lambda \in E_a^0(T)$ then λ is isolated in $\sigma_a(T)$ and hence $a(T - \lambda) = d(T - \lambda) < \infty$. As $\alpha(T - \lambda)$ is finite, it follows by Theorem 3.4 of [1] that $\beta(T - \lambda)$ is also finite, thus $\lambda \in \pi^0(T)$. This shows that $E_a^0(T) \subseteq \pi^0(T)$, since we have always $\pi^0(T) \subseteq E_a^0(T)$. Hence $E_a^0(T) = \pi^0(T)$, and so $\pi^0(T) = \pi_a^0(T)$. Therefore,

$$\pi_a(T) = \sigma_a(T) \setminus \sigma_{ub}(T) = \sigma(T) \setminus \sigma_{ub}(T).$$

That is, T satisfies property (m). □

Theorem 3.2. *Suppose that $T \in \mathcal{L}(\mathcal{X})$ is a -polaroid and $\sigma(T) = \sigma_a(T)$. Then T satisfies property (gm).*

Proof. Note first that if T is a -polaroid then $E_a(T) = \pi(T)$. In fact, if $\lambda \in E_a(T)$ then λ is isolated in $\sigma_a(T)$ and hence $a(T - \lambda) = d(T - \lambda) < \infty$ and so $\lambda \in \pi(T)$. Since the other inclusion is always verified, we then have $E_a(T) = \pi(T)$ and hence $\pi(T) = \pi_a(T)$. Therefore,

$$\pi^a(T) = \sigma_a(T) \setminus \sigma_{LD}(T) = \sigma(T) \setminus \sigma_{LD}(T).$$

That is, T satisfies property (gm). □

Let $H(\sigma(T))$ denote the set of analytic functions defined on an open neighborhood of $\sigma(T)$, such that f is non constant on each of the components of its domain.

Theorem 3.3. *Suppose that $T \in \mathcal{L}(\mathcal{X})$ is polaroid and $f \in H(\sigma(T))$.*

- (i) *If T^* has SVEP, then property (m) holds for $f(T)$, or equivalently, property (R), property (w), Weyl's theorem and a -Weyl's theorem hold for $f(T)$.*
- (ii) *If T has SVEP, then property (m) holds for $f(T^*)$, or equivalently, property (R), property (w), Weyl's theorem and a -Weyl's theorem hold for $f(T^*)$.*

Proof. (i) It follows from Theorem 3.4 of [9] that property (R) holds for $f(T)$, or equivalently, property (w) , Weyl's theorem and a -Weyl's theorem hold for $f(T)$. Since the SVEP of T^* implies by Corollary 2.45 that $\sigma(T) = \sigma_a(T)$. So, the result follows now by Theorem 2.4.

(ii) It follows from Theorem 3.4 of [9] that property (R) holds for $f(T^*)$, or equivalently, property (w) , Weyl's theorem and a -Weyl's theorem hold for $f(T^*)$. Since the SVEP of T implies by Corollary 2.45 that $\sigma(T^*) = \sigma_a(T^*)$. So, the result follows now by Theorem 2.4. \square

Theorem 3.4. *Suppose that $T \in \mathcal{L}(\mathcal{X})$ is polaroid and $f \in H(\sigma(T))$.*

- (i) *If T^* has SVEP, then property (gm) holds for $f(T)$, or equivalently, property (gw) , generalized Weyl's theorem and generalized a -Weyl's theorem hold for $f(T)$.*
- (ii) *If T has SVEP, then property (gm) holds for $f(T^*)$, or equivalently, property (gw) , generalized Weyl's theorem and generalized a -Weyl's theorem hold for $f(T^*)$.*

Proof. (i) By Lemma 3.11 of [8] we know that $f(T)$ is polaroid. By Theorem 2.40 of [1] $f(T^*)$ has SVEP, hence $f(T)$ is a -polaroid by equivalence (3.2). Since $f(T^*)$ has SVEP, then it follows from [1, Corollary 2.45] that $\sigma(f(T)) = \sigma_a(f(T))$. So, it follows from Theorem 3.2 that $f(T)$ satisfies property (gm) and this is equivalent by Theorem 2.14 to saying that property (gw) , generalized Weyl's theorem and generalized a -Weyl's theorem hold for $f(T)$.

(ii) It follows from equivalence (3.3) that T^* is polaroid and hence it then follows by Lemma 3.11 of [8] that $f(T^*)$ is polaroid. Moreover, always by Theorem 2.40 of [1], $f(T)$ has SVEP and so it follows from Corollary 2.45 of [1] that $\sigma(f(T)) = \sigma_a(f(T^*))$. So, it follows from equivalence (3.3) that $f(T^*)$ is a -polaroid. Therefore, it follows by Theorem 3.2 that $f(T^*)$ satisfies property (gm) and this by Theorem 2.14 is

equivalent to saying that property (gw), generalized Weyl's theorem and generalized a -Weyl's theorem hold for $f(T^*)$. \square

A bounded operator $T \in \mathcal{L}(\mathcal{X})$ is said to be left polaroid if every isolated point of $\sigma_a(T)$ is a left pole of the resolvent of T . $T \in \mathcal{L}(\mathcal{X})$ is said to be right polaroid if every isolated point of $\sigma_a(T)$ is a right pole of the resolvent of T .

Trivially,

$$(3.4) \quad T \text{ } a\text{-polaroid} \Rightarrow T \text{ left polaroid.}$$

Furthermore,

$$(3.5) \quad T \text{ left and right polaroid} \Rightarrow T \text{ polaroid.}$$

Theorem 3.5. *Suppose that $T \in \mathcal{L}(\mathcal{X})$ and $f \in H(\sigma(T))$. Then the following assertions hold:*

- (i) *If T^* has SVEP and T is left polaroid, then property (m) holds for $f(T)$, or equivalently, property (R), property (w), Weyl's theorem and a -Weyl's theorem hold for $f(T)$.*
- (ii) *If T has SVEP and T is right polaroid, then property (gm) holds for $f(T^*)$, or equivalently, property (R), property (w), Weyl's theorem and a -Weyl's theorem hold for $f(T^*)$.*

Proof. (i) It follows from Theorem 3.7 of [9] that property (R) holds for $f(T)$, or equivalently, property (w), Weyl's theorem and a -Weyl's theorem hold for $f(T)$. Since the SVEP of T^* implies by Corollary 2.45 that $\sigma(T) = \sigma_a(T)$. So, the result follows now by Theorem 2.4.

(ii) It follows from Theorem 3.7 of [9] that property (R) holds for $f(T^*)$, or equivalently, property (w), Weyl's theorem and a -Weyl's theorem hold for $f(T^*)$. Since the SVEP of T implies by Corollary 2.45 that $\sigma(T^*) = \sigma_a(T^*)$. So, the result follows now by Theorem 2.4. \square

Theorem 3.6. *Suppose that $T \in \mathcal{L}(\mathcal{X})$ and $f \in H(\sigma(T))$. Then the following assertions hold:*

- (i) *If T^* has SVEP and T is left polaroid, then property (gm) holds for $f(T)$, or equivalently, property (gw), generalized Weyl's theorem and generalized a-Weyl's theorem hold for $f(T)$.*
- (ii) *If T has SVEP and T is right polaroid, then property (gm) holds for $f(T^*)$, or equivalently, property (gw), generalized Weyl's theorem and generalized a-Weyl's theorem hold for $f(T^*)$.*

Proof. (i) Let $\lambda \in \text{iso}\sigma(T)$. Since T^* has SVEP, it then follows by Corollary 2.5 of [1] that $\sigma(T) = \sigma_a(T)$, so $\lambda \in \text{iso}\sigma_a(T)$ and hence a left pole for T . In particular, $T - \lambda$ is left Drazin invertible and hence $a(T - \lambda) < \infty$. By [7, Theorem 2.5] we know that $\lambda - f(T)$ is semi B-Fredholm, i.e., there exists a natural number $n \in \mathbb{N}$ such that $\Re(T - \lambda)^n$ is closed and the restriction $T - \lambda|_{\Re(T - \lambda)^n}$ is semi-Fredholm, in particular $T - \lambda$ is quasi-Fredholm. The SVEP for T^* implies that $d(T - \lambda) < \infty$. Hence λ is a pole of the resolvent of T . This proves that T is polaroid. By Lemma 3.11 of [8], $f(T)$ is polaroid and since by [1, Theorem 2.40], $f(T^*)$ has SVEP, the assertion follows now from part (i) of Theorem 3.4.

(ii) Suppose that T has SVEP and T is right polaroid. The SVEP of T entails $\sigma_s(T) = \sigma(T)$, see Corollary 2.5 of [1]. Let $\lambda \in \text{iso}\sigma(T)$. Then $\lambda \in \sigma_s(T)$ and hence is a right pole of T . Therefore, $d(T - \lambda) < \infty$. On the other hand, since $T - \lambda$ is right Drazin invertible, then $T - \lambda$ is semi B-Fredholm, in particular $T - \lambda$ is quasi-Fredholm. The SVEP for T at λ entails that $a(t - \lambda) < \infty$. Consequently, λ is a pole of the resolvent of T and hence T is polaroid. By Lemma 3.11 of [8], $f(T^*)$ is polaroid and since $f(T)$ has SVEP, the assertion follows now from part (ii) of Theorem 3.4. □

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