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NONUNIFORM WAVELET PACKET BASES FOR THE SPACES $L^p(\mathbb{R})$ AND $H^1(\mathbb{R})$

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Abstract: In this paper, we prove the results on the existence of unconditional nonuniform wavelet packet bases for the spaces $L^p(\mathbb{R})$, $1 < p < \infty$ and $H^1(\mathbb{R})$ based on the approach similar to that of Meyer and Coifman. Certain results are obtained in this direction by assuming only that the nonuniform wavelet packets ω_n and its derivatives ω_n' have a common radial decreasing L^1–majorant function.

1. Introduction

In his pioneering paper, Mallat [15] first formulated a new and remarkable idea of multiresolution analysis (MRA) which deals with a general formalism for the construction of an orthonormal basis of wavelet bases. Mathematically, an MRA consist of a sequence of embedded closed subspaces, $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ such that $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$. Furthermore, there exists an element $\varphi \in V_0$ such that the collection of integer translates of function φ, $\{\varphi(x-k) : k \in \mathbb{Z}\}$ represents a complete orthonormal system for V_0. The function φ is called the *scaling function* or the *father wavelet*. Recently, the idea of MRA and wavelets have been generalized in many different settings, for example, one can replace the dilation factor 2 by an

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integer $M \geq 2$ and in higher dimensions, it can be replaced by a dilation matrix A, in which case the number of wavelets required is $|detA| - 1$. But in all these cases, the translation set is always a group. In the two papers [8,9], Gabardo and Nashed considered a generalization of Mallat's [15] celebrated theory of MRA based on spectral pairs, in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. More precisely, this set is of the form $\Lambda = \{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N - 1, r$ is an odd integer relatively prime to N. They call this a *nonuniform multiresolution analysis* (NUMRA). Moreover, they have provided the necessary and sufficient conditions for the existence of associated wavelets in $L^2(\mathbb{R})$. Later on, Gabardo and Yu [10,11] continued their research in this non-standard setting and gave the characterization of all nonuniform wavelets associated with nonuniform multiresolution analysis.

It is well-known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^\wedge$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman *et al.*[7] constructed univariate orthogonal wavelet packets. The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbb{R})$, which can be searched in real time for the best expansion with respect to a given application. Chui and Li [5] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. Later on, Behera [2] constructed nonuniform wavelet packets associated with nonuniform multiresolution analysis. He proved lemmas on the so-called splitting trick and several theorems concerning the construction of nonuniform wavelet packets on $\mathbb{R}$. Other notable generalizations are the orthogonal p-wavelet

packets and the p-wavelet frame packets related to the Walsh polynomials [17,18] and the M-band framelet packets [19].

It is a part of the general wisdom that wavelet bases are unconditional bases in $L^p(\mathbb{R})$ for $1 < p < \infty$. Results of this type are proven in almost every book on wavelets, see, eg., [14,16]. In all those places, however, some assumptions on the smoothness of the wavelets are used. But, Gripenberg [12] introduced the subject of unconditionality of wavelet bases for Lebesgue spaces $L^p(\mathbb{R}),\ 1 < p < \infty$ which uses no smoothness of the wavelets. Later on, Wojtaszczyk [22] improved the results of Gripenberg [12] and Chang [4], and proved that some unimodular wavelets also yields unconditional bases in $L^p(\mathbb{R}),\ 1 < p < \infty$. The constructive proofs of the unconditional basis for $H^1(\mathbb{R})$ have been given by Carleson [3] and Wojtaszczyk [21], where the later author has given an example of an unconditional basis for the Hardy space $H^1(\mathbb{R})$ as the Franklin system. In fact, a large class of wavelets which have unconditional basis for the Hardy spaces $H^1(\mathbb{R})$ was discovered by Meyer [16] and recently, Khalil *et al.*[1] have generalized these results to the stationary wavelet packets. It was Strömberg [20] who first discovered unconditional bases for spaces $H^1(\mathbb{R})$ and $L^p(\mathbb{R}),\ 1 < p < \infty$, and they are spline systems of higher order.

Motivated and inspired by the importance of nonuniform wavelet packets, in the present paper, we prove the results on the existence of unconditional nonuniform wavelet packet bases for spaces $H^1(\mathbb{R})$ and $L^p(\mathbb{R}),\ 1 < p < \infty$ based on the approach similar to that of Meyer [16] and Coifman [6].

2. Preliminaries

Definition 2.1. Let N be an integer, $N \geq 1$, and $\Lambda = \{0, r/N\} + 2\mathbb{Z}$, where r is an odd integer relatively prime to N with $1 \leq r \leq 2N-1$. A sequence $\{V_j : j \in \mathbb{Z}\}$ of

closed subspaces of $L^2(\mathbb{R})$ is called a nonuniform multiresolution analysis (NUMRA) associated with Λ if the following conditions are satisfied:

(2.1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(2.2) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(2.3) $f \in V_j$ if and only if $f(2N.) \in V_{j+1}$,

(2.4) there exists a function φ in V_0 such that the system of functions $\{\varphi(. - \lambda)\}_{\lambda\in\Lambda}$ forms an orthonormal basis for subspace V_0.

The function φ whose existence is asserted in (2.4) is called a *scaling function* of the given NUMRA.

It is worth noting that, when $N = 1$, one recovers from the definition above the standard definition of a one-dimensional multiresolution analysis with dilation factor equal to 2. When, $N > 1$, the dilation factor of $2N$ ensures that $2N\Lambda \subset 2\mathbb{Z} \subset \Lambda$. Equation (2.3) implies that

$$\varphi\left(\frac{x}{2N}\right) = \sum_{\lambda\in\Lambda} a_\lambda\, \varphi(x - \lambda), \tag{2.5}$$

where $\sum_{\lambda\in\Lambda} |a_\lambda|^2 < \infty$.

Now, we consider W_0 the orthogonal complement of V_0 on V_1, i.e.,

$$V_1 = V_0 \oplus W_0.$$

If $\psi_1, \psi_2, ..., \psi_{2N-1}$ are the functions in W_0, then for $\ell = 0, 1, ..., 2N - 1$, there exists sequences $\left\{a_\lambda^\ell\right\}_{\lambda\in\Lambda}$ satisfying $\sum_{\lambda\in\Lambda} |a_\lambda^\ell|^2 < \infty$ such that

$$\frac{1}{2N}\psi_\ell\left(\frac{x}{2N}\right) = \sum_{\lambda\in\Lambda} a_\lambda^\ell\, \varphi(x - \lambda). \tag{2.6}$$

Now, consider

$$\hat{\psi}_\ell(2N\xi) = m_\ell(\xi)\,\hat{\varphi}(\xi) \tag{2.7}$$

where the functions $m_\ell(\xi) = \sum_{\lambda\in\Lambda} a_\lambda^\ell\, e^{-2\pi i\lambda\xi}$ are locally L^2. Since $\Lambda = \{0, r/N\}+2\mathbb{Z}$, we can write that

$$m_\ell(\xi) = m_\ell^1(\xi) + e^{-2\pi i\xi r/N}\, m_\ell^2(\xi),\ \ell = 0, 1, ..., 2N-1, \tag{2.8}$$

where m_ℓ^1 and m_ℓ^2 are locally L^2, $1/2-$periodic functions.

In this case $\{\psi_1, \psi_2, ..., \psi_{2N-1}\}$ is a set of basic wavelets associated with a scaling function φ. It is easy to show that $\{\psi_\ell(x-\lambda) : 1 \le \ell \le 2N-1\}$ is an orthonormal basis of W_0. An obvious rescaling shows that

$$\big\{\psi_{\ell,j,\lambda} = (2N)^{j/2}\psi_\ell\big((2N)^j x - \lambda\big) : 1 \le \ell \le 2N-1,\, \lambda \in \Lambda\big\}$$

is an orthonormal basis of W_j. Since $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$, the collection

$$\{\psi_{\ell,j,\lambda} : j \in \mathbb{Z},\, \lambda \in \Lambda,\ 1 \le \ell \le 2N-1,\}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

We, now, define ω_n for each integer $n \ge 0$ as follows. Suppose that for $p \ge 0$, ω_p is already defined. Then, define basic nonuniform wavelet packets ω_{q+2Np}, $0 \le q \le 2N-1$, by

$$\omega_{q+2Np}(x) = \sum_{\lambda\in\Lambda} (2N) a_\lambda^q \omega_p(2Nx - \lambda), \tag{2.9}$$

where $\sum_{\lambda\in\Lambda} |a_\lambda^q|^2 < \infty$.

Clearly, the set $\{\omega_n(x-\lambda) : \lambda \in \Lambda,\ n = 0, 1, ...\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Corresponding to some orthonormal scaling function $\varphi = \omega_0$, the family of nonuniform wavelet packets ω_n defines a family of subspaces of $L^2(\mathbb{R})$ as follows:

$$(2.10)\ U_j^n = \text{span}\Big\{(2N)^{j/2}\omega_n\big((2N)^j x - \lambda\big) :\ \lambda \in \Lambda\Big\};\ j \in \mathbb{Z},\ n = 0, 1, 2, ...$$

Since $\omega_0 = \varphi$ is the scaling function and ω_n, $1 \leq n \leq 2N-1$, are the nonuniform wavelet packets, we observe that

$$U_j^0 = V_j, U_j^1 = W_j = \bigoplus_{r=1}^{2N-1} U_j^r,\ j \in \mathbb{Z}$$

so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$, can be written as

$$(2.11) \qquad U_{j+1}^0 = U_j^0 \oplus U_j^1 = \bigoplus_{r=0}^{2N-1} U_j^r.$$

A generalization of this result for other values of $n = 1, 2, ..., $ can be written as

$$(2.12)\ U_{j+1}^n = \bigoplus_{r=0}^{2N-1} U_j^{r+2Nn},\ j \in \mathbb{Z}.$$

Lemma 2.2[2]. If $j \geq 0$, then

$$W_j = \bigoplus_{r=1}^{2N-1} U_j^r = \bigoplus_{r=2N}^{(2N)^2-1} U_{j-1}^r = ... = \bigoplus_{r=(2N)^m}^{(2N)^{m+1}-1} U_{j-m}^r,\ m \leq j$$

$$= \bigoplus_{r=(2N)^j}^{(2N)^{j+1}-1} U_0^r,$$

where U_j^n is defined in (2.10). Using this decomposition, we get the nonuniform wavelet packets $\omega_{r,j,\lambda}$, decomposition of subspaces W_j, $j \geq 0$. Therefore, for any

function $f \in L^2(\mathbb{R})$, we have

$$f(x) = \sum_{n=(2N)^m}^{(2N)^{m+1}-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} C_{r,n,\lambda}\, \omega_{r,j,\lambda}(x),$$

where $r = j - m$, $m = 0$ if $j < 0$ and $m = 0, 1, 2, ..., j$ if $j \geq 0$; will be a nonuniform wavelet packet expansion of f and $C_{r,n,\lambda}$ the wavelet packet coefficients, defined as

$$C_{r,n,\lambda} = \langle f, \omega_{r,j,\lambda}\rangle.$$

Let P_j and Q_j, respectively be the orthogonal projections onto the spaces V_j and W_j with the kernels $P_j(x, y)$ and $Q_j(x, y)$, defined as follows:

$$P_j(x, y) = \sum_{\lambda\in\Lambda} \varphi_{j,\lambda}(x)\, \overline{\varphi_{j,\lambda}(y)}, \tag{2.13}$$

where $\varphi_{j,\lambda}(x) = (2N)^{j/2}\varphi\big((2N)^j x - \lambda\big)$ and

$$Q_j(x, y) = \sum_{\ell=0}^{2N-1} \sum_{\lambda\in\Lambda} \psi_{\ell,j,\lambda}(x)\, \overline{\psi_{\ell,j,\lambda}(y)}. \tag{2.14}$$

In the light of $V_{j+1} = V_j \oplus W_j$, $P_j(x, y)$ can be written as

$$P_j(x, y) = \sum_{m<j} Q_m(x, y) = \sum_{\ell=0}^{2N-1} \sum_{m<j} \sum_{\lambda\in\Lambda} \psi_{\ell,j,\lambda}(x)\, \overline{\psi_{\ell,j,\lambda}(y)}. \tag{2.15}$$

Now, we consider a projection Q_j^n onto U_j^n with kernel $Q_j^n(x, y)$ defined as

$$Q_j^n(x, y) = \sum_{\lambda\in\Lambda} \omega_{j,n,\lambda}(x)\, \overline{\omega_{j,n,\lambda}(y)}\ ;\ j \in \mathbb{Z},\ n = 0, 1, 2, ... \tag{2.16}$$

where $\omega_{j,n,\lambda}$ are the nonuniform wavelet packets.

Lemma 2.3[14]. For a basis $\{x_j : j \in \mathbb{N}\}$ of a Banach space $(\mathbb{B}, \|.\|)$ the following statements are equivalent:

(i): $\{x_j : j \in \mathbb{N}\}$ is an unconditional basis for $\mathbb{B}$.

(ii): There exists a constant $C > 0$ such that $\|S_\beta(x)\| \leq C\|x\|$ for all sequences $\beta = \{\beta_j\}_{j\in\mathbb{N}}$ with $|\beta_j| \leq 1$, where $S_\beta(x) = \sum_{j\in\mathbb{N}} \beta_j f_j(x)x_j$, for all $x \in \mathbb{B}$, and $f_j's$ are the coefficient functionals.

(iii): There exists a constant $C > 0$ such that $\|S_\beta(x)\| \leq C\|x\|$ for all finitely non-zero sequences $\beta = \{\varepsilon_j\}_{j\in\mathbb{N}}$ with $\varepsilon_j = \pm 1$.

(iv): There exists a constant $C > 0$ such that $\|S_\varepsilon(x)\| \leq C\|x\|$ for all sequences $\varepsilon = \{\beta_j\}_{j\in\mathbb{N}}$ with $\beta_j = 1$ or 0.

Definition 2.4[14]. For a function f defined on $\mathbb{R}$, we say that a bounded function $E : [0,\infty) \longrightarrow \mathbb{R}^+$ is a radial decreasing L^1 -majorant of f if $|f(x)| \leq E(|x|)$ and E satisfies the following conditions:

$$\begin{cases} (i) E \in L^1[0,\infty), \\ (ii) E \text{ is decreasing}, \\ (iii) E(0) < \infty. \end{cases} \tag{2.17}$$

Lemma 2.5[13]. Let E be the function on $[0,\infty)$ satisfying the conditions of (2.17). Then

$$\sum_{\lambda\in\Lambda} E(|x-\lambda|)E(|y-\lambda|) \leq CE\left(\frac{|x-y|}{2N}\right), \text{ for all } x, y \in \mathbb{R}, \text{ and } N \geq 1.$$

where C is a constant depending on E.

Lemma 2.6[13]. A Calderon-Zygmund operator T is a bounded linear operator on $L^2(\mathbb{R})$ such that

$$(Tf)(x) = \int_{\mathbb{R}} K(x,y)f(y)\,dy,$$

where $x \notin \operatorname{supp}(f)$ and the kernel K is a jointly measurable function satisfying

$$\big|K(x,y)\big| \leq \frac{C_1}{|x-y|}; \tag{2.18}$$

$$\big|K(x_0,y) - K(x,y)\big| \leq \frac{C_2|x-x_0|}{|x-y|^2} \text{ if } |x-x_0| \leq \frac{1}{2}|x-y|; \tag{2.19}$$

$$\big|K(x,y_0) - K(x,y)\big| \leq \frac{C_3|y-y_0|}{|x-y|^2} \text{ if } |y-y_0| \leq \frac{1}{2}|x-y|. \tag{2.20}$$

Lemma 2.7[13]. Let T be a Calderon-Zygmund operator such that

$$\int_{\mathbb{R}} Tf(x)\,dx = 0 \text{ and } \int_{\mathbb{R}} T^*f(x)\,dx = 0,$$

whenever $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} f(x)\,dx = 0$, where T^* is the dual of T. Then, T extends to a bounded operator on $H^1(\mathbb{R}), BMO(\mathbb{R})$ and on $L^p(\mathbb{R}),\ 1 < p < \infty$, with operator norm depending only on $\|T\|_{L^2(\mathbb{R})}$ and the constants involved in the inequalities (2.18), (2.19) and (2.20).

3. Main Results

To study that the nonuniform wavelet packets forms an unconditional basis for $H^1(\mathbb{R})$ and $L^p(\mathbb{R}),\ 1 < p < \infty$, we first define an operator T_β by

$$T_\beta f = \sum_{n=(2N)^m}^{(2N)^{m+1}-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \beta_{r,n,\lambda} \langle f, \omega_{r,n,\lambda}\rangle\, \omega_{r,n,\lambda}(x) \tag{3.1}$$

in $H^1(\mathbb{R})$ and in $L^p(\mathbb{R}),\ 1 < p < \infty$; where $r = j - m$, $m = 0$ if $j \leq 0$ and $m = 0, 1, 2, ..., j$ if $j > 0$. Suppose $\beta = \{\beta_{r,n,\lambda}\}$ is a sequence such that $\beta_{r,n,\lambda} = 1$ for finite

number of indices and $\beta_{r,n,\lambda} = 0$ for remaining indices. Then in order to study the boundedness of this operator T_β, we write it in the integral form as

$$T_\beta f(x) = \int_{\mathbb{R}} K_\beta(x,y) f(y)\, dy, \tag{3.2}$$

where

$$K_\beta(x,y) = \sum_{n=(2N)^m}^{(2N)^{m+1}-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \beta_{r,n,\lambda}\, \omega_{r,n,\lambda}(x), \overline{\omega_{r,n,\lambda}(y)}. \tag{3.3}$$

Suppose that the wavelet packets $\{\omega_n : n \geq 0\}$ are bounded by a radial decreasing L^1-majorant E, then by Lemma 2.5, we obtain

$$\begin{aligned}
\big|K_\beta(x,y)\big| &\leq \sum_{n=(2N)^m}^{(2N)^{m+1}-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} (2N)^r \big|\omega_n\big((2N)^r x - \lambda\big)\, \omega_n\big((2N)^r y - \lambda\big)\big| \\
&\leq \sum_{n=(2N)^m}^{(2N)^{m+1}-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} (2N)^r E\big(\big|(2N)^r x - \lambda\big|\big)\, E\big(\big|(2N)^r y - \lambda\big|\big) \\
&\leq \sum_{n=(2N)^m}^{(2N)^{m+1}-1} \sum_{j\in\mathbb{Z}} (2N)^r C E\left(\frac{(2N)^r |x-y|}{2N}\right) \\
&= C \sum_{j\in\mathbb{Z}} (2N)^m (2N)^r E\left((2N)^{r-1}|x-y|\right) \\
&= C \sum_{j\in\mathbb{Z}} (2N)^j E\left((2N)^{r-1}|x-y|\right)
\end{aligned} \tag{3.4}$$

where C depends only on E and where $r = j - m$, $m = 0$ if $j \leq 0$ and $m = 0, 1, 2, ..., j$ if $j > 0$.

Theorem 3.1. Let ω_n be nonuniform wavelet packets such that ω_n and ω_n' (derivative of ω_n) have a common radial decreasing L^1-majorant E, for all n, satisfying

$$\int_{\mathbb{R}} sE(s)\,ds < \infty.$$

Then, the operator T_β defined by (3.2) and (3.3) is bounded in $H^1(\mathbb{R})$ and $L^p(\mathbb{R})$ $1 < p < \infty$, with norm bounded by a constant independent of the finitely non-zero sequence β consisting of zeros and ones.

Proof. Since the system $\{\omega_{r,n,\lambda}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, so it easy to verify that the operator T_β is bounded, i.e.,

$$\|T_\beta f\|^2_{L^2(\mathbb{R})} = \sum_{n=(2N)^m}^{(2N)^{m+1}-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\beta_{r,n,\lambda} \langle f, \omega_{r,n,\lambda}\rangle\right|^2$$

$$\leq \sum_{n=(2N)^m}^{(2N)^{m+1}-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\langle f, \omega_{r,n,\lambda}\rangle\right|^2 = \|f\|^2_{L^2(\mathbb{R})},$$

where $r = j - m$, $m = 0$ if $j \leq 0$ and $m = 0, 1, 2, ..., j$ if $j > 0$.

For any $f \in L^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} T_\beta f(x)\,dx = 0 = \int_{\mathbb{R}} T^*_\beta f(x)\,dx,$$

since $T_\beta f$ and $T^*_\beta f$ are finite linear combinations of the $\omega_{r,n,\lambda}$'s. Moreover, we have

$$0 = \hat{\omega}_n(0) = \int_{\mathbb{R}} \omega_n(x)\,dx, \text{for } n \geq 1. \text{(see [1])}$$

Now, in order to prove the theorem, it is required to show that T_β is a Calderon-Zygmund operator and then the theorem will follow by using Lemma 2.7. Therefore,

it is sufficient to show that K_β satisfies the conditions (2.18), (2.19) and (2.20). To prove (2.18), we use (3.4) and obvious estimates, to obtain

$$\left|K_\beta(x,y)\right| \leq C\sum_{j\in\mathbb{Z}}(2N)^j E\left((2N)^{j-p-1}|x-y|\right)$$

$$\leq C\sum_{j=-\infty}^{0}(2N)^j E\left((2N)^{j-1}|x-y|\right) + C\sum_{j=1}^{\infty}(2N)^j E\left((2N)^{j-m-1}|x-y|\right). \tag{3.5}$$

Now, we decompose W_j spaces for some $j = M$, for sufficiently large M. Then, all W_j spaces, for which $j \leq M$, will decompose up to the last formula in Lemma 2.2 and other W_j spaces, for which $j > M$, will decompose according to intermediate formula in the same Lemma 2.2. So inequality (3.5) takes the form

$$\left|K_\beta(x,y)\right| \leq C\sum_{j=-\infty}^{0}(2N)^j E\left((2N)^{j-1}|x-y|\right) + C\sum_{j=1}^{M}(2N)^j E\left((2N)^{j-1}|x-y|\right)$$

$$+C\sum_{j=M+1}^{\infty}(2N)^j E\left((2N)^{j-M-1}|x-y|\right)$$

$$\leq C\sum_{j=-\infty}^{0}(2N)^j E\left((2N)^{j-M-1}|x-y|\right) + C\sum_{j=1}^{M}(2N)^j E\left((2N)^{j-M-1}|x-y|\right)$$

$$+C\sum_{j=M+1}^{\infty}(2N)^j E\left((2N)^{j-M-1}|x-y|\right)$$

$$= C\sum_{j=-\infty}^{\infty}(2N)^j E\left((2N)^{j-M-1}|x-y|\right)$$

$$\leq (2N)C\int_0^\infty E\left((2N)^{-M-1}t|x-y|\right)\,dt$$

$$= \frac{C(2N)^{M+2}}{|x-y|}\|E\|_{L^1(0,\infty)}.$$

To prove (2.19), we assume that $x < x_0$, then we shall show

$$\Big|\frac{\partial}{\partial x}K_\beta(x,y)\Big| \le \frac{C}{|x-y|^2} \text{for } y \in \mathbb{R}. \tag{3.6}$$

It is easy to see that the inequality (3.6) implies (2.19). To see this we apply Mean value Theorem to obtain a point $x' \in (x_0, x)$ such that

$$\big|K_\beta(x_0,y) - K_\beta(x,y)\big| \le |x_0 - x|\Big|\frac{\partial K_\beta(x',y)}{\partial x}\Big| \le \frac{C|x_0-x|}{|x'-y|^2}.$$

Observe that (2.19) implies that $y \notin (x_0, x)$. If $y \ge x$, it is clear that

$$|x'-y| \ge |x-y| \ge \frac{1}{2}|x-y|.$$

If $y \le x_0$, we use (2.19) to obtain

$$|x'-y| \ge |x_0-y| \ge |x-y| - |x-x_0| \ge \frac{1}{2}|x-y|.$$

Hence,

$$\big|K_\beta(x_0,y) - K_\beta(x,y)\big| \le \frac{4C|x_0-x|}{|x-y|^2}$$

provided

$$|x-x_0| \le \frac{1}{2}|x-y|.$$

Thus, we need to show (3.6), we use Lemma 2.5 and the fact that E is decreasing to obtain

$$\begin{aligned}\Big|\frac{\partial}{\partial x}K_\beta(x,y)\Big| &= \left|\sum_{n=(2N)^m}^{(2N)^{m+1}-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\beta_{r,n,\lambda}(2N)^{2r}\omega_n'\big((2N)^r x-\lambda\big)\,\overline{\omega_n\big((2N)^r y-\lambda\big)}\right| \\ &\le \sum_{n=(2N)^m}^{(2N)^{m+1}-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}(2N)^{2r}E\big(\big|(2N)^r x-\lambda\big|\big)\,E\big(\big|(2N)^r y-\lambda\big|\big) \\ &\le C\sum_{j\in\mathbb{Z}}(2N)^{2r+m}\sum_{\lambda\in\Lambda}E\big(\big|(2N)^r x-\lambda\big|\big)\,E\big(\big|(2N)^r y-\lambda\big|\big)\end{aligned}$$

$$\leq C\sum_{j\in\mathbb{Z}}(2N)^{2j-m}E\left((2N)^{j-m-1}|x-y|\right)$$

$$\leq C\sum_{j=-\infty}^{0}(2N)^{2j-M}E\left((2N)^{j-1}|x-y|\right)+C\sum_{j=1}^{M}(2N)^{2j}E\left((2N)^{-1}|x-y|\right)$$

$$+C\sum_{j=M+1}^{\infty}(2N)^{2j-M}E\left((2N)^{j-M-1}|x-y|\right)$$

$$\leq C\sum_{j=-\infty}^{0}(2N)^{2j}E\left((2N)^{j-M-1}|x-y|\right)+C\sum_{j=1}^{M}(2N)^{2j}E\left((2N)^{j-M-1}|x-y|\right)$$

$$+C\sum_{j=M+1}^{\infty}(2N)^{2j}E\left((2N)^{j-M-1}|x-y|\right)$$

$$=C\sum_{j=-\infty}^{\infty}(2N)^{2j}E\left((2N)^{j-M-1}|x-y|\right)$$

$$\leq (2N)C\int_0^{\infty}tE\left((2N)^{-M-1}t|x-y|\right)\,dt$$

$$=(2N)C\int_0^{\infty}\frac{(2N)^{M+1}}{|x-y|}sE(s)\frac{(2N)^{M+1}}{|x-y|}\,ds$$

$$=\frac{(2N)^{2M+3}}{|x-y|^2}\int_0^{\infty}sE(s)\,ds.$$

This proves (3.6) and, consequently, (2.19) follows. Inequality (2.20) follows from a similar argument as in (2.19). Hence T_β is a Calderon-Zygmund operator. By using Lemma 2.7, the proof of the theorem follows. □

□

Theorem 3.2. Let ω_n be nonuniform wavelet packets for $L^2(\mathbb{R})$ such that ω_n and ω_n' have a common radial decreasing L^1-majorant E satisfying

$$\int_0^{\infty}sE(s)\,ds<\infty.$$

Then, the system

$$\left\{\omega_{r,n,\lambda} : r = j - m;\ n = (2N)^m, (2N)^m + 1, ..., (2N)^{m+1} - 1,\ j \in \mathbb{Z}, \lambda \in \Lambda\right\}$$

is an unconditional basis for $L^p(\mathbb{R})$, $1 < p < \infty$, $m = 0$ if $j \leq 0$ and $m = 0, 1, 2, ..., j$ if $j > 0$.

Proof. We start by showing that the system considered is a basis for $L^p(\mathbb{R})$, $1 < p < \infty$. To this let, $S_{u,v}f$ be the "rectangular" partial sum of the nonuniform wavelet packet expansions of f, i.e.,

$$S_{u,v}f = \sum_{n=(2N)^m}^{(2N)^{m+1}-1} \sum_{|j|<u} \sum_{|\lambda|<v} \langle f, \omega_{r,n,\lambda}\rangle\, \omega_{r,n,\lambda}, \tag{3.7}$$

where $f \in L^p(\mathbb{R})$, $1 < p < \infty$. This operator is well defined in view of Theorem 3.1. Now, we show that for given $f \in L^p(\mathbb{R})$, $1 < p < \infty$ and $\varepsilon > 0$, we can find u and v large enough so that

$$\|f - S_{u,v}f\|_{L^p(\mathbb{R})} < \varepsilon.$$

Let $C = \sup \|T_\beta\| < \infty$, where T_β's are the operators defined in Theorem 3.1 and the supremum is taken over all admissible sequence $\beta = \{\beta_{r,n,\lambda}\}$ considered in Theorem 3.1. Since $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, we can find $g \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ such that

$$\|f - g\|_{L^p(\mathbb{R})} < \frac{\varepsilon}{C+3}.$$

Thus, we can write

$$\|f - S_{u,v}f\|_{L^p(\mathbb{R})} \leq \|f - g\|_{L^p(\mathbb{R})} + \|g - S_{u,v}g\|_{L^p(\mathbb{R})} + \|S_{u,v}(g - f)\|_{L^p(\mathbb{R})}. \tag{3.8}$$

The last summand on the right hand side of (3.8) is smaller than $\frac{\varepsilon C}{C+3}$ in view of Theorem 3.1. Now, we estimate $\|g - S_{u,v}g\|_{L^p(\mathbb{R})}$ for $g \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$. By duality,

and the density of $L^2(\mathbb{R})\cap L^{p'}(\mathbb{R})$ in $L^{p'}(\mathbb{R})$, where $(\frac{1}{p}+\frac{1}{p'})$, we can find $h \in L^2(\mathbb{R})\cap L^{p'}$ such that

$$\|g - S_{u,v}f\|_{L^p(\mathbb{R})} \leq \left| \int_{\mathbb{R}} \{g(x) - S_{u,v}g(x)\}\, \overline{h(x)}\, dx \right| + \frac{\varepsilon}{C+3}. \tag{3.9}$$

Using the Schwarz inequality, we deduce that

$$\left| \int_{\mathbb{R}} \{g(x) - S_{u,v}g(x)\}\, \overline{h(x)}\, dx \right| = \left| \int_{\mathbb{R}} g(x) \left\{ \overline{h(x)} - \overline{S_{u,v}h(x)} \right\} dx \right|$$

$$\leq \|g\|_{L^2(\mathbb{R})} < \|h - S_{u,v}h\|_{L^2(\mathbb{R})}.$$

Since $\{\omega_{r,n,\lambda}\}$ is an unconditional basis for $L^2(\mathbb{R})$, we can find u and v large enough so that

$$\|h - S_{u,v}h\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{\|g\|_{L^2(\mathbb{R})}(C+3)}.$$

Hence

$$\|f - S_{u,v}f\|_{L^p(\mathbb{R})} \leq \frac{\varepsilon}{C+3} + \frac{\varepsilon}{C+3} + \frac{\varepsilon}{C+3} + \frac{\varepsilon C}{C+3} = \varepsilon.$$

From the orthonormality of the system $\{\omega_{r,n,\lambda}\}$, it follows that the representation

$$f = \sum_{n=(2N)^m}^{(2N)^{m+1}-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} C_{r,n,\lambda}\, \omega_{r,n,\lambda} \tag{3.10}$$

with convergence in $L^p(\mathbb{R})$, $1 < p < \infty$ is unique. Now, multiplying both sides by $\overline{\omega_{r,n,\lambda}}$ and integrating, we obtain $C_{r,n,\lambda} = \langle f, \omega_{r,n,\lambda}\rangle$. The unconditionality of the basis follows from Theorem 3.1 and Lemma 2.3. □ □

Theorem 3.3. Let ω_n be nonuniform wavelet packets for $L^2(\mathbb{R})$ such that ω_n and ω_n' have a common radial decreasing L^1-majorant E satisfying

$$\int_0^\infty sE(s)\,ds < \infty.$$

Then, the system

$$\left\{\omega_{r,n,\lambda} : r = j-m;\ n = (2N)^m, (2N)^m+1, ..., (2N)^{m+1}-1,\ j \in \mathbb{Z}, \lambda \in \Lambda\right\}$$

is an unconditional basis for $H^1(\mathbb{R})$, where $m = 0$ if $j \leq 0$ and $m = 0, 1, 2, ..., j$ if $j > 0$.

Proof. The proof of this theorem is similar to that of Theorem 3.2. For this, we need to show that the system under consideration is a basis for $H^1(\mathbb{R})$. Inequality (3.8) is true with L^p-norm replaced by H^1-norm and choosing g to be finite linear combination of atoms. Since $(H^1(\mathbb{R}))^* = BMO(\mathbb{R})$, we can find a bounded function $h \in BMO(\mathbb{R})$ such that (3.9) is true by replacing $\|.\|_{L^p(\mathbb{R})}$ by $\|.\|_{H^1(\mathbb{R})}$. By choosing sufficiently large M, we have

$$\|g - S_{u,v}g\|_{H^1(\mathbb{R})} \leq \left|\int_{\mathbb{R}} \{g(x) - S_{u,v}g(x)\}\,\chi_{[-M,M]}(x)\overline{h(x)}\,dx\right| + \frac{\varepsilon}{C+3}.$$

Observe that $\chi_{[-M,M]}h \in L^2(\mathbb{R})$ and, thus, the proof follows from the proof of Theorem 3.2.□

□

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